Numerical Simulation of Anderson Localization on a Randomly Disordered Rectangle

Landon Lehman

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1 Introduction

Anderson localization is a phenomenon that was first characterized by Philip Anderson in 1958. Anderson later won a Nobel Prize for his work. On a intuitive level, Anderson localization occurs when classical waves described by the wave equation or quantum-mechanical wavefunctions described by the Schrodinger equation become "trapped" or localized in a random medium. There is a certain level of randomness at which this occurs - if the randomness is below this level than Anderson localization will not be observed. A rigorous understanding of the mathematics behind Anderson localization has been elusive despite a large amount of research in the area.

2 Description of System

The purpose of this research is to gain a better understanding of Anderson localization through the use of a finite elements numerical simulation. We are specifically interested in describing the time-independent Schrodinger operator in $\mathbb{R}^2$ with Dirichlet boundary conditions:

$$-\Delta u + V(x)u = \lambda u$$

$$u\big|_{\partial\Omega} = 0$$

where Ω is an $a \times b$ rectangle in $\mathbb{R}^2$, and $V(x)$ is a random function of x in $[0, 1]$.

We split up the rectangle into an array of smaller rectangles (in this case $40 \times 40 = 1600$ rectangles). Each of these smaller rectangles is randomly assigned a value between 0 and V_{max}. V_{max} thus acts as an upper bound on the amount of disorder on the rectangular domain. V_{max} can be interpreted as a random potential matrix for the Schrodinger operator.

We define $N(\lambda)$ as the number of eigenvalues of the operator less than or equal to λ. We also define the localization coefficient $\alpha(u)$ of an eigenfunction u as follows:

$$\alpha(u) = \frac{\left(\int_\Omega u^2 d\Omega\right)^2}{\int_\Omega u^4 d\Omega}$$

The localization coefficient $\alpha(u)$ provides a measure of the existence area of the eigenfunction u.

3 Case When $V_{max} = 0$: Laplacian

When $V_{max} = 0$ we have the case of a Laplacian:

$$\Delta u = \lambda u$$

$$u\big|_{\partial\Omega} = 0$$

with eigenvalues

$$\lambda_{m,n} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2.$$

where m and n are positive integers. The corresponding eigenfunctions are

$$u_{m,n}(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right).$$

3.1 Asymptotic Behavior of N(λ)

For any eigenvalue λ, we can rearrange the expression for an eigenvalue to get

$$\left(\frac{n\pi}{a\sqrt{\lambda}}\right)^2 + \left(\frac{m\pi}{b\sqrt{\lambda}}\right)^2 = 1.$$

This is the equation for an ellipse in the mn plane with area

$$A_e = \frac{ab\lambda}{\pi}.$$

Now the eigenvalues less than or equal to λ must be within this ellipse. More precisely, they are the positive integer lattice points contained within A_e. The number of positive integer lattice points contained within the ellipse is less than or equal to $\frac{1}{4}A_e$ (because we are only interested in the quadrant with $m, n > 0$), i.e.

$$N(\Lambda) \leq \frac{ab\lambda}{4\pi}.$$

We have thus placed an upper bound on $N(\lambda)$. For a lower bound, notice that the ellipse intersects the m axis at $m = \frac{b\sqrt{\lambda}}{\pi}$ and intersects the n axis at $n = \frac{a\sqrt{\lambda}}{\pi}$. Now,

$$N(\lambda) = \sum_{n=1}^{\lfloor \frac{a\sqrt{\lambda}}{\pi}\rfloor} \left\lfloor b\sqrt{\frac{\lambda}{\pi^2} - \frac{n^2}{a^2}}\right\rfloor.$$

Compare $N(\lambda)$ to the area under the shape formed by moving the shape of $N(\lambda)$ up one unit on the m axis and to the right one unit on the n axis. This area A_+ is greater than or equal to A_e. More specifically,

$$A_+ = N(\lambda) + \left\lceil\frac{b\sqrt{\lambda}}{\pi}\right\rceil + \left\lceil\frac{a\sqrt{\lambda}}{\pi}\right\rceil - 1 \geq A_e.$$

Rearranging,

$$N(\lambda) \geq A_e - \left\lceil \frac{b\sqrt{\lambda}}{\pi} \right\rceil - \left\lceil \frac{a\sqrt{\lambda}}{\pi} \right\rceil + 1.$$

Since the ceiling function is always greater than or equal to its argument, we can write

$$N(\lambda) \geq A_e - \frac{b\sqrt{\lambda}}{\pi} - \frac{a\sqrt{\lambda}}{\pi} + 1.$$

$N(\lambda)$ is now bounded from above and below:

$$\frac{ab\lambda}{4\pi} \geq N(\lambda) \geq \frac{ab\lambda}{4\pi} - \frac{b\sqrt{\lambda}}{\pi} - \frac{a\sqrt{\lambda}}{\pi} + 1.$$

