Localization of eigenfunctions of a one-dimensional elliptic operator

Marcel Filoche, Svitlana Mayboroda, and Brandon Patterson

ABSTRACT. The localization of vibrations is a widely observed, but little understood physical phenomenon. Roughly speaking, the effect of localization is a confinement of some eigenfunctions of an elliptic operator to a small portion of the original domain in the presence of irregularities of the boundary or of the coefficients of the underlying operator. Until recently, there have been essentially no mathematical results explaining such a behavior.

In the present paper the authors establish an asymptotic formula for the localization of eigenfunctions of the elliptic operator $L = -\frac{d}{dx}A(x)\frac{d}{dx}$ associated to a piecewise constant function A. Quite unexpectedly, this formula expresses the strength of localization purely as a function of $f(x_0)$, the value of the corresponding eigenfunction at the discontinuity point of the coefficients.

1. Introduction

Essentially any vibrating system, whether in acoustics, optics, mechanics, or quantum dynamics, displays localization of vibrations due to inhomogeneity of medium or geometric singularities of the underlying domain. In mathematical terms, *localization* means that some eigenfunctions may have a large amplitude in a small portion of the original domain and almost vanish in the remaining part. This effect is believed to be a consequence of the irregularities of the boundary of the domain or of the underlying elliptic operator, although it is not clear how particular irregularities affect the shape of the localization subregions, the difference of amplitude in different subregions, or the eigenvalues of the localized eigenfunctions.

A striking example of localization behavior has been observed in [**SHR97**] and ultimately was used to design a new type of noise abatement walls [**Wal**]. For illustration we display here two eigenfunctions of the Laplacian in two dimensional domains: with Dirichlet boundary conditions in Figure 1 and Neumann boundary data in Figure 2, taken from the works in [**SGM91, SG93, ERR$^+$99**] and [**SHR97, FAFS07**], respectively.

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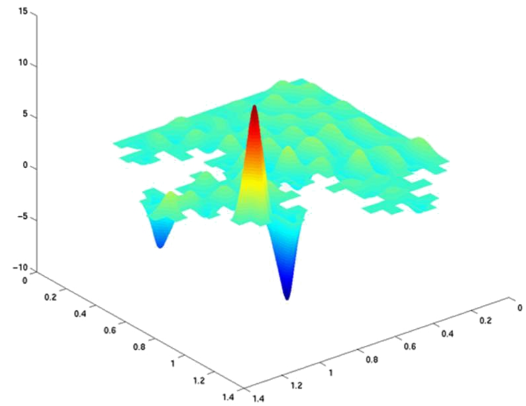

FIGURE 1. Eigenfunction of the Laplacian with Dirichlet data in a domain with fractal boundary and Dirichlet boundary condition [**ERR**$^{+}$**99**]. One observes that the eigenfunction is essentially localized to a small subregion of the entire domain.

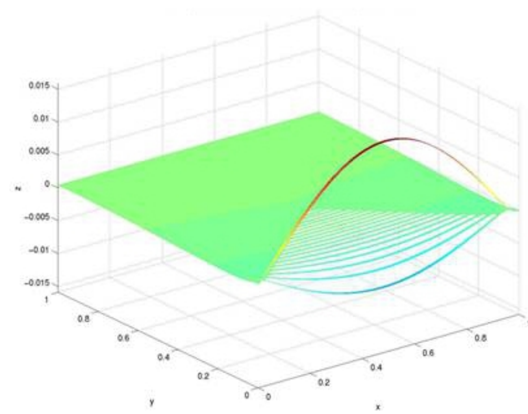

FIGURE 2. One localized eigenfunction of the Laplacian with Neumann data in a domain with parallel slits arranged in a triangular subregion [**FAFS07**]. The eigenfunction is almost entirely localized in one slit region.

This is just one of many instances of the localization phenomenon. In this paper we do not aim to survey an extended list of its appearances in science. One has to mention though the famous Anderson localization of quantum states of electrons [**And58**], which, in a certain form, can be seen as an outcome of an extremely rough inhomogeneous medium [**FM11**].

Despite wide interest and an enormous body of related work in physics and in engineering, from mathematical point of view the phenomenon of localization remains largely a mystery (see, e.g., [**FM09**] or the review of R. Strichartz in [**HS10**]). For instance, if one inspects the geometry in Figures 1 and 2 some naturally arising questions are: How does a eigenfunction choose its principal subregion? Why only

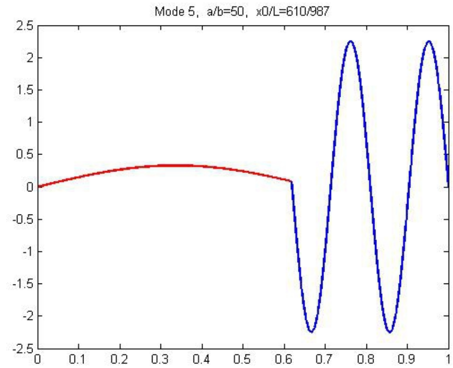

FIGURE 3. Localization in an inhomogeneous vibrating string: the eigenfunction is almost entirely concentrated in one subregion.

selected eigenfunctions are localized and can one predict which ones? What is the strength of localization?

In search of insight into localization phenomenon, in the present work we investigate the eigenfunctions of an inhomogeneous vibrating string comprised of two uniform pieces. The eigenvalue problem then reads

$$-\frac{d}{dx}A(x)\frac{d}{dx}f = \mu^2 f \text{ on } (0,L), \quad f(0)=0, \quad f(L)=0, \tag{1.1}$$

where $A = A(x)$, $x \in [0, L]$, is a piecewise constant function equal to a^2 on $[0, x_0)$ and equal to b^2 on $(x_0, L]$, $a, b \in \mathbb{R}$, $x_0 \in (0, L)$, (understood, as usually, in the weak sense). A sample eigenfunction of such a system exhibiting localization is displayed in Figure 3.

The focus of our interest is a degree to which $f = f_\mu$ defined by (1.1) is (or is not) uniformly distributed throughout interval $[0, L]$. It would be desirable to predict from the knowledge of a, b, x_0 which eigenfunctions will be essentially comprised to $(0, x_0)$ or (x_0, L) and which will be essentially uniform on $(0, L)$, in some quantitative way.

A standard measure of the strength of localization is the localization coefficient

$$\alpha_f := \left(\int_0^L f^2\,dx\right)^2 \left(\int_0^L f^4\,dx\right)^{-1}. \tag{1.2}$$

The coefficient α_f, sometimes also called *participation ratio*, is a measure of the degree of the confinement of an eigenfunction f to a subinterval of $[0, L]$. In a sense, it measures the size of a subinterval of $[0, L]$ where f is significantly different from zero. For instance, for a function f that is constant on a subinterval of $[0, L]$ of length ℓ and is equal to zero outside the subinterval, the localization coefficient is equal to ℓ.

The realistic situation is, of course, much more involved, but roughly speaking, α_f is the size of the "existence area" (in our case, the length of the interval) where f is "lives".

