Square function estimates on layer potentials for higher-order elliptic equations

Ariel Barton[1,*], **Steve Hofmann**[2,**], and **Svitlana Mayboroda**[3,***]

[1] Department of Mathematical Sciences, 309 SCEN, University of Arkansas, Fayetteville, AR 72701
[2] Mathematics Department, 202 Math Sciences Bldg., University of Missouri, Columbia, MO 65211
[3] Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words Elliptic equation, higher-order differential equation, square function estimates, layer potentials
MSC (2010) Primary 35J30, Secondary 31B10, 35C15

In this paper we establish square-function estimates on the double and single layer potentials for divergence form elliptic operators, of arbitrary even order $2m$, with variable t-independent coefficients in the upper half-space. This generalizes known results for variable-coefficient second-order operators, and also for constant-coefficient higher-order operators.

1 Introduction

In this paper we continue towards the goal of resolving the Dirichlet and Neumann problems for general divergence form higher order elliptic operators with L^p data. The investigation of the second-order case has spanned the past three decades in the subject, drawing from the field of harmonic analysis and giving back to it many tools, and by now the real coefficient case is relatively well understood. However, there are still open problems in the theory of even the simplest higher order operators, such as the bilaplacian; for instance, the sharp range of p for which the Dirichlet problem for the bilaplacian is well-posed in L^p is still not known in high dimensions. Even less is known in the case of more complicated operators; indeed, we are not aware of any L^p well-posedness results that are currently available in the general variable coefficient case.

In this work we aim to develop the method of layer potentials for general divergence form higher order elliptic operators. The main results of the present paper are square function estimates for single and double layer potentials in L^2 and the corresponding Sobolev spaces. We remark that one of the key difficulties in this context lies in the definition of suitable layer potentials and, more generally, of Dirichlet and Neumann boundary data, as in the higher order case there is considerable ambiguity, some choices leading to ill-posed problems. Our approach is new even in the constant coefficient context, but is carefully crafted to handle the general case.

Let us discuss the background and the results in more detail.

In this project we study elliptic differential operators of the form

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u), \tag{1.1}$$

for $m \geq 2$, with general bounded measurable coefficients. As mentioned above, contrary to the second order case, most of the known well-posedness results for higher order boundary value problems have been established only

* Corresponding author E-mail: aeb019@uark.edu, Phone: +1 510 502 8326, Fax: +1 479 575 8630
** E-mail: hofmanns@missouri.edu. Steve Hofmann is partially supported by the NSF grant DMS-1361701.
*** E-mail: svitlana@math.umn.edu. Svitlana Mayboroda is partially supported by the Alfred P. Sloan Fellowship, the NSF CAREER Award DMS 1056004, the NSF INSPIRE Award DMS 1344235, and the NSF Materials Research Science and Engineering Center Seed Grant.

in the case of constant coefficients (see, for example, [1–7], or the survey paper [8]), or concern boundary-value problems with data in fractional smoothness spaces, such as the Dirichlet problem

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta}\partial^\beta u) = 0 \text{ in } \Omega, \qquad \nabla^{m-1} u = \dot{\boldsymbol{f}} \text{ on } \partial\Omega \tag{1.2}$$

where Ω is a Lipschitz domain and where $\dot{\boldsymbol{f}}$ lies in a boundary Besov space with smoothness parameter between zero and one. See [9–11]. We are interested in the Dirichlet problem (1.2), with variable coefficients, in the classical case where the boundary data $\dot{\boldsymbol{f}}$ lies in $L^2(\partial\Omega)$.

1.1 The method of layer potentials, general framework

Classic tools for solving second-order boundary-value problems are the double and single layer potentials, given by

$$\mathcal{D}_\Omega^{\boldsymbol{A}} f(X) = \int_{\partial\Omega} \overline{\nu \cdot \boldsymbol{A}^*(Y)\nabla_Y E^{L^*}(Y,X)}\, f(Y)\, d\sigma(Y), \tag{1.3}$$

$$\mathcal{S}_\Omega^L g(X) = \int_{\partial\Omega} E^L(X,Y)\, g(Y)\, d\sigma(Y) \tag{1.4}$$

for all $X \in \mathbb{R}^{n+1} \setminus \partial\Omega$, where $\Omega \subset \mathbb{R}^{n+1}$ is a Lipschitz domain with boundary surface measure $d\sigma$, ν is the unit outward normal to Ω, and where $E^L(X,Y)$ is the fundamental solution for the operator $L = -\operatorname{div} \boldsymbol{A}\nabla$. Making sense of formulas (1.3) and (1.4) in the context of higher-order operators is one of the key tasks in its own right. In the case where Ω is the half-space $\mathbb{R}^{n+1}_+$, we will return to it in Section 2.4 below; in more general Lipschitz domains we refer the interested reader to [8, 11].

It may be shown that, for any nice functions f and g defined on $\partial\Omega$, the functions $u = \mathcal{D}_\Omega^{\boldsymbol{A}} f$ or $u = \mathcal{S}_\Omega^L g$ satisfy $Lu = 0$ away from $\partial\Omega$. The classic method of layer potentials for solving (for example) the Dirichlet problem with data in the boundary Sobolev space $\dot{W}_1^2(\partial\Omega)$ is to show that, for all $g \in L^2(\partial\Omega)$, the boundary value $\mathcal{S}_\Omega^L g\big|_{\partial\Omega}$ exists in some sense and lies in $\dot{W}_1^2(\partial\Omega)$, and moreover $g \mapsto \mathcal{S}_\Omega^L g\big|_{\partial\Omega}$ is invertible from $L^2(\partial\Omega)$ to $\dot{W}_1^2(\partial\Omega)$. Then the function $u = \mathcal{S}_\Omega^L\big((\mathcal{S}_\Omega^L\big|_{\partial\Omega})^{-1} f\big)$ is a solution to the Dirichlet problem

$$Lu = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega. \tag{1.5}$$

Furthermore, if $\mathcal{S}_\Omega^L$ satisfies some estimate, then solutions to the Dirichlet problem may be shown to satisfy a corresponding estimate; for example, if

$$\int_\Omega |\nabla \partial_{n+1} \mathcal{S}_\Omega^L g(X)|^2 \operatorname{dist}(X, \partial\Omega)\, dX \leq C\|g\|_{L^2(\partial\Omega)}^2,$$

then the solution u to the Dirichlet problem (1.5) satisfies

$$\int_\Omega |\nabla \partial_{n+1} u(X)|^2 \operatorname{dist}(X, \partial\Omega)\, dX \leq C\|f\|_{\dot{W}_1^2(\partial\Omega)}^2.$$

This method has been used in [12–16] in the case of harmonic functions (that is, the case $\boldsymbol{A} = \boldsymbol{I}$ and $L = -\Delta$). This method has also been used to study more general second order problems in [17–23] under various assumptions on the coefficients $\boldsymbol{A}$. Layer potentials have been used in other ways in [24–29]. In particular, the second-order double and single layer potentials have been used to study higher-order differential equations in [30, 31].