If we take the limit as λ approaches infinity and consider the ratio $N(\lambda)/\lambda$, we get

$$\frac{ab}{4\pi} \geq \frac{N(\lambda)}{\lambda} \geq \frac{ab}{4\pi}.$$

In other words, $N(\lambda)/\lambda$ is asympotically equal to $\frac{ab}{4\pi}$.
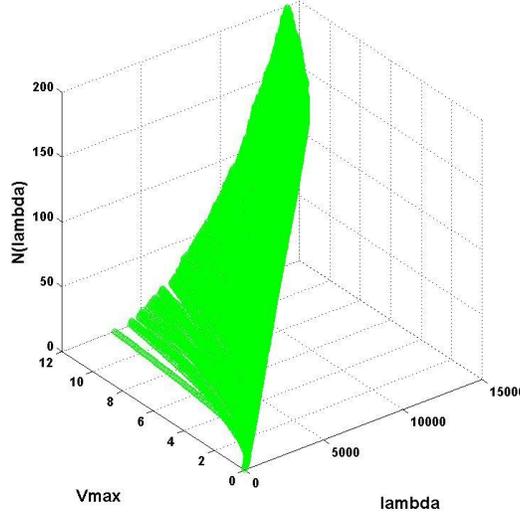
Figure 1: Plot 1

3.2 Localization Coefficient of Laplacian Eigenfunctions

The localization coefficient for eigenfunctions of the lapalcian on the $a \times b$ rectangle is a constant for all of the eigenfunctions:

$$\alpha(u) = \frac{\left(\int_0^b \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) \sin^2\left(\frac{m\pi y}{b}\right) dx dy \right)^2}{\int_0^b \int_0^a \sin^4\left(\frac{n\pi x}{a}\right) \sin^4\left(\frac{m\pi y}{b}\right) dx dy} = \frac{(ab/4)^2}{9ab/64} = \frac{4}{9}ab.$$

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The fact that the localization coefficents are all equal for the Laplacian means that there is no localization when $V_{max} = 0$.

4   Plots and Discussion

Plot 1 is a 3-dimensional plot of the Schrodinger operator on a 1×1.1 domain with eigenvalues (λ) on the x-axis, V_{max} on the y-axis, and $N(\lambda)$ on the z-axis. Code 1 (file included) is the Matlab code used to generate Plot 1.

Plot 2 is a 3-dimensional plot of the Schrodinger operator on a 1×1.1 domain with eigenvalues (λ) on the x-axis, V_{max} on the y-axis, and $\alpha(u)$ on the z-axis. Code 2 (file included) is the Matlab code used to generate Plot 2.
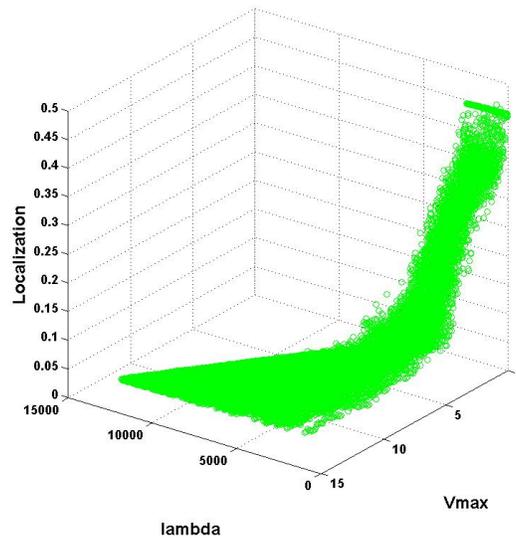
Figure 2: Plot 2

For both plots, the program used to generate the plot looped 120 times. Each time, the value of V_{max} is increased by 1/10. So the value of V_{max} reaches a maximum of 12 on each plot. The value of V_{max} multiplies each element in the random potential matrix. The random matrix has values between 0 and 2997.311. Also, each plot has the first 200 eigenvalues for each value of V_{max}.

4.1   Behavior at $V_{max} = 0$

The behavior of the plots at $V_{max} = 0$ verifies the calculations for $N(\lambda)$ and $\alpha(u)$ for the Laplacian. When $V_{max} = 0$ on Plot 1, the plot of $N(\lambda)$ vs. λ is a straight line. When $V_{max} = 0$ on Plot 2, α remains approximately a constant at a value of 0.45.

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4.2   Evolution with Increasing Disorder

As V_{max} increases on Plot 1, the plot of $N(\lambda)$ vs. λ looks increasingly like an increasing exponential function rather than a straight line. As V_{max} increases on Plot 2, the value of the localization constant α drops off very quickly to almost zero.

4.3   Conjecture of Behavior at Infinite Disorder

As V_{max} goes to infinity, it is expected that the localization coefficient $\alpha(u)$ will go to zero. α might actually approach zero more closely in reality than it does in the plot, because the use of the finite elements method to solve the boundary value problem The minimum value of α on the plot is about 0.0025, which may be close to a limiting value because the localization area cannot be smaller than then area of one of the finite element squares.