At first sight, an explicit formula for eigenfunctions of this boundary value problem and thus, the formula for their localization ratio seems a straightforward exercise. Its completion, however, yields essentially no insight or conjecture regarding the general pattern: the formula is an entangled trigonometric system, involving in a fairly complicated implicit fashion many parameters: the eigenvalue itself, the coefficients a and b, the juncture point x_0 etc. To be precise, modulo some exceptional cases, a direct computation yields

$$\alpha_f = \left(\frac{f^4(x_0)}{32} \left[\left(\frac{x_0}{\sin^4 X_a} \right) \frac{12X_a - 8\sin(2X_a) + \sin(4X_a)}{X_a} + \left(\frac{L - x_0}{\sin^4 X_b} \right) \frac{12X_b - 8\sin(2X_b) + \sin(4X_b)}{X_b} \right] \right)^{-1}. \tag{1.3}$$

where X_a and X_b are defined by

$$X_a = \frac{\mu x_0}{a}, \qquad X_b = \frac{\mu}{b}(L - x_0), \tag{1.4}$$

and the eigenvalues μ are determined by a relation

$$a \cot\left(\frac{\mu x_0}{a}\right) = -b \cot\left(\frac{\mu(L - x_0)}{b}\right)$$

(see Proposition 2.2).

Note that, in general, μ, f_μ and, thus, α_f satisfying the equations above can only be found numerically. Moreover, α_f depends on μ through an involved trigonometric relation (observe that $f(x_0) = f_\mu(x_0)$ is also linked to μ). Therefore, such results ultimately yield solely numerical solutions, which unfortunately do not provide much insight into dependence of localization from parameters of the initial string. Our goal is, instead, to describe the properties of α_f without calculating the eigenvalues of the boundary problem.

However, further numerical experiments revealed a greatly surprising property: independently of the eigenvalue, for any given eigenfunction f its localization strength α_f must be a simple function of $f(x_0)$ only, the value of the eigenfunction at the juncture point! (See, e.g., Figure 4).

The present manuscript provides a proof of this remarkable and quite unpredictable phenomenon. It turns out that the strength of localization of *any* eigenfunction f in $[0, L]$ can be indeed determined purely from the knowledge of the value of f at the juncture point, $f(x_0)$, *without knowledge of the corresponding eigenvalue* or any information about the behavior of f in the remaining interval. In other words, the value $f(x_0)$ essentially *entirely* determines how uniformly f is distributed on $[0, L]$.

The main result of the present paper is as follows.

THEOREM 1.1. *Let $L > 0$, $x_0 \in (0, L)$, and denote by $\mu > 0$ and $f = f_\mu$ the eigenvalues and eigenfunctions of $-\frac{d}{dx}A(x)\frac{d}{dx}$ with Dirichlet boundary data, i.e., the solutions of the boundary problem* (1.1). *Furthermore, let*

$$J_f := \frac{3}{2L}\left[1 + \rho(Lf^2(x_0) - 2)^2\right], \tag{1.5}$$

where ρ is a positive constant depending on L, a, b and x_0 only.

Then

$$\lim_{\mu \to +\infty} \alpha_f \times J_f = 1. \tag{1.6}$$

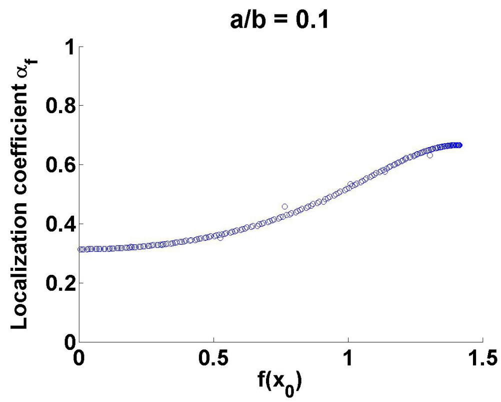

FIGURE 4. Plot of the points $(f(x_0), \alpha_f)$ for a large number of eigenfunctions, in the case where $a/b = 0.3$. One can observe that almost all these points gather around a simple curve. It suggests that α_f, the localization coefficient is almost entirely determined by the value of f at the point x_0.

In other words, asymptotically

$$\alpha_f \sim \frac{2L}{3}\left[1+\rho(Lf^2(x_0)-2)^2\right]^{-1}. \tag{1.7}$$

Note that the asymptotic relationship found between α_f and $f(x_0)$ only involves a simple fourth order polynomial.

Let us remark that even the existence of a simple asymptotic curve is quite remarkable. When one investigates the formula (1.3) in an attempt to connect α_f to $f(x_0)$, the obstacle is the presence of μ. In principle, the eigenvalue μ determines $f(x_0)$, but expressing μ as a function of $f(x_0)$ seems impossible, and moreover, one certainly can not a priori guarantee that α_f is uniquely defined by $f(x_0)$, i.e., that it can be viewed as a *function*.

Furthermore, we show in the end of Section 4 that the error of the asymptotic formula (1.6) is or the order C/μ, where C is a constant depending on a, b, x_0, L only. While the explicit value of C is not given, μ is growing fast and so the error is essentially negligible for practical applications. That is, for most practically relevant values of the parameters, α_f can be very well approximated by (1.7), for almost *all* eigenvalues μ.

The proof requires minimal mathematical background. However, it raises a series of conjectures for higher dimensional vibration systems which quickly lead into deep questions in advanced harmonic analysis and theory of elliptic PDE. For instance, given a two dimensional domain (for simplicity, a square), and an elliptic PDE with piecewise constant coefficients, can the localization properties of the eigenfunctions be deduced from the knowledge of the function at the "juncture surfaces" where the jumps of coefficients are located? What are the responsible "juncture surfaces" for variable coefficients which are not simply piecewise constant? What role plays the geometry of the domain?

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2. Preliminaries: solution of the eigenvalue problem

This section can be considered as a preparatory material. It presents a formal, weak and strong, statement of the boundary problem, the formula for eigenfunctions and calculation of the localization coefficients (1.2). This is done by the standard methods. However, we display the course of argument for the sake of completeness and to be able to mark some intermediate steps for future reference.

The study of the solution to the wave equation can be reduced to the study of the "stationary waves", i.e., the eigenfunctions of the corresponding elliptic operator. In the present setting the operator in question is $\mathcal{L} = -\frac{d}{dx}A\frac{d}{dx}$, where $A = A(x)$ is a piecewise constant function equal to a^2 on $[0, x_0)$ and equal to b^2 on $(x_0, L]$. Since A is discontinuous, the eigenvalue problem is to be understood in the weak sense, that is, one is looking for $\mu \in \mathbb{R}$ and $f \in \mathring{H}^1((0, L))$ such that

$$\int_0^L A(x)f'(x)\,\varphi'(x)\,dx = \mu^2 \int_0^L f(x)\,\varphi(x)\,dx, \ \forall\ \varphi \in C_0^\infty((0, L)), \tag{2.1}$$

where $\mathring{H}^1((0, L))$ is the Sobolev space of functions given by the closure of $C_0^\infty((0, L))$ in the norm $\|f\|_{H^1((0,L))} = \|f'\|_{L^2((0,L))}$. Note that the classical arguments (see, e.g., [**Eva10**]) guarantee that the eigenvalues of $\mathcal{L}$ are positive, which allows us to denote the eigenvalue by μ^2 in (2.1). We shall slightly abuse the terminology and refer to μ, $\mu > 0$, as eigenvalues of $\mathcal{L}$.