1.2 Outline of the main results

In this paper we begin to generalize this method to the case of higher-order equations by defining the double and single layer potentials $\mathcal{D}^{\boldsymbol{A}}$ and $\mathcal{S}^L$ for higher-order equations in the half-space (see Section 2.4), and then by establishing some bounds on these potentials under certain conditions on the coefficients $\boldsymbol{A}$. We hope in future work to establish invertibility of layer potentials for some variable coefficients $\boldsymbol{A}$, thus establishing existence of solutions to the corresponding boundary value problems.

Even in the case of second-order equations, some regularity assumption must be imposed on the coefficients $\boldsymbol{A}$ in order to ensure well-posedness of boundary-value problems. See the classic example of Caffarelli, Fabes, and Kenig [32], in which real, symmetric, bounded, continuous, elliptic coefficients $\boldsymbol{A}$ are constructed for which the Dirichlet problem with L^p boundary data is not well-posed in the unit disk for any $1 < p < \infty$. A common starting regularity condition is t-independence, that is,

$$\boldsymbol{A}(x,t) = \boldsymbol{A}(x,s) = \boldsymbol{A}(x) \quad \text{for all } x \in \mathbb{R}^n \text{ and all } s,\, t \in \mathbb{R}. \tag{1.6}$$

Boundary value problems for such coefficients have been investigated extensively in domains Ω where the distinguished t-direction is always transverse to the boundary, that is, $\Omega = \{(x,t) : t > \varphi(x)\}$ for some Lipschitz function φ. See, for example, [17,20,23,29,33–40]. (In two dimensions some well-posedness results are available even if the distinguished direction is not transverse to the boundary; see [18,25,41].)

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose that L is an elliptic operator associated with coefficients $\boldsymbol{A}$ that are t-independent in the sense of formula* (1.6) *and satisfy the ellipticity conditions* (2.4) *and* (2.5).

Then the double and single layer potentials $\mathcal{D}^{\boldsymbol{A}}$ and $\mathcal{S}^L$ in the half-space, as defined by formulas (2.25) *and* (2.32)*, satisfy the bounds*

$$\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} |\nabla^m \partial_t \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}}(x,t)|^2\,|t|\,dt\,dx \leq C\|\dot{\boldsymbol{f}}\|^2_{\dot{W}^2_1(\mathbb{R}^n)} = C\|\nabla_\| \dot{\boldsymbol{f}}\|^2_{L^2(\mathbb{R}^n)}, \tag{1.7}$$

$$\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} |\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}}(x,t)|^2\,|t|\,dt\,dx \leq C\|\dot{\boldsymbol{g}}\|^2_{L^2(\mathbb{R}^n)} \tag{1.8}$$

for all $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ and all $\dot{\boldsymbol{f}} = \operatorname{Tr} \nabla^{m-1}\varphi$ for some $\varphi \in C^\infty_0(\mathbb{R}^{n+1})$, where C depends only on the dimension $n+1$ and the ellipticity constants λ and Λ in the bounds (2.4) *and* (2.5).

We conjecture that this theorem may be generalized from the half-space to Lipschitz graph domains, but the method of proof at the moment requires the extra structure of $\mathbb{R}^{n+1}_+$. In the case of second-order operators (the case $m = 1$), bounds in the upper half-space may be immediately extended to bounds in domains above Lipschitz graphs via a change of variables, and so extra arguments are not necessary. In the higher-order case, this is not true, as the divergence form (1.1) is not preserved under changes of variables. (A different form of higher-order operator is preserved under changes of variables; such operators were investigated in [31].)

1.3 Boundedness of layer potentials for second order elliptic operators

We now turn to the history of this problem. A reader familiar with the second order case may skip this subsection. As discussed above, layer potentials have been used extensively in the theory of second-order and constant coefficient boundary value problems. Thus, boundedness results for layer potentials have long been of interest. The celebrated result of Coifmann, McIntosh and Meyer [42] established boundedness of the Cauchy integral on a Lipschitz curve; this implies that the operators $f \mapsto \mathcal{D}^{\boldsymbol{I}}_\Omega f\big|_{\partial\Omega}$ and $g \mapsto \nu \cdot \nabla \mathcal{S}^{-\Delta}_\Omega g\big|_{\partial\Omega}$ are bounded $L^2(\partial\Omega) \mapsto L^2(\partial\Omega)$, where Ω is a Lipschitz domain and where $\boldsymbol{A} = \boldsymbol{I}$ is the identity matrix (that is, where $L = -\Delta$ is the Laplace operator). From there many other bounds on harmonic layer potentials may be derived. For example, boundedness $L^p(\partial\Omega) \mapsto L^p(\partial\Omega)$, for $1 < p < \infty$, follows from classical Calderón-Zygmund theory. Also, bounds on $\mathcal{D}^{\boldsymbol{I}}_\Omega f$ and $\mathcal{S}^{-\Delta}_\Omega$ in Ω (rather than on $\partial\Omega$) were established in [12] in the case where Ω is C^1; as observed in [13], these results may be extended to the case of Lipschitz domains using boundedness of the Cauchy integral.

In the case of second-order equations with variable t-independent coefficients, a number of boundedness results have been established. In [24], Kenig and Rule established that in dimension $n+1 = 2$, layer potentials for operators with real-valued coefficients are bounded on $L^p(\partial\Omega)$ for Ω the domain above a Lipschitz graph, and in [25] this result was extended to bounded Lipschitz domains and Lipschitz graph domains with arbitrary orientation. In [17], boundedness of layer potentials on $L^2(\partial\Omega)$ was established in arbitrary dimensions, in the domain above a Lipschitz graph, for coefficients that are real-valued and symmetric. A stability result was also established; that is, if layer potentials for some operator L_0 have certain boundedness and invertibility properties on $L^2(\partial\Omega)$, and if corresponding boundary value problems are well-posed, then the same is true for any operator L_1 whose coefficients $\boldsymbol{A}_1$ are t-independent and near (in L^∞) to those of L_0. (This result required a local Hölder continuity estimate for solutions to $L_0 u = 0$; this estimate is always valid if $\boldsymbol{A}_0$ is real-valued but may not be valid for complex $\boldsymbol{A}_0$.)

More generally, in [19] Rosén showed that layer potentials are always bounded on $L^2(\partial\Omega)$, for Ω the domain above a Lipschitz graph, provided that the coefficients of the associated operator are t-independent, and also that solutions to $Lu = 0$ are continuous and satisfy the local bound

$$|u(X)| \leq C\left(\fint_{B(X,r)} |u|^2\right)^{1/2}$$

whenever $Lu = 0$ in $B(X, r)$. (The local Hölder continuity requirement, used in [17] and in many other papers, is a stronger requirement than this local bound. The local boundedness estimate is necessary for Rosén's construction of the fundamental solution $E^L(X, Y)$, and thus for the formulas (1.3) and (1.4) to be meaningful; he also showed that, even without local boundedness, the double and single layer potentials may be continued analytically to bounded operators for t-independent coefficients $\boldsymbol{A}$.) Rosén's results built on an alternative approach to boundary-value problems involving semigroups [36, 37]; essentially he established that layer potentials are equal to certain operators studied in [37], and thus the boundedness results therein apply. The results of [19, 37] extend to the case of elliptic systems.