We shall not elaborate on weak solutions any further, as the explicit form of A readily provides a *strong* analogue of this boundary value problem, as follows. Find $\mu > 0$ and corresponding functions f, continuous on $[0, L]$, piecewise defined by

$$f(x) = \begin{cases} f_a(x), & x \in [0, x_0], \\ f_b(x), & x \in [x_0, L], \end{cases} \tag{2.2}$$

where $f_a \in C^2((0, x_0)) \cap C([0, x_0])$ and $f_b \in C^2((x_0, L)) \cap C([x_0, L])$ are such that

$$-a^2 f_a''(x) = \mu^2 f_a(x) \qquad 0 < x < x_0, \tag{2.3}$$

$$-b^2 f_b''(x) = \mu^2 f_b(x) \qquad x_0 < x < L, \tag{2.4}$$

subject to boundary conditions

$$f_a(0) = 0 \quad \text{and} \quad f_b(L) = 0, \tag{2.5}$$

and moreover, f_a has a left-hand side derivative at x_0, f_b has a right-hand side derivative at x_0 and they coincide:

$$-a^2\,\frac{df_a}{dx}(x_0) = -b^2\,\frac{df_b}{dx}(x_0). \tag{2.6}$$

One can easily check that (2.3)–(2.4) together with the compatibility condition (2.6) yield (2.1). Since f is clearly defined modulo a multiplicative constant, we shall impose in addition a customary normalization condition

$$\int_0^L f^2(x)\,dx = 1. \tag{2.7}$$

Proposition 2.1. Fix $L > 0$ and $x_0 \in (0, L)$. The eigenvalues and the eigenfunctions of $\mathcal{L}$ defined via (2.2)–(2.7) satisfy the following formulas.

All eigenvalues μ such that $\frac{\mu x_0}{\pi a} \notin \mathbb{Z}$ and $\frac{\mu(L-x_0)}{\pi b} \notin \mathbb{Z}$ can be determined as the roots of the equation

$$a \cot\left(\frac{\mu x_0}{a}\right) = -b \cot\left(\frac{\mu(L-x_0)}{b}\right). \tag{2.8}$$

The corresponding eigenfunctions are given by

$$f(x) = \begin{cases} f_a(x) = C_{2a} \sin\left(\frac{\mu x}{a}\right), & x \in [0, x_0], \\ f_b(x) = C_{2b} \sin\left(\frac{\mu(L-x)}{b}\right), & x \in [x_0, L], \end{cases} \tag{2.9}$$

where the coefficients C_{2a}, C_{2b} can be expressed explicitly via:

$$\begin{aligned} C_{2a}^2 =& \frac{4\mu}{\sin^2\left(\frac{\mu x_0}{a}\right)} \left[\frac{a}{\sin^2\left(\frac{x_0\mu}{a}\right)} \left(\frac{2x_0\mu}{a} - \sin\left(\frac{2x_0\mu}{a}\right)\right)\right. \\ & \left. + \frac{b}{\sin^2\left(\frac{\mu}{b}(L-x_0)\right)} \left(\frac{(L-x_0)2\mu}{b} - \sin\left(\frac{(L-x_0)2\mu}{b}\right)\right)\right]^{-1}, \end{aligned} \tag{2.10}$$

$$C_{2b} = C_{2a} \frac{\sin\left(\frac{x_0\mu}{a}\right)}{\sin\left(\frac{\mu}{b}(L-x_0)\right)}. \tag{2.11}$$

If $\frac{x_0}{L-x_0}\,\frac{b}{a}$ is irrational, the solutions of (2.8) exhaust the eigenvalues of $\mathcal{L}$.

In the case when $\frac{x_0}{L-x_0}\,\frac{b}{a}$ is a rational number, in addition to eigenvalues given by (2.8) one has the set of eigenvalues $\mu = \dfrac{ak\pi}{x_0} = \dfrac{bm\pi}{L-x_0}$, where $k, m \in \mathbb{Z}$ are such that $\dfrac{x_0}{L-x_0}\,\dfrac{b}{a} = \dfrac{k}{m}$. In that case, $f_a(x_0) = f_b(x_0) = 0$ and the corresponding eigenfunctions are given by (2.9) with

$$C_{2a}^2 = 2\left[x_0 + \frac{a^2}{b^2}(L-x_0)\right]^{-1}, \tag{2.12}$$

$$C_{2b} = (-1)^{k+m+1}\ \frac{a}{b}\ C_{2a}. \tag{2.13}$$

REMARK 2.1. Recall that f is only defined modulo a multiplicative constant, and while we normalized the size in (2.7), we did not normalize the sign. Based on (2.10), throughout the paper we shall employ

$$\begin{aligned} C_{2a} =& \frac{2\sqrt{\mu}}{\sin\left(\frac{\mu x_0}{a}\right)} \left[\frac{a}{\sin^2\left(\frac{x_0\mu}{a}\right)} \left(\frac{2x_0\mu}{a} - \sin\left(\frac{2x_0\mu}{a}\right)\right)\right. \\ & \left. + \frac{b}{\sin^2\left(\frac{\mu}{b}(L-x_0)\right)} \left(\frac{(L-x_0)2\mu}{b} - \sin\left(\frac{(L-x_0)2\mu}{b}\right)\right)\right]^{-\frac{1}{2}}, \end{aligned} \tag{2.14}$$

(note that by (2.10) the expression in brackets is non-negative, which justifies taking the square root in (2.14)).

Similarly, based on (2.12)

$$C_{2a} = \sqrt{2}\left[x_0 + \frac{a^2}{b^2}(L-x_0)\right]^{-1/2}. \tag{2.15}$$

PROOF. The fact that a solution has the form (2.9) follows immediately from resolving (2.3) and (2.4) with boundary conditions (2.5) . Then the normalization condition (2.7) further gives

$$\frac{1}{2}C_{2a}^2\left(x_0-\frac{a}{2\mu}\sin\left(\frac{2\mu x_0}{a}\right)\right)+\frac{1}{2}C_{2b}^2\left[(L-x_0)-\frac{b}{2\mu}\sin\left(\frac{2\mu\,(L-x_0)}{b}\right)\right]=1, \tag{2.16}$$

and the requirements of continuity of f at x_0 and continuity of the conormal derivative at point x_0 (2.6) lead to

$$C_{2a}\sin\left(\frac{\mu x_0}{a}\right)=C_{2b}\sin\left(\frac{\mu\,(L-x_0)}{b}\right), \tag{2.17}$$

$$C_{2a}\ \mu a\cos\left(\frac{\mu x_0}{a}\right)=-\,C_{2b}\ \mu b\cos\left(\frac{\mu(L-x_0)}{b}\right). \tag{2.18}$$

If $\frac{\mu x_0}{\pi a}\notin\mathbb{Z}$, dividing (2.18) by (2.17) yields the desired relationship for μ (2.8) and (2.16), (2.17) give (2.10)–(2.11).