In the case of two dimensions, or of smooth coefficients, standard Calderón-Zygmund theory allows for straightforward generalization of L^2 bounds to L^p bounds, $1 < p < \infty$. In the case of rough coefficients in higher dimensions, new arguments are necessary to bound the layer potentials (1.3) and (1.4) on $L^p(\partial\Omega)$. Some such arguments are presented in various papers, in particular in [21, 22, 40].

In the case of scalar equations, Rosén's L^2 boundedness result was later established another way, without semigroups, by Grau de la Herran and Hofmann in [43]. As in [17], they required that solutions to $Lu = 0$ be locally Hölder continuous, and in particular locally bounded. In this paper we will closely follow their approach. We will need to confront a number of additional difficulties that arise in the case of higher-order equations. However, one significant advantage of the higher-order setting is that local Hölder continuity is automatic in the case of operators of very high order, and there are established techniques to generalize to operators of low or moderate order (see [44, 45] or Section 11); thus, our Theorem 1.1 is valid without any assumptions on local boundedness or Hölder continuity of solutions.

1.4 Layer potentials for higher order operators: known approaches and new ideas

Turning to the history of higher order problems, we recall that an interesting first step lies in even defining layer potentials in the higher order case. In particular, the prototypical higher order operator, the bilaplacian Δ^2, can be viewed in two ways: either as an operator in the divergence form (1.1), $\Delta^2 = \sum_{j,k=1}^{n+1} \partial_{jk}(\partial_{jk})$, or as a composition of two second order operators (two copies of $-\Delta$). Many papers have used potentials based on a formulation of fourth order operators as compositions; see [1, 30, 31, 46]. A somewhat different approach is necessary for the operators studied in this paper; we instead view Δ^2 as a divergence form operator, and seek to generalize to other such operators.

We begin by defining Neumann boundary values. This is a necessary precursor to defining layer potentials; notice that the Neumann boundary values $\nu \cdot \boldsymbol{A}^* \nabla E^{L^*}$ of the fundamental solution appear in the definition (1.3) of the second order double layer potential. In fact, layer potentials are deeply connected to Dirichlet and Neumann boundary values of solutions in other ways; for example, if u is a reasonably nice solution to $Lu = 0$ in Ω for some second-order operator L and Lipschitz domain Ω, then u satisfies the Green's formula

$$u(X) = -\mathcal{D}^{\boldsymbol{A}}_\Omega(u\big|_{\partial\Omega})(X) + \mathcal{S}^L_\Omega(\nu \cdot \boldsymbol{A}\nabla u)(X) \qquad \text{for all } X \in \Omega. \tag{1.9}$$

That is, we have a formula for u in Ω involving only the Dirichlet and Neumann boundary values of u on $\partial\Omega$, mediated by the layer potentials.

The formulation of Neumann boundary data for higher-order equations is an interesting question in its own right. It has often been based on an integration by parts: for sufficiently nice domains Ω, operators L given by (1.1), and test functions w and v, there exist functions $B_j^{\boldsymbol{A}} v$ defined on $\partial\Omega$ such that

$$\int_\Omega w\, Lv = \sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha w\, A_{\alpha\beta}\, \partial^\beta v + \sum_{j=0}^{m-1} \int_{\partial\Omega} \partial_\nu^j w\, B_j^{\boldsymbol{A}} v\, d\sigma \tag{1.10}$$

where ∂_ν^j is the jth derivative in the direction normal to the boundary. (If desired, exact formulas for the functions $B_j^{\boldsymbol{A}} v$ in terms of the higher derivatives of v may be computed; if $L = \Delta^2$ then formulas for $B_j^{\boldsymbol{A}} v$ may be found in [47, 48] or in Section 2.2.1 below, and in the case of general constant-coefficient operators an explicit formula may be found in [7, Proposition 4.3].)

It is very natural to regard the array $\{\partial_\nu^j w\}_{j=0}^{m-1}$ as the Dirichlet boundary values of w. Then the array of functions $\{B_j^{\boldsymbol{A}} v\}_{j=0}^{m-1}$ may be regarded as the Neumann boundary values of v. The Neumann problem for the biharmonic function Δ^2, with this formulation of boundary data, was studied in [47–50]. The Neumann problem for more general constant-coefficient operators was studied in [7, 51], and for some classes of variable coefficient operators in [9]. We remark that a given operator L may be associated to more than one coefficient matrix $\boldsymbol{A}$, and that each choice of $\boldsymbol{A}$ will give rise to different boundary operators $B_j^{\boldsymbol{A}}$. We will provide more details and a specific example of these different boundary operators (for $L = \Delta^2$) in Section 2.2.1; several of them are physically relevant (in different contexts) and some even lead to ill-posed problems.

Going further, if $Lu = 0$ in a Lipschitz domain Ω and $E^{L^*}(X, Y)$ is the fundamental solution to L^* (so that $L_Y^* E^{L^*}(Y, X) = \delta_X(Y)$), then for any $X \in \Omega$,

$$u(X) = \int_\Omega u\, \overline{L^* E^{L^*}(\,\cdot\,, X)} = \sum_{j=0}^{m-1} \int_{\partial\Omega} \partial_\nu^j u\, \overline{B_j^{\boldsymbol{A}^*} E^{L^*}(\,\cdot\,, X)}\, d\sigma - \int_{\partial\Omega} B_j^{\boldsymbol{A}} u\, \overline{\partial_\nu^j E^{L^*}(\,\cdot\,, X)}\, d\sigma.$$

This naturally suggests the two multiple layer potentials

$$\mathcal{D}_\Omega^{\boldsymbol{A}} \dot{\boldsymbol{f}}(X) = \sum_{j=0}^{m-1} \int_{\partial\Omega} \overline{B_j^{\boldsymbol{A}^*} E^{L^*}(\,\cdot\,, X)}\, f_j\, d\sigma, \qquad \mathcal{S}_\Omega^L \dot{\boldsymbol{g}}(X) = \sum_{j=0}^{m-1} \int_{\partial\Omega} \overline{\partial_\nu^j E^{L^*}(\,\cdot\,, X)}\, g_j\, d\sigma.$$

Notice that in the higher-order case, layer potentials take as input an array of several functions. Also, this formulation of layer potentials generalizes the Green's formula (1.9). Layer potentials constructed in this way, from an integration by parts against the fundamental solution, have been used in [7, 47, 48, 50, 52] to study the biharmonic operator Δ^2 (and in particular the associated Neumann problem), and in [7, 53] to study more general constant-coefficient operators; therein certain boundedness and invertibility results were established for such potentials.