Observe that neither C_{2a} nor C_{2b} can be identically zero, and hence, either both $\frac{\mu x_0}{\pi a}\notin\mathbb{Z}$ and $\frac{\mu(L-x_0)}{\pi b}\notin\mathbb{Z}$, or both these quantities are integer due to (2.17). Then $\mu=\frac{ak\pi}{x_0}$, $k\in\mathbb{Z}$, is automatically equivalent to the statement that there exists $m\in\mathbb{Z}$ such that $\mu=\frac{bm\pi}{L-x_0}$, which is only possible when $\frac{x_0}{L-x_0}\frac{b}{a}=\frac{k}{m}$ from the beginning that is, $\frac{x_0}{L-x_0}\frac{b}{a}$ is rational. In this case, evidently, (2.18) yields

$$C_{2b}=-\,C_{2a}\frac{a\cos\left(k\pi\right)}{b\cos\left(m\pi\right)} \tag{2.19}$$

and hence, (2.13) while (2.16) reduces to

$$C_{2a}^2x_0+C_{2b}^2\left(L-x_0\right)=2, \tag{2.20}$$

which ultimately yields (2.12). □

REMARK 2.2. For future reference, we underline the observation made above that either both $\frac{\mu x_0}{\pi a}\notin\mathbb{Z}$ and $\frac{\mu(L-x_0)}{\pi b}\notin\mathbb{Z}$, or both these quantities are integer. Thus, there is no ambiguity in splitting simply into cases $\frac{\mu x_0}{\pi a}\in\mathbb{Z}$ and $\frac{\mu x_0}{\pi a}\notin\mathbb{Z}$.

At this point, let us briefly discuss why Proposition 2.1 provides all solutions to the eigenvalue problem, formulated in the strong or weak sense. First of all, the weak solution must be continuous on $[0,L]$ by Sobolev embedding. Also, by interior regularity theorems a weak solution to (2.3) is $C^2((0,x_0))$, and the same holds on the complimentary interval. Furthermore, every C^2 solution to (2.3) or (2.4) must be a linear combination of two linearly independent solutions, thus, (taking into account (2.5)), formulas (2.9) completely describe possible solutions on $(0,x_0)$ and (x_0,L). Since, evidently, all such solutions have well-defined one-sided first derivatives at the end-points of their respective intervals, integration by parts is justified and yields (2.6). Hence, Proposition 2.1 identifies all solutions of (2.1).

PROPOSITION 2.2. Let $L > 0$, $x_0 \in (0, L)$, and denote by f a solution to the boundary problem (2.2)–(2.6). If the eigenvalue μ is such that $\dfrac{\mu x_0}{\pi a} \notin \mathbb{Z}$, then the corresponding localization coefficient is given by

$$\alpha_f = \left(\frac{f^4(x_0)}{32} \left[\left(\frac{x_0}{\sin^4 X_a} \right) \frac{12X_a - 8\sin(2X_a) + \sin(4X_a)}{X_a} \right. \right.$$
$$\left. \left. + \left(\frac{L - x_0}{\sin^4 X_b} \right) \frac{12X_b - 8\sin(2X_b) + \sin(4X_b)}{X_b} \right] \right)^{-1}, \tag{2.21}$$

where X_a and X_b are defined by

$$X_a = \frac{\mu x_0}{a}, \qquad X_b = \frac{\mu}{b}(L - x_0). \tag{2.22}$$

If μ is such that $\dfrac{\mu x_0}{\pi a} \in \mathbb{Z}$, then the corresponding localization coefficient is given by

$$\alpha_f = \frac{2}{3} \, \frac{\left[x_0 + (L - x_0)\dfrac{a^2}{b^2} \right]^2}{x_0 + (L - x_0)\dfrac{a^4}{b^4}}. \tag{2.23}$$

PROOF. The solutions to the boundary problem (2.2)–(2.6) are completely described in Proposition 2.1. Given the normalization chosen in (2.7), the corresponding $\alpha_f = \left(\int_0^L f^4 \, dx \right)^{-1}$.

First case: $\left(\dfrac{\mu x_0}{\pi a} \right) \notin \mathbb{Z}$.

Due to (2.17), one also has $\dfrac{\mu(L - x_0)}{\pi b} \notin \mathbb{Z}$. With this in mind, a few common terms and expressions will be redefined as follows:

$$G_a = \frac{2X_a - \sin(2X_a)}{\sin^2(X_a)}, \qquad G_b = \frac{2X_b - \sin(2X_b)}{\sin^2(X_b)}, \tag{2.24}$$

and finally,

$$T = a\, G_a + b\, G_b. \tag{2.25}$$

Using these expressions in (2.14), (2.11), we can write

$$C_{2a} = \frac{2}{\sin X_a} \sqrt{\frac{\mu}{T}}, \quad C_{2b} = \frac{2}{\sin X_b} \sqrt{\frac{\mu}{T}}. \tag{2.26}$$

For later reference, note that the displacement of the system at the discontinuity as

$$f(x_0) = f_a(x_0) = f_b(x_0) = 2\sqrt{\frac{\mu}{T}}, \tag{2.27}$$

and hence,

$$C_{2a} = \frac{f(x_0)}{\sin X_a}, \qquad C_{2b} = \frac{f(x_0)}{\sin X_b}. \tag{2.28}$$

Now (2.9) gives

$$
\begin{aligned}
\int_0^L f^4\,dx = & C_{2a}^4\left(\frac{3}{8}x_0 - \frac{a}{4\mu}\sin X_a + \frac{a}{32\mu}\sin X_a\right) \\
& + C_{2b}^4\left(\frac{3}{8}(L-x_0) - \frac{b}{4\mu}\sin X_b + \frac{b}{32\mu}\sin X_b\right),
\end{aligned} \tag{2.29}
$$

so that substituting formulas (2.28) yields (2.21).

Second case: $\left(\dfrac{\mu x_0}{\pi a}\right)$ is integer.

By (2.13) one has $C_{2b}^2 = \dfrac{a^2}{b^2}\ C_{2a}^2$. Furthermore, (2.29) implies that in the present case

$$
\int_0^L f^4\,dx = \frac{3}{8}\left[\ x_0 C_{2a}^4 + (L-x_0)C_{2b}^4\ \right] = \frac{3}{8}\ C_{2a}^4\left[\ x_0 + (L-x_0)\frac{a^4}{b^4}\ \right], \tag{2.30}
$$

and by (2.12), one finally obtains (2.23). □

3. Numerical results

At this point we turn to the study of dependence of the coefficient of localization, α_f on the amplitude of the eigenfunction at the juncture point, $f(x_0)$. To be precise, given a physical system (or, mathematically, having fixed a, b, L, and x_0), we would like to investigate if there is a connection between $f(x_0)$ and the value of α_f. Note that in the formula (2.21) parameters X_a, X_b depend on μ (which is, in turn, connected to the eigenfunction f) and thus, implicitly, X_a, X_b depend on $f(x_0)$. So $f(x_0)$, in fact, enters α_f in a non-trivial way, and a priori it is not evident that there is any relatively simple description of the dependence of α_f on $f(x_0)$. For that reason, we proceeded with numerical experiments first.