Our formulation of the Neumann boundary values of a solution, and thus layer potentials, is different. Specifically, observe that the different terms $\partial_\nu^j u$ exhibit different degrees of smoothness; if $\nabla^{m-1} u \in L^2(\partial\Omega)$, for example, and $\partial\Omega$ is sufficiently smooth, then we expect $\partial_\nu^j u$ to lie in the Sobolev space $\dot{W}^2_{m-1-j}(\partial\Omega)$ of functions with gradients of order $m-1-j$. Furthermore, if $\nabla^m u \in L^2(\partial\Omega)$, then we generally expect the Neumann boundary terms $B_j^{\boldsymbol{A}} u$ to lie in *negative* smoothness spaces (specifically, we expect $B_j^{\boldsymbol{A}} u \in \dot{W}^2_{j+1-m}(\partial\Omega)$, and so only $B_{m-1}^{\boldsymbol{A}} u$ lies in $L^2(\partial\Omega)$). See Section 2.2.1 for an example.

This is somewhat problematic in the case of Lipschitz domains and other non-smooth domains, as higher smoothness spaces and negative smoothness spaces are difficult to formulate. Furthermore, dealing with mixed orders of smoothness is difficult even in smooth domains. To avoid these difficulties, we will prefer to regard $\nabla^{m-1} u$, rather than $\{\partial_\nu^j u\}_{j=0}^{m-1}$, as the Dirichlet boundary values of u; this will allow us to formulate a similarly homogeneous notion of Neumann boundary data. The latter has the advantage of working with elements of the same degree of smoothness and being naturally adaptable to the general context of Lipschitz domains (see [11]). However, explicit formulas for Neumann boundary data can only rarely be obtained; we treat the entire package of Neumann data as a linear functional on a suitable Sobolev space. See a detailed discussion and an example in Section 2.2.1. We have also formulated layer potentials based on this notion of boundary data; see Section 2.4. Our potentials thus take as input arrays of functions in homogeneous spaces; notice the L^2 norms on the right-hand sides of the bounds (1.8) and (1.7).

A Green's formula involving homogeneous boundary data has been used in [3, 54]. However, this Green's formula was formulated in terms of derivatives of order $2m-1$, and as such does not lend itself to formulation of Neumann boundary data or the natural division into double and single layer potentials. Furthermore, their construction used some delicate integrations by parts not available in the variable coefficient case, and so our formulation of layer potentials is of necessity somewhat different and more abstract.

1.5 Our method and outline of the paper

The remainder of this paper will be devoted to a proof of Theorem 1.1. Specifically, we will define our terminology in Section 2. We will provide a few preliminary arguments, mainly involving the theory of solutions to higher-order equations, in Section 3. We will show that the bounds (1.7) and (1.8) follow from more convenient bounds (specifically, bounds involving derivatives in the t-direction only) in Section 4; we will also define new operators Θ_t^D and Θ_t^S that are somewhat easier to work with. The proof of Theorem 1.1 will make extensive use of $T1$ and Tb theorems; we will state the theorems we will need (taken from [55] and [43]) in Section 5. The remaining sections of the paper will be devoted to showing that Θ_t^D and Θ_t^S satisfy the conditions of Theorems 5.2 and 5.4, and thus satisfy appropriate estimates; a more detailed outline of Sections 6–11 is provided in Section 5.

2 Definitions

Throughout we work with an elliptic operator L in the divergence form (1.1), of order $2m$, acting on functions defined in $\mathbb{R}^{n+1}$.

We will reserve the letters α, β, γ, ζ and ξ to denote multiindices in $\mathbb{N}^{n+1}$. (Here $\mathbb{N}$ denotes the nonnegative integers.) If $\zeta=(\zeta_1,\zeta_2,\dots,\zeta_{n+1})$ is a multiindex, then we define $|\zeta|$, ∂^ζ and $\zeta!$ in the usual ways, as $|\zeta|=\zeta_1+\zeta_2+\dots+\zeta_{n+1}$, $\partial^\zeta=\partial_{x_1}^{\zeta_1}\partial_{x_2}^{\zeta_2}\cdots\partial_{x_{n+1}}^{\zeta_{n+1}}$, and $\zeta!=\zeta_1!\,\zeta_2!\cdots\zeta_{n+1}!$. If ζ and ξ are two multiindices, then we say that $\xi\leq\zeta$ if $\xi_i\leq\zeta_i$ for all $1\leq i\leq n+1$, and we say that $\xi<\zeta$ if in addition the strict inequality $\xi_i<\zeta_i$ holds for at least one such i.

We will routinely deal with arrays $\dot{\boldsymbol{F}}=\big(F_\zeta\big)$ of numbers or functions indexed by multiindices ζ with $|\zeta|=k$ for some k. In particular, if φ is a function with weak derivatives of order up to k, then we view $\nabla^k\varphi$ as such an array.

The inner product of two such arrays of numbers $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ is given by

$$\big\langle\dot{\boldsymbol{F}},\dot{\boldsymbol{G}}\big\rangle=\sum_{|\zeta|=k}\overline{F_\zeta}\,G_\zeta.$$

If $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ are two arrays of functions defined in an open set Ω or on its boundary, then the inner product of $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ is given by

$$\big\langle\dot{\boldsymbol{F}},\dot{\boldsymbol{G}}\big\rangle_\Omega=\sum_{|\zeta|=k}\int_\Omega\overline{F_\zeta}\,G_\zeta\qquad\text{or}\qquad\big\langle\dot{\boldsymbol{F}},\dot{\boldsymbol{G}}\big\rangle_{\partial\Omega}=\sum_{|\zeta|=k}\int_{\partial\Omega}\overline{F_\zeta}\,G_\zeta\,d\sigma$$

where σ denotes surface measure.

If $\dot{\boldsymbol{G}}$ is an array of functions defined in Ω and indexed by multiindices α with $|\alpha|=m$, then $\operatorname{div}_m\dot{\boldsymbol{G}}$ is the distribution given by

$$\big\langle\varphi,\operatorname{div}_m\dot{\boldsymbol{G}}\big\rangle_\Omega=(-1)^m\big\langle\nabla^m\varphi,\dot{\boldsymbol{G}}\big\rangle_\Omega \tag{2.1}$$

for all smooth test functions φ supported in Ω. In particular, if the right-hand side is zero for all such φ then we say that $\operatorname{div}_m\dot{\boldsymbol{G}}=0$.

We let $\vec{e}_j$ be the unit vector in $\mathbb{R}^{n+1}$ in the jth direction; notice that $\vec{e}_j$ is a multiindex with $|\vec{e}_j|=1$. We let $\dot{\boldsymbol{e}}_\zeta$ be the "unit array" corresponding to the multiindex ζ; thus, $\langle\dot{\boldsymbol{e}}_\zeta,\dot{\boldsymbol{F}}\rangle=F_\zeta$. We will often distinguish the $n+1$th direction; we let $\gamma_\perp=(m-1)\vec{e}_{n+1}=(0,\dots,0,m-1)$ and $\alpha_\perp=m\vec{e}_{n+1}$, and let the array $\dot{\boldsymbol{e}}_\perp$ denote either $\dot{\boldsymbol{e}}_{\gamma_\perp}$ or $\dot{\boldsymbol{e}}_{\alpha_\perp}$. Which is meant should be clear from context.

We let $L^p(U)$ and $L^\infty(U)$ denote the standard Lebesgue spaces with respect to either Lebesgue measure (if U is a domain) or surface measure (if U is a subset of the boundary of a domain). We say that $u\in L^p_{loc}(U)$ if

$u \in L^p(V)$ for every bounded set V with $\overline{V} \subset U$. In particular, if U is a set and $\overline{U}$ is its closure, then functions in $L^p_{loc}(\overline{U})$ are required to be locally integrable even near the boundary ∂U; if U is open this is not true of $L^p_{loc}(U)$.

We let the homogeneous Sobolev space $\dot W^p_k(U)$ be the set of all equivalence classes of functions $u \in L^1_{loc}(U)$, modulo adding polynomials of degree $k-1$, that have weak derivatives of order up to k in U, and for which the Sobolev norm $\|u\|_{\dot W^p_k(U)} = \|\nabla^k u\|_{L^p(U)}$ is finite. (Notice that if p is a polynomial of degree $k-1$ then $\nabla^k p = 0$ and so $\|p\|_{\dot W^p_k(U)} = 0$.) We define $\dot W^p_{k,loc}(U)$ analogously to $L^p_{loc}(U)$, as the set of all (equivalence classes of) functions $u \in L^1_{loc}(U)$ such that $\nabla^k u \in L^p(V)$ for all V bounded with $\overline{V} \subset U$.

If μ is a measure and E is a μ-measurable set, with $\mu(E) < \infty$, we let $\fint_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu$. If $E \subset \mathbb{R}^{n+1}$ is a set, we let $\mathbf{1}_E$ denote the characteristic function of E; in particular, we will let $\mathbf{1}_\pm$ denote the characteristic function of the half-space $\mathbb{R}^{n+1}_\pm$. If f is a function defined on E, we will often let $\mathbf{1}_E f$ denote the extension of f to $\mathbb{R}^{n+1}$ by zero.

If $Q \subset \mathbb{R}^n$ is a cube, we let $\ell(Q)$ be its side-length. We let rQ be the concentric cube of side-length $r\ell(Q)$. We will make frequent use of "dyadic annuli" defined as follows. We let

$$A_0(Q) = 2Q, \qquad A_j(Q) = 2^{j+1}Q \setminus 2^j Q \quad \text{for all } j \geq 1. \tag{2.2}$$

If $i \geq 0$, let

$$A_{j,i}(Q) = \bigcup_{\ell=j-i}^{j+i} A_\ell(Q) \qquad \text{where } A_\ell(Q) = \emptyset \text{ whenever } \ell < 0. \tag{2.3}$$

Throughout the paper we will work mainly in the domain $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$. We will also need to consider $\mathbb{R}^{n+1}_- = \{(x,t) : x \in \mathbb{R}^n, t < 0\}$. We will often identify $\mathbb{R}^n$ with $\partial\mathbb{R}^{n+1}_\pm$.

If φ is a function defined on an open subset of $\mathbb{R}^{n+1}$, we will let $\nabla_\| \varphi = (\partial_1\varphi, \partial_2\varphi, \dots, \partial_n\varphi)$ denote the gradient only in the first n variables; we will also use $\nabla_\| f$ to denote the gradient of a function f defined on $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_\pm$. We will view $\nabla^k_\|\varphi$ as an array of functions indexed by multiindices $\zeta \in \mathbb{N}^{n+1}$ with $|\zeta| = k$ and $\zeta_{n+1} = 0$; equivalently we may view $\nabla^k_\|\varphi$ as an array of functions indexed by multiindices $\zeta \in \mathbb{N}^n$ with $|\zeta| = k$.

2.1 Elliptic operators

Let $\boldsymbol{A} = \big(A_{\alpha\beta}\big)$ be measurable coefficients defined on $\mathbb{R}^{n+1}$, indexed by multtiindices α, β with $|\alpha| = |\beta| = m$. If $\dot{\boldsymbol{F}}$ is an array, then $\boldsymbol{A}\dot{\boldsymbol{F}}$ is the array given by

$$(\boldsymbol{A}\dot{\boldsymbol{F}})_\alpha = \sum_{|\beta|=m} A_{\alpha\beta} F_\beta.$$

Throughout we consider coefficients that satisfy the Gårding inequality

$$\operatorname{Re}\big\langle \nabla^m\varphi, \boldsymbol{A}\nabla^m\varphi\big\rangle_{\mathbb{R}^{n+1}} \geq \lambda\|\nabla^m\varphi\|^2_{L^2(\mathbb{R}^{n+1})} \quad \text{for all } \varphi \in \dot W^2_m(\mathbb{R}^{n+1}) \tag{2.4}$$

and the bound

$$\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^{n+1})} \leq \Lambda \tag{2.5}$$

for some $\Lambda > \lambda > 0$.

We let L be the $2m$th-order divergence form operator associated with $\boldsymbol{A}$. That is, we say that $Lu = \operatorname{div}_m \dot{\boldsymbol{F}}$ in Ω in the weak sense if, for every φ smooth and compactly supported in Ω, we have that

$$\big\langle \nabla^m\varphi, \boldsymbol{A}\nabla^m u\big\rangle_\Omega = (-1)^m \big\langle \nabla^m\varphi, \dot{\boldsymbol{F}}\big\rangle_\Omega, \tag{2.6}$$

that is, we have that

$$\sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha\bar\varphi\, A_{\alpha\beta}\,\partial^\beta u = (-1)^m \sum_{|\alpha|=m} \int_\Omega \partial^\alpha\bar\varphi\, F_\alpha.$$

In particular, if the left-hand side is zero for all such φ then we say that $Lu = 0$.

We remark that the coefficients $\boldsymbol{A}$ are not uniquely determined by the elliptic operator L; for example, if $\boldsymbol{M}$ is constant and $M_{\alpha\beta} = -M_{\beta\alpha}$, then the coefficients $\boldsymbol{A}$ and $\boldsymbol{A}+\boldsymbol{M}$ are associated to the same elliptic operator.

We let $\boldsymbol{A}^*$ be the adjoint matrix, that is, $A^*_{\alpha\beta} = \overline{A_{\beta\alpha}}$. We let L^* be the associated elliptic operator.

In this paper we will focus exclusively on operators L that are t-independent, that is, whose coefficients satisfy formula (1.6).

Throughout the paper we will let C denote a constant whose value may change from line to line, but which depends only on the dimension $n+1$, the ellipticity constants λ and Λ in the bounds (2.4) and (2.5), and the order $2m$ of our elliptic operators. Any other dependencies will be indicated explicitly.

2.2 Dirichlet and Neumann boundary data

Our goal in the present paper is to bound the double and single layer potentials; in future work we hope to use the results of this paper to solve the Dirichlet and Neumann boundary value problems. Thus, in this section, we will define higher-order Dirichlet and Neumann boundary data.