First, one can try to observe the behavior of both quantities, $f(x_0)$ and α_f, as a function of the eigenvalue μ, or of the rank of the eigenvalues since they all can be ordered a increasing way. For instance, we display in Figure 5 $f(x_0)$, the value of the eigenfunction at the interface x_0 (graph on the left), and α_f, the localization coefficient (graph on the right), both plotted against the rank of the corresponding eigenvalue (the parameters here are such that $a^2/b^2 = 0.5$ and $x_0/L = 1/3$). A careful analysis of the two pictures reveals similar patterns, suggesting there might exist a correlation between the two quantities, $f(x_0)$ and α_f.

We therefore computed more generally the localization coefficient α_f against $f(x_0)$ for several values of the parameters a^2/b^2 and x_0/L. In Figures 6 and 7, we have displayed two general cases, **Case I**: $x_0/L = 1/3$, and **Case II**: x_0/L is irrational and, more specifically, $x_0/L = 1/\phi$, where ϕ is the golden ratio, $\phi = (1+\sqrt{5})/2$. For both cases, calculations were performed for a range of wave speed ratios, $a^2/b^2 = 10^{-5}, 1/10, 1/2, 2, 10$, and 10^5.

The experiments in which a^2/b^2 was either very small or very large represent a physical system of a string made of two very different materials such as fishing line and a steel rod. For this extreme case, it makes physical sense that the vibrations in this example are going to be almost entirely confined on one side of the juncture for the vast majority of eigenfunctions. Thus $f(x_0)$ is very close to zero, which is observed in the graphs, and also the localization coefficient is going to be mostly constant, corresponding to a localization of the sine wave in the region having the smallest coefficient a or b.

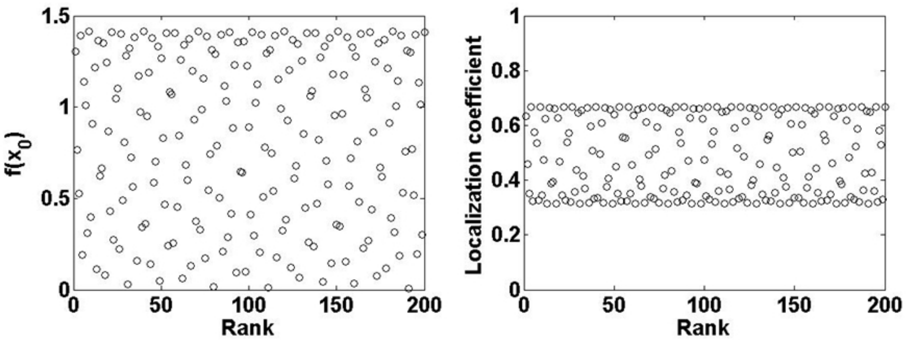

FIGURE 5. Left: Value of $f(x_0)$ for the first 200 eigenfunctions solving the problem (2.2)–(2.6) (here $a^2/b^2 = 0.5$ and $x_0/L = 1/3$). Right: Localization coefficient for the same eigenfunctions. One can observe the same pattern in both figures, indicating that both quantities are strongly correlated.

At the same time, when a^2/b^2 becomes close to 1 (0.5 and 2), then one can observe that the localization coefficients become almost constant and close to 2/3, as predicted by the exact calculation for $a = b$. Indeed, if $a = b$ the boundary problem (2.2)–(2.6) reduces one single ODE whose solution is a sine function with zero values at both ends of the interval $[0, L]$. The point x_0 does not play any specific role and one can chose any location. If one chooses x_0 such that $f(x_0) = 0$, then the value of the localization coefficient α_f can be derived from (2.23), which yields $\alpha_f = \dfrac{2L}{3}$.

Far more surprisingly, in the general case, the figures lead to an interesting and unexpected observation that the points $(\alpha_f, f(x_0))$ seem to fall mostly on a very defined simple curve (see, e.g., $\dfrac{a^2}{b^2} = 0.1$ or 10). A few points (usually, less than five) which fall slightly off of the curve correspond to the first eigenfunctions, i.e., the eigenfunctions with a small eigenvalue.

This suggests that there must be a fairly simple and explicit asymptotic formula expressing α_f in terms of $f(x_0)$, possibly not covering several first eigenfunctions. In the next Section we derive such a formula in the precise mathematical terms. It has an asymptotic nature, which is consistent with the aforementioned observation that the first few eigenfunctions do not fall under its scope. However, the error of asymptotics becomes insignificant very quickly.

4. Dependence of the localization coefficient on the amplitude of an eigenfunction at the juncture point: explicit asymptotic formula

In all statements throughout this section we assume the hypotheses and notation of Theorem 1.1.

PROPOSITION 4.1. Let μ be such that $\dfrac{\mu x_0}{\pi a} \notin \mathbb{Z}$. Define the quantity I_f as

$$I_f = \frac{3}{8}\, f^4(x_0) \left[\frac{x_0}{\sin^4 X_a} + \frac{L - x_0}{\sin^4 X_b} \right]. \tag{4.1}$$