We define higher-order Dirichlet boundary data as follows. Suppose that $\nabla^m u \in L^1_{loc}(\overline{\mathbb{R}^{n+1}_+})$ or $\nabla^m u \in L^1_{loc}(\overline{\mathbb{R}^{n+1}_-})$. Then $\partial^\gamma u \in \dot W^1_{1,loc}(\overline{\mathbb{R}^{n+1}_\pm})$ for any γ with $|\gamma| = m-1$. We define $\dot{\mathbf{Tr}}^\pm_{m-1} u$ as the array given by

$$\big(\dot{\mathbf{Tr}}^\pm_{m-1} u\big)_\gamma = \operatorname{Tr} \partial^\gamma u \quad \text{for all } |\gamma| = m-1 \tag{2.7}$$

where Tr denotes the standard trace operator on the Sobolev space $\dot W^1_{1,loc}(\overline{\mathbb{R}^{n+1}_\pm})$. Notice that if $\nabla^m u$ is locally integrable in all of $\mathbb{R}^{n+1}$, then $\dot{\mathbf{Tr}}^+_{m-1} u = \dot{\mathbf{Tr}}^-_{m-1} u$. We will sometimes omit the $\pm$ superscript, either because $u \in \dot W^1_{1,loc}(\mathbb{R}^{n+1})$ or because the space under consideration is clear from context.

With some care we may define boundary values of certain higher-order derivatives. If $u \in \dot W^1_{m,loc}(\overline{\mathbb{R}^{n+1}_\pm})$ is such that $\dot{\mathbf{Tr}}^\pm_{m-1} u \in \dot W^1_{1,loc}(\mathbb{R}^n)$, then for each β with $\beta_{n+1} < |\beta| = m$, we define

$$(\dot{\mathbf{Tr}}^\pm_{m,\|} u)_\beta = \partial_{x_j} \operatorname{Tr} \partial^{\beta-\vec e_j} u \quad \text{for all } j \text{ with } 1 \leq j \leq n \text{ and } \beta_j > 0. \tag{2.8}$$

Note the requirement that $j \neq n+1$. This is well-defined; that is, if $\beta_j > 0$ and $\beta_k > 0$ for some $j < n+1$ and $k < n+1$, then it does not matter whether we choose x_j or x_k as our distinguished representative.

We define higher-order Neumann boundary data as follows. Let $\dot W^2_{m,0}(\mathbb{R}^{n+1}_\pm)$ be the closure in $\dot W^2_m(\mathbb{R}^{n+1}_\pm)$ of the set of all smooth functions supported in $\mathbb{R}^{n+1}_\pm$. It is well known (see, for example, the proof of [56, Theorem 5.5.2]) that we may alternatively characterize $\dot W^2_{m,0}(\mathbb{R}^{n+1}_\pm)$ by

$$\dot W^2_{m,0}(\mathbb{R}^{n+1}_\pm) = \{v \in \dot W^2_m(\mathbb{R}^{n+1}_\pm) : \dot{\mathbf{Tr}}^\pm_{m-1} v = 0\}.$$

Observe that if $v \in \dot W^2_{m,0}(\mathbb{R}^{n+1}_\pm)$, $u \in \dot W^2_m(\mathbb{R}^{n+1}_\pm)$ and $Lu = 0$ in $\mathbb{R}^{n+1}_\pm$, then by the definition (2.6) of Lu,

$$\langle \nabla^m v, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_\pm} = 0.$$

Thus, if φ and η are two functions in $\dot W^2_m(\mathbb{R}^{n+1}_\pm)$, and if $\dot{\mathbf{Tr}}^\pm_{m-1} \varphi = \dot{\mathbf{Tr}}^\pm_{m-1} \eta$, then $\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_\pm} = \langle \nabla^m \eta, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_\pm}$. Hence, if $\varphi \in \dot W^2_m(\mathbb{R}^{n+1}_\pm)$, then the inner product $\langle \nabla^m \varphi, A\nabla^m u\rangle_{\mathbb{R}^{n+1}_\pm}$ depends only on $\dot{\mathbf{Tr}}^\pm_{m-1} \varphi$. We define the Neumann boundary values $\dot{\mathbf{M}}^\pm_{\boldsymbol{A}} u$ by

$$\big\langle \dot{\mathbf{Tr}}^\pm_{m-1} \varphi, \dot{\mathbf{M}}^\pm_{\boldsymbol{A}} u\big\rangle_{\partial\mathbb{R}^{n+1}_\pm} = \big\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m u\big\rangle_{\mathbb{R}^{n+1}_\pm} \qquad \text{for all } \varphi \in \dot W^2_m(\mathbb{R}^{n+1}_\pm). \tag{2.9}$$

$\dot{\mathbf{M}}^\pm_{\boldsymbol{A}} u$ is then a linear operator on the space of traces of $\dot W^2_m(\mathbb{R}^{n+1}_\pm)$-functions.

Recall that the elliptic operator L may be associated with more than one set of coefficients $\boldsymbol{A}$; the Neumann boundary operator $\dot{\mathbf{M}}^\pm_{\boldsymbol{A}}$ depends on our particular choice of associated coefficients $\boldsymbol{A}$, not only on the associated operator L. This phenomenon also occurs in the (smooth) second-order case, when the Neumann boundary data for solutions to $Lu = -\operatorname{div} \boldsymbol{A}\nabla u = 0$ is given by $\nu \cdot \boldsymbol{A}\nabla u$ and thus depends on the particular choice of $\boldsymbol{A}$.

2.2.1 Historical remarks and context

We now provide some further discussion and history of Dirichlet and Neumann boundary data.

We remark that we have three ways to refer to derivatives of order m at $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_\pm$, namely $\nabla^m_\| u(x,0)$, $\dot{\mathbf{Tr}}_{m,|}\, u(x)$, and $\nabla^m u(x,0)$. All three are arrays indexed by multiindices β with $|\beta| = m$. To give the reader some intuition, let us discuss the simplest case, where $m = 2$ and u is defined in $\mathbb{R}^2_+ = \{(x,t) : x \in \mathbb{R},\ t > 0\}$ and smooth up to the boundary. In this case, $\nabla^m_\| u(x,0) = \nabla^2_\| u(x,0) = \partial^2_{xx} u(x,0)$, $\dot{\mathbf{Tr}}_{m,|}\, u(x) = \dot{\mathbf{Tr}}_{2,|}\, u(x)$ is the array $(\partial^2_{xx} u, \partial^2_{xt} u)$ containing $\partial^2_{xx} u = \partial_x \operatorname{Tr} \partial_x u$ and $\partial^2_{xt} u = \partial_x \operatorname{Tr} \partial_t u$ on $\partial\mathbb{R}^2_+$, while $\nabla^m u(x,0) = \nabla^2 u(x,0)$ is the array $(\partial^2_{xx} u, \partial^2_{xt} u, \partial^2_{tt} u)$ of all second derivatives.