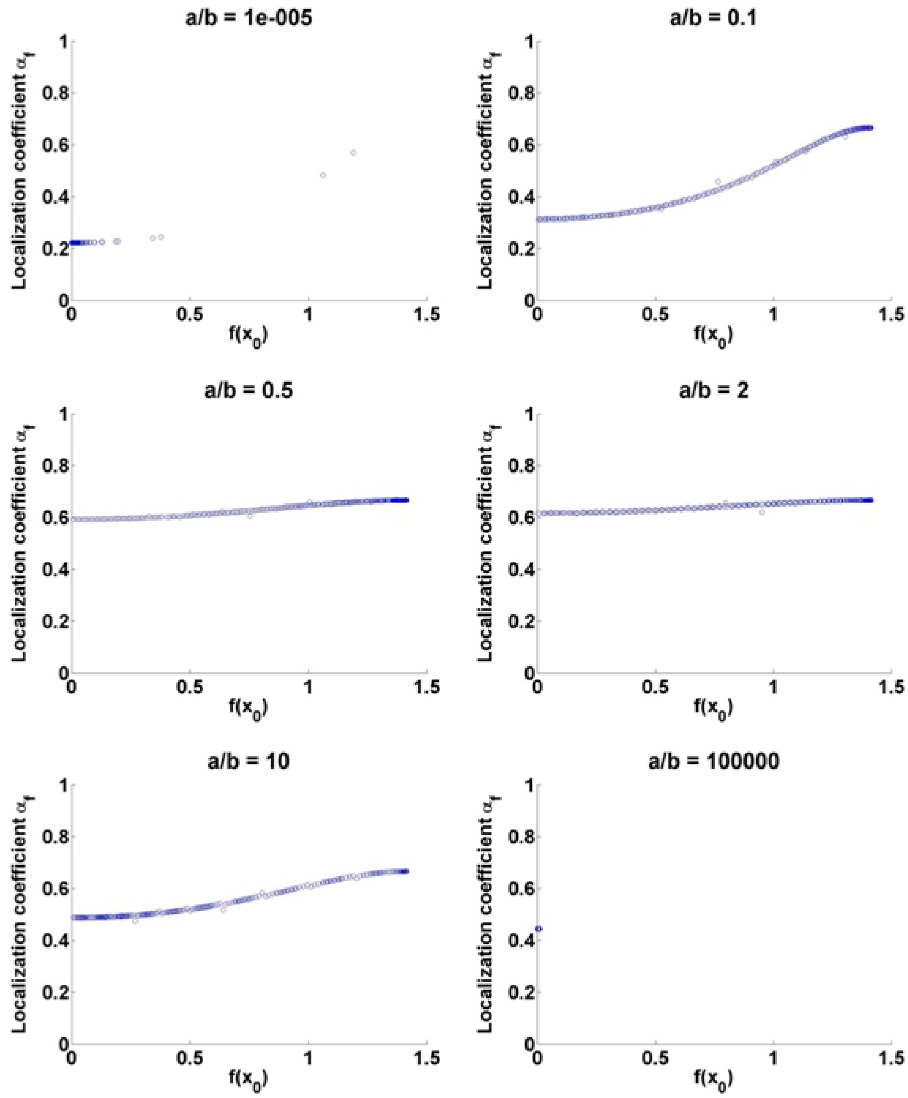

FIGURE 6. **Case 1:** $x_0 = \frac{1}{3}L$

Then

$$\lim_{\mu \to +\infty} I_f \times \alpha_f = 1, \tag{4.2}$$

where the limit is taken over all the eigenvalues of $\mathcal{L}$ such that $\dfrac{\mu x_0}{\pi a} \notin \mathbb{Z}$.

PROOF. Because X_b and X_a increase linearly with μ

$$\lim_{\mu \to +\infty} X_a = +\infty \qquad \text{and} \qquad \lim_{\mu \to +\infty} X_b = +\infty$$

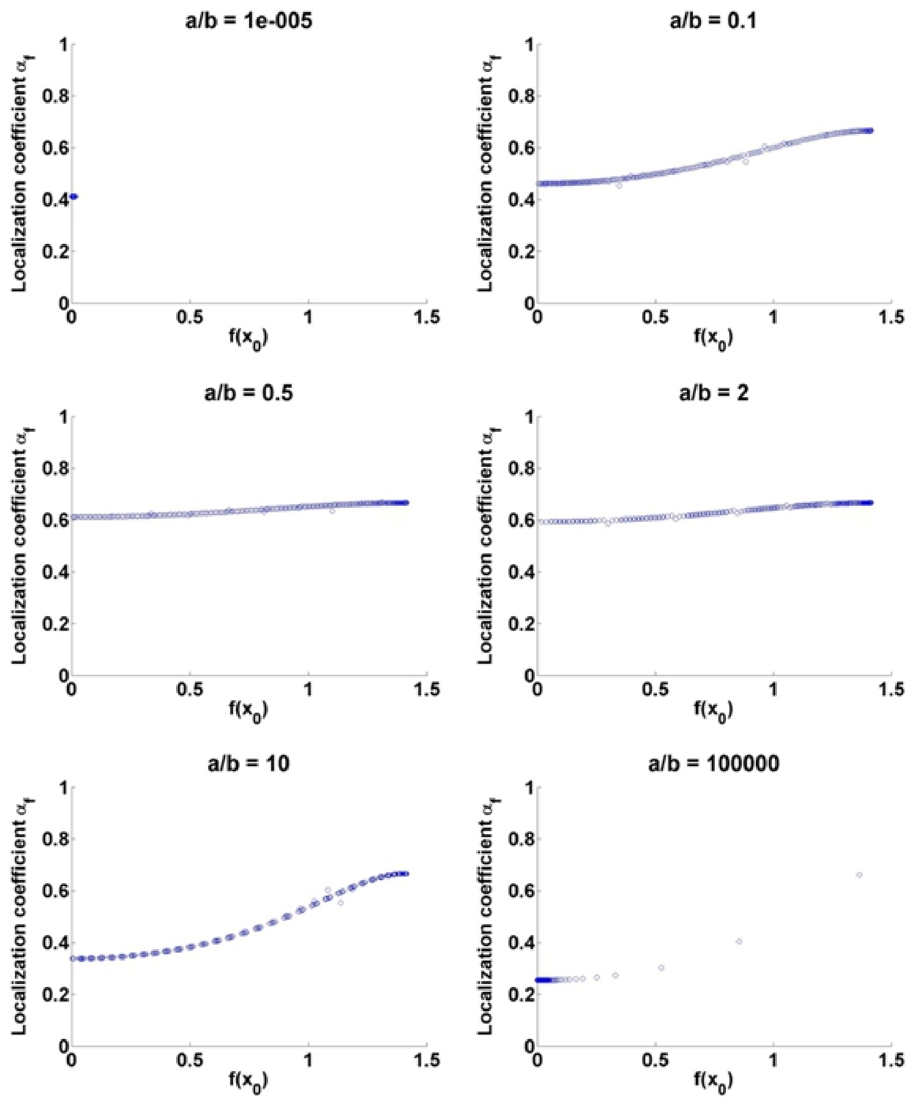

FIGURE 7. **Case 2:** $x_0 = \frac{1}{g}L$

Consequently,

$$\lim_{\mu\to+\infty} \frac{12X_a - 8\sin(2X_a) + \sin(4X_a)}{X_a} = 12, \; \lim_{\mu\to+\infty} \frac{12X_b - 8\sin(2X_b) + \sin(4X_b)}{X_b} = 12 \tag{4.3}$$

Thus, for all ε, there exists μ_0 such that

$$\forall \mu > \mu_0 \qquad 12\ (1-\varepsilon) < \frac{12X_a - 8\sin(2X_a) + \sin(4X_a)}{X_a} < 12\ (1+\varepsilon), \tag{4.4}$$

and the same holds for X_b. Inserting this into equation (2.21), one obtains

$$(1-\varepsilon)\ I_f < \alpha_f^{-1} < (1+\varepsilon)\ I_f \tag{4.5}$$

which proves Proposition 4.1 □

Using (4.1) and (4.2), we now exhibit an asymptotic expression for the localization coefficient α_f when μ goes to infinity.

LEMMA 4.1. *Assume that $\mu > 0$ is such that $\frac{\mu x_0}{\pi a} \notin \mathbb{Z}$ and recall the expression for T in* (2.25). *Define further*

$$T_1 = 2\mu \left(\frac{x_0}{\sin^2(X_a)} + \frac{L - x_0}{\sin^2(X_b)} \right) \tag{4.6}$$

Then

$$\lim_{\mu \to +\infty} \frac{T}{T_1} = 1, \tag{4.7}$$

where the limit is taken over all the eigenvalues of $\mathcal{L}$ such that $\frac{\mu x_0}{\pi a} \notin \mathbb{Z}$.

PROOF. Using (2.22) and (2.24),

$$T = 2\mu \left[\frac{x_0}{\sin^2(X_a)} \left(1 - \frac{\sin(2X_a)}{2X_a} \right) + \frac{L - x_0}{\sin^2(X_b)} \left(1 - \frac{\sin(2X_b)}{2X_b} \right) \right]. \tag{4.8}$$

Both terms $\frac{x_0}{\sin^2(X_a)}$ and $\frac{L - x_0}{\sin^2(X_b)}$ are positive, which means that

$$\forall \varepsilon > 0, \exists\, \mu_0, \forall \mu > \mu_0, \qquad (1 - \varepsilon)\, T_1 < T < (1 + \varepsilon)\, T_1 \tag{4.9}$$

which proves the lemma. □

PROPOSITION 4.2. Let $\delta = \frac{x_0}{L}$, $\gamma = \frac{a^2}{b^2}$, and let $\rho = \rho_{a,b,x_0,L}$ be a positive constant defined by

$$\rho = \frac{\delta(1 - \delta)(1 - \gamma)^2}{4\left[\delta + (1 - \delta)\gamma\right]^2}. \tag{4.10}$$

Then

$$\lim_{\mu \to +\infty} I_f - J_f \;\; = 0, \tag{4.11}$$

where the limit is taken over all the eigenvalues of $\mathcal{L}$ such that $\frac{\mu x_0}{\pi a} \notin \mathbb{Z}$ and J_f has been defined in (1.5).

PROOF. Let us denote $u = \sin^{-2}(X_a)$. From (2.8) one knows that $a^2 \text{cotan}^2(X_a) = b^2 \text{cotan}^2(X_b)$. We thus have

$$a^2\,(u - 1) = b^2 \left(\sin^{-2}(X_b) - 1 \right) \qquad \text{or} \qquad \sin^{-2}(X_b) = 1 + \gamma(u - 1), \tag{4.12}$$

$$T_1 = 2\mu \left(\frac{x_0}{\sin^2(X_a)} + \frac{L - x_0}{\sin^2(X_b)} \right) = 2\mu \,\left[x_0 u + (L - x_0)\,(\gamma u + 1 - \gamma) \right], \tag{4.13}$$

$$I_f = \frac{3}{8} f^4(x_0) \,\left[x_0 u^2 + (L - x_0)\,(\gamma u + 1 - \gamma)^2 \right]. \tag{4.14}$$

To simplify the calculations, we introduce the parameter A such that

$$A = \delta + \gamma(1 - \delta) \tag{4.15}$$

From (4.13) one deduces

$$u = \frac{\frac{T_1}{2\mu} - (L - x_0)(1-\gamma)}{x_0 + \gamma(L - x_0)} = \frac{\frac{2T_1}{Tf^2(x_0)} - L(1-\delta)(1-\gamma)}{AL} \tag{4.16}$$

which allows us to express

$$\frac{f^4(x_0)}{\sin^4(X_a)} = f^4(x_0)\ u^2 = \frac{1}{A^2L^2}\left[\frac{2T_1}{T} - (1-\delta)(1-\gamma)\ Lf^2(x_0)\right]^2. \tag{4.17}$$

In the same way, one can express

$$\frac{f^4(x_0)}{\sin^4(X_b)} = f^4(x_0)\ \ (\gamma u + 1 - \gamma)^2 = \frac{1}{A^2L^2}\left[\gamma\frac{2T_1}{T} + \delta(1-\gamma)\ Lf^2(x_0)\right]^2. \tag{4.18}$$

Finally, I_f can be computed by using these last two expressions

$$I_f = \frac{3}{8A^2L}\left[\delta\left(\frac{2T_1}{T} - (1-\delta)(1-\gamma)\ Lf^2(x_0)\right)^2\right. \tag{4.19}$$

$$\left. + (1-\delta)\left(\gamma\frac{2T_1}{T} + \delta(1-\gamma)\ Lf^2(x_0)\right)^2\right]. \tag{4.20}$$

In the above expression, only T_1/T and $f(x_0)^2$ depend on μ. If one now considers I_f as a function $P(y,z)$ of the two variables y and z, where $y = T_1/T$ and $z = Lf(x_0)^2$, then

$$P(y,z) = \frac{3}{8A^2L}\left[\ \delta\,(2y - Bz)^2 + (1-\delta)\ \ (2\gamma y + \delta(1-\gamma)z)^2\ \right]. \tag{4.21}$$

This can be considered as a second order polynomial in z that one can develop. The coefficient near z^2 in the bracket is

$$\delta(1-\gamma)^2(1-\delta)^2 + (1-\delta)\delta^2(1-\gamma)^2 = \delta(1-\delta)(1-\gamma)^2. \tag{4.22}$$

The coefficient near z is

$$-4\delta(1-\delta)(1-\gamma)^2\ y. \tag{4.23}$$

The zero order coefficient in z is

$$4y^2\left(\delta + (1-\delta)\gamma^2\right). \tag{4.24}$$

Putting these coefficients back into (4.21), one finally gets

$$P(y,z) = \frac{3}{8L}\left(\frac{\delta(1-\delta)(1-\gamma)^2}{A^2}\right)\ \left[\ z^2 - 4yz + y^2\left(\frac{4\left(\delta + (1-\delta)\gamma^2\right)}{\delta(1-\delta)(1-\gamma)^2}\right)\ \right]. \tag{4.25}$$

Recall now the constant ρ defined in (4.10). Thus

$$P(y,z) = \frac{3\rho}{2L}\left[(z-2y)^2 + By^2\ \right]. \tag{4.26}$$

where B writes:

$$B = \frac{4\left(\delta + (1-\delta)\gamma^2\right)}{\delta(1-\delta)(1-\gamma)^2} - 4 = \frac{4\,(\delta + (1-\delta)\gamma)^2}{\delta(1-\delta)(1-\gamma)^2} = \frac{1}{\rho}. \tag{4.27}$$

Finally, one finds

$$P(y,z) = \frac{3}{2L}\left[\ y^2 + \rho\ (z-2y)^2\ \right]. \tag{4.28}$$

We recall here that $y = T_1/T$ and $z = Lf^2(x_0)$. As a consequence, I_f writes

$$I_f = \frac{3}{2L}\left[\left(\frac{T_1}{T}\right) + \rho\left(Lf^2(x_0) - 2\frac{T_1}{T}\right)^2\right]. \tag{4.29}$$

Finally, using the limit found in (4.7) and the fact that $f^2(x_0)$ is bounded uniformly in μ (see (2.27) and (2.25), (2.24)), we arrive at (4.11). □

Proof. Given the fact that α_f is bounded uniformly in μ (which can be seen from the definition of μ and the Cauchy-Schwarz inequality), Proposition 4.1 together with Proposition 4.2 yield (4.2) for μ with $\frac{\mu x_0}{\pi a} \notin \mathbb{Z}$. As far as the case $\frac{\mu x_0}{\pi a} \in \mathbb{Z}$ is concerned, it corresponds to $f(x_0) = 0$ (see (2.9)) and α_f independent of μ, given by (2.23). It is not difficult to see that for $f(x_0) = 0$,

$$J_f^{-1} = \frac{2L}{3}\,\frac{A^2}{\delta + (1-\delta)\gamma^2} = \frac{2L}{3}\,\frac{(\ \delta + (1-\delta)\gamma\ )^2}{\delta + (1-\delta)\gamma^2} \tag{4.30}$$

which is equal to α_f in (2.23). Therefore, (4.2) is valid for all eigenvalues μ, as desired. □