The reader should compare our choice of representation of the Dirichlet data $\dot{\mathbf{Tr}}_1\, u = (\partial_x u(x,0), \partial_t u(x,0))$ to the traditional choice $(u(x,0), \partial_t u(x,0))$. This is, of course, a question of representation; $(u(x,0), \partial_t u(x,0))$ determines $\dot{\mathbf{Tr}}_1\, u(x)$, and the reverse is true up to adding a constant (or, for $m-1 \geq 2$, a polynomial). What matters is the function spaces for the data. Working with $\dot{\mathbf{Tr}}_1$ in place of $(u(x,0), \partial_t u(x,0))$ brings considerable advantage and clarity mainly because both components of $\dot{\mathbf{Tr}}_1\, u$ belong to a function space of the same level of smoothness. For example, $\dot{\mathbf{Tr}}_1$ is a vector with both components in $L^2(\mathbb{R})$ precisely when $(u(x,0), \partial_t u(x,0))$ lies in $\dot W^2_1(\mathbb{R}) \times L^2(\mathbb{R})$. This makes things much clearer when dealing with divergence form operators of arbitrary higher order. This also allows us to properly define Neumann data.

Neumann data is somewhat intricate even for the simple case of the bilaplacian and some seemingly natural formulations can make the Neumann problem ill-posed. Let us discuss this in some detail, starting with the bilaplacian on a Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$. We will translate to the half-space below.

The Neumann boundary values of a solution are traditionally given by an integration by parts (formula (1.10)) or less explicitly as an inner product (formula (2.9)). In the case of the biharmonic equation, Neumann boundary values also have applications in the theory of elasticity. The principal physical motivation for the inhomogeneous biharmonic equation $\Delta^2 u = h$ is that it describes the equilibrium position of a thin elastic plate subject to a vertical force h. The Dirichlet problem $u\big|_{\partial\Omega} = f$, $\nabla u\big|_{\partial\Omega} = \vec g$ describes an elastic plate whose edges are clamped, that is, held at a fixed position in a fixed orientation. The Neumann problem, on the other hand, corresponds to the case of a free boundary. Guido Sweers has written an excellent short paper [57] discussing the boundary conditions that correspond to these and other physical situations.

More precisely, if a thin two-dimensional plate is subject to a force h and the edges are free to move, then its displacement u satisfies the boundary value problem

$$\begin{cases} \Delta^2 u = h & \text{in } \Omega \subset \mathbb{R}^2, \\ \rho\Delta u + (1-\rho)\partial^2_\nu u = 0 & \text{on } \partial\Omega, \\ \partial_\nu \Delta u + (1-\rho)\partial_\tau\left(\partial_{\nu\tau} u\right) = 0 & \text{on } \partial\Omega. \end{cases}$$

Here ρ is a physical constant, called the Poisson ratio, and ν and τ are the unit outward normal and unit tangent vectors to the boundary. This formulation goes back to Kirchoff and is well known in the theory of elasticity; see, for example, Section 3.1 and Chapter 8 of the classic engineering text [58].

This suggests the following homogeneous boundary value problem in a Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$ of arbitrary dimension. We say that the L^p-Neumann problem is well-posed if there exists a constant $C > 0$ such that, for every $f_0 \in L^p(\partial\Omega)$ and $\Lambda_0 \in W^p_{-1}(\partial\Omega)$, there exists a function u such that

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ M_\rho u := \rho\Delta u + (1-\rho)\partial^2_\nu u = f_0 & \text{on } \partial\Omega, \\ K_\rho u := \partial_\nu \Delta u + (1-\rho)\dfrac{1}{2}\displaystyle\sum_{j,k=1}^{n+1} \partial_{\tau_{jk}}\left(\partial_{\nu\tau_{jk}} u\right) = \Lambda_0 & \text{on } \partial\Omega, \\ \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \leq C\|f_0\|_{W^p_1(\partial\Omega)} + C\|\Lambda_0\|_{W^p_{-1}(\partial\Omega)} & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Here $\tau_{jk} = \nu_j \vec e_k - \nu_k \vec e_j$ is a vector orthogonal to the outward normal ν and lying in the $x_j x_k$-plane, and $N(\nabla^2 u)$ denotes the nontangential maximal function common in the theory of elliptic boundary value problems.

The boundary operators M_ρ and K_ρ, derived from the theory of elasticity, are the same as the Neumann boundary operators discussed in Section 1.4. Specifically, for any $\rho \in \mathbb{R}$, the equation

$$\int_\Omega w\,\Delta^2 v = \int_\Omega \left(\rho\,\Delta w\,\Delta v + (1-\rho)\sum_{j,k=1}^{n+1} \partial_{jk} w\,\partial_{jk} v\right) + \int_{\partial\Omega} w\,K_\rho v - \partial_\nu w\,M_\rho v\,d\sigma \tag{2.11}$$

is valid for arbitrary smooth functions. Comparing to formula (1.10), we see that $B_0^{\boldsymbol{A}} = K_\rho$ and $B_1^{\boldsymbol{A}} = -M_\rho$, where $\boldsymbol{A} = \boldsymbol{A}_\rho$ is an appropriate choice of coefficients such that $L = \Delta^2 = \operatorname{div}_2 \boldsymbol{A}_\rho \nabla^2$; observe that the coefficients $\boldsymbol{A}_\rho$ are different for each value of ρ.

Thus, there is a *family* of coefficients and relevant Neumann data for the biharmonic equation. Moreover, different values (or, rather, ranges) of ρ correspond to different natural physical situations. We refer the reader to [48] for a detailed discussion.

Recall that our formulation of Neumann data is somewhat different; we use the array $\dot{\mathbf{M}}_{\boldsymbol{A}}^{\pm} u$ of formula (2.9) rather than the functions $B_j^{\boldsymbol{A}} u$ of formula (1.10). As an example, in the case $L = \Delta^2$ and $\Omega = \mathbb{R}^{n+1}_+$, we will provide an explicit formula for one representative of $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u$. On the boundary of the half-space,

$$K_\rho u = -\partial_{n+1}\Delta u - (1-\rho)\sum_{j=1}^{n} \partial_j(\partial^2_{j(n+1)} u), \qquad M_\rho u = \rho\Delta u + (1-\rho)\partial^2_{n+1} u.$$

If Δu is harmonic in $\mathbb{R}^{n+1}_+$, then $\partial_{n+1}\Delta u = \sum_{j=1}^n \partial_j R_j(\Delta u)$, where R_j denotes the jth Riesz transform. Thus, if $\vec\varphi : \mathbb{R}^n \mapsto \mathbb{C}^n$ is any divergence-free vector field, then

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} w\,\Delta^2 u = \int_{\mathbb{R}^{n+1}_+} &\left(\rho\,\Delta w\,\Delta u + (1-\rho)\sum_{j,k=1}^{n+1} \partial_{jk} w\,\partial_{jk} u\right) \\ &+ \int_{\partial\mathbb{R}^{n+1}_+} \partial_{n+1} w\,M_\rho u + \sum_{j=1}^{n} \partial_j w\big(R_j(\Delta u) + (1-\rho)\partial^2_{j(n+1)} u + \varphi_j\big)\,d\sigma\end{aligned} \tag{2.12}$$

and so

$$(\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u)_{\vec e_j} = R_j(\Delta u) + (1-\rho)\partial^2_{j(n+1)} u + \varphi_j \quad \text{for } 1 \le j \le n, \qquad (\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u)_{\vec e_{n+1}} = M_\rho u. \tag{2.13}$$

We comment on several aspects of this formula. First, observe that we still have a family of Neumann boundary data indexed by the parameter ρ. Next, observe that we did use the fact that $\Delta^2 u = 0$; formula (2.12), unlike formula (2.11), is not valid for arbitrary smooth functions. Furthermore, observe the presence of the vector field $\vec\varphi$ in $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u$; our explicit representation gives a natural normalization $\vec\varphi = 0$, but for more general operators the divergence-free vector field cannot be neglected. Finally, observe that our formula for $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u$ is not a local one: it involves the Riesz transforms of derivatives of u rather than simply linear combinations.