Remark 4.2. Carefully tracing the arguments above, one can show that the order of the error in the asymptotic formula (1.5)–(1.6) is $1/\mu$. That is, there exist positive constants $C = C_{a,b,x_0,L}$ and $M = M_{a,b,x_0,L}$ depending on a, b, x_0, L only such that

$$|\alpha_f \times J_f - 1| \le C/\mu \quad \text{for all} \quad \mu > M.$$

Indeed, the proof of Proposition 4.1 shows that

$$(I_f \times \alpha_f)^{-1} - 1 = \frac{\frac{C_{a,x_0,L}(\mu)}{\mu}\,\frac{x_0}{\sin^4 X_a} + \frac{C_{b,x_0,L}(\mu)}{\mu}\,\frac{L-x_0}{\sin^4 X_b}}{\frac{x_0}{\sin^4 X_a} + \frac{L-x_0}{\sin^4 X_b}},$$

where $C_{a,x_0,L}(\mu), C_{b,x_0,L}(\mu)$ are uniformly bounded from above independently of μ. Hence, $\left|(I_f \times \alpha_f)^{-1} - 1\right| \le C_{a,b,x_0,L}/\mu$ for all μ and therefore,

$$|I_f \times \alpha_f - 1| \le C_{a,b,x_0,L}/\mu \quad \text{for all} \quad \mu > M, \tag{4.31}$$

for some $M = M_{a,b,x_0,L}$.

Furthermore, due to (4.8),

$$\frac{T}{T_1} = 1 - \frac{\frac{x_0}{\sin^2(X_a)}\,\frac{\sin(2X_a)}{2X_a} + \frac{L-x_0}{\sin^2(X_b)}\,\frac{\sin(2X_b)}{2X_b}}{\frac{x_0}{\sin^2(X_a)} + \frac{L-x_0}{\sin^2(X_b)}}. \tag{4.32}$$

Then, as above, $\left|\frac{T}{T_1} - 1\right| \le C_{a,b,x_0,L}/\mu$ for all μ and therefore,

$$\left|\frac{T_1}{T} - 1\right| \le C_{a,b,x_0,L}/\mu \quad \text{for all} \quad \mu > M, \tag{4.33}$$

for some $M = M_{a,b,x_0,L}$.

Finally, according to formulas (4.29) and (1.5),

$$I_f - J_f = \frac{3}{2L}\left(\frac{T_1}{T} - 1\right)\left[\ 1 - 4\rho L f^2(x_0) + 4\rho\left(\frac{T_1}{T} + 1\right)\right],$$

so that by (4.33)

$$|I_f - J_f| \le C_{a,b,x_0,L}/\mu \quad \text{for all} \quad \mu > M, \tag{4.34}$$

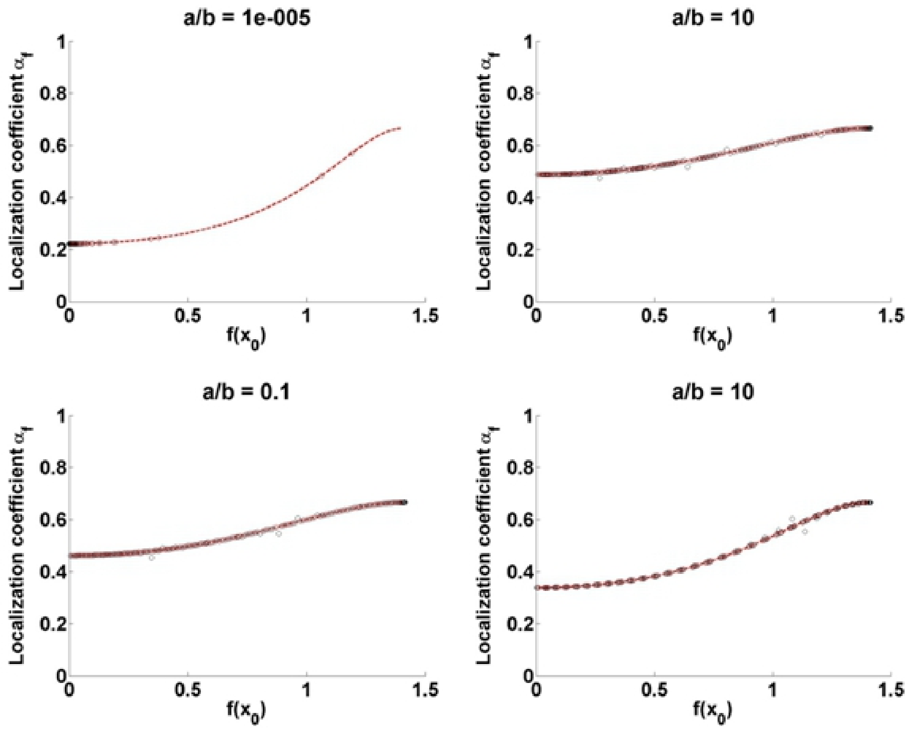

Figure 8. Comparisons between the computed values of α_f (circles) and the asymptotic approximation of Theorem 1.1 (dotted line). Top graphs: $x_0 = \frac{1}{3}L$, for two different values of a/b. Bottom graphs: $x_0 = \frac{1}{g}L$, for two different values of a/b.

for some $M = M_{a,b,x_0,L}$. Now, since by definition and Cauchy-Schwarz inequality $\alpha_f \leq L$, combining (4.31) and (4.34),

$$|\alpha_f \times J_f - 1| \leq |\alpha_f \times (J_f - I_f) + \alpha_f \times I_f - 1| \leq C_{a,b,x_0,L}/\mu \quad \text{for all} \quad \mu > M,$$

for some $M = M_{a,b,x_0,L}$, as desired.

Remark 4.3. The quality of the approximation of α_f using the asymptotic expression J_f from Theorem 1.1 has been tested in all numerical examples discussed above. Figure 8 presents 4 cases, two for a $x_0/L = 1/3$ and two for $x_0/L = 1/g$, g being the golden ratio. In all cases, one can indeed witness that the analytical asymptotic expression depending only on $f(x_0)$ is a nearly perfect approximation of the value of α_f.

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Physique de la Matière Condensée, Ecole Polytechnique, CNRS, 91128 Palaiseau, France

E-mail address: marcel.filoche@polytechnique.edu

School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. SE, Minneapolis, MN 55455, USA

E-mail address: svitlana@math.umn.edu

Department of Mechanical Engineering, University of Michigan, Walter E. Lay Automotive Engineering Laboratory, 1231 Beal Ave., Ann Arbor, MI 48109, USA

E-mail address: brandon.patterson@gmail.com