However, notice one significant advantage of our formulation (2.13) over the operators K_ρ and M_ρ. The term $M_\rho u$ involves second derivatives of u, while the term $K_\rho u$ involves third derivatives; we have expressed all components of $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u$ using the second derivatives of u. As discussed in Section 1.4, this means that we expect the different components of the Neumann boundary data to lie in a single smoothness space; furthermore, using boundedness of the Riesz transform, we may control $\|\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u\|_{L^p(\mathbb{R}^n)}$ by $\|\nabla^2 u\|_{L^p(\mathbb{R}^n)}$, for $1 < p < \infty$.

Our formulation of Neumann boundary data for general operators will display most of these issues. The existence of a family of Neumann data may be eliminated by specifying the matrix of coefficients $\boldsymbol{A}$ in formula (2.9), but our formulation of $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u$ does require that u be a solution, is well-defined only up to adding divergence-free terms, and need not have a local representation. Indeed, in this article, the estimate $\|\dot{\mathbf{M}}_{\boldsymbol{A}}^{+} u\|_{L^p(\mathbb{R}^n)} \le C\|\nabla^m u\|_{L^p(\mathbb{R}^n)}$, while apparently plausible, is still only a conjecture.

We now discuss the history of the L^p-Neumann problem (2.10). In [47], Cohen and Gosselin showed that this problem was well-posed in C^1 domains contained in $\mathbb{R}^2$ for for $1 < p < \infty$, provided in addition that $\rho = -1$. In [48], Verchota investigated the Neumann problem (2.10) in full generality. He considered Lipschitz domains with compact, connected boundary contained in $\mathbb{R}^{n+1}$, $n+1 \ge 2$. He showed that if $-1/n \le \rho < 1$, then the Neumann problem is well-posed provided $2 - \varepsilon < p < 2 + \varepsilon$. That is, the solutions exist, satisfy the desired estimates, and are unique either modulo functions of an appropriate class, or (in the case where Ω is unbounded) when subject to an appropriate growth condition. See [48, Theorems 13.2 and 15.4]. The Neumann problem is ill-posed for $\rho \geq 1$ and $\rho < -1/n$; see [48, Section 21]. More recently, in [59], Shen improved upon Verchota's results by extending the range on p (in bounded simply connected Lipschitz domains) to $2n/(n+2) - \varepsilon < p < 2 + \varepsilon$ if $n+1 \geq 4$, and $1 < p < 2 + \varepsilon$ if $n+1 = 2$ or $n+1 = 3$. All of the aforementioned results rely on the method of layer potentials. Finally, in [7, Section 6.5], I. Mitrea and M. Mitrea showed that if $\Omega \subset \mathbb{R}^{n+1}$ is a simply connected domain whose unit outward normal ν lies in $VMO(\partial\Omega)$ (for example, if Ω is a C^1 domain), then the acceptable range of p is $1 < p < \infty$; this may be seen as a generalization of the result of Cohen and Gosselin to higher dimensions, to other values of ρ, and to slightly rougher domains. The question of the sharp range of p for which the L^p-Neumann problem is well-posed in a Lipschitz domain is still open.

Very few well-posedness results for the Neumann problem, beyond the case of the bilaplacian, are known. Even defining Neumann boundary values for more general operators is complicated; consider the traditional definition of Neumann boundary data of (1.10) and our formulation (2.9). Some further issues are discussed in [51]. While some progress has been made (see [7,9,11]), at present there are no well-posedness results for the Neumann problem with L^p boundary data.

Recall that [7,9] have investigated the Neumann problem for boundary data formulated as in formula (1.10). Specifically, Agranovich has established some well-posedness results for the inhomogeneous problem $Lu = h$ with homogeneous Neumann boundary data, and has provided some brief discussion of the conditions needed to resolve the Neumann problem with inhomogeneous boundary data; see [9, Section 5.2]. The book [7] considers the case of constant-coefficient operators at length; therein they establish well-posedness results for the Neumann problem, with boundary data in certain fractional smoothness spaces, for homogeneous constant-coefficient operators that satisfy a certain very strong ellipticity condition.

2.3 The Newton potential and the fundamental solution

The main purpose of the present paper is to define and bound the double and single layer potentials for higher-order elliptic operators of the form specified in Section 2.1. Recall from formulas (1.3) and (1.4) that the second-order layer potentials are built from the second-order fundamental solution.

The main result of the paper [45] was a construction of the fundamental solution E^L in the case of higher-order operators. E^L was constructed as an order-m antiderivative of the kernel to the operator Π^L, the Newton potential for L, defined as follows. For any $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$, by the Lax-Milgram lemma there is a unique function $v = \Pi^L \dot{\boldsymbol{H}}$ in $\dot{W}^2_m(\mathbb{R}^{n+1})$ that satisfies

$$\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m v\rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \Pi^L \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \varphi, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} \tag{2.14}$$

for all $\varphi \in \dot{W}^2_m(\mathbb{R}^{n+1})$. The Newton potential is a bounded operator on $L^2(\mathbb{R}^{n+1})$ and satisfies the bound

$$\|\nabla^m \Pi^L \dot{\boldsymbol{H}}\|_{L^2(\mathbb{R}^{n+1})} \leq C\|\dot{\boldsymbol{H}}\|_{L^2(\mathbb{R}^{n+1})}. \tag{2.15}$$

Notice the resemblance of formula (2.14) to formula (2.9). Recall that any elliptic operator L is associated to more than one choice of coefficients $\boldsymbol{A}$. Because the inner product on the right-hand side of formula (2.9) is taken over a half-space, the choice of coefficients $\boldsymbol{A}$ affects the value of the Neumann boundary values $\dot{\mathrm{M}}^{\pm}_{\boldsymbol{A}} u$ defined by formula (2.9). However, because the inner products in formula (2.14) are taken over the whole space $\mathbb{R}^{n+1}$, this dependency disappears; the unique function $\Pi^L \dot{\boldsymbol{H}}$ that satisfies formula (2.15) is independent of our choice of coefficients $\boldsymbol{A}$.

We will need two additional properties of the Newton potential from [45]. First, we will need the symmetry relation

$$\langle \dot{\boldsymbol{G}}, \nabla^m \Pi^L \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \Pi^{L^*} \dot{\boldsymbol{G}}, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} \tag{2.16}$$

for all $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^n)$ and all $\dot{\boldsymbol{G}} \in L^2(\mathbb{R}^n)$. Second, we will need the identity

$$\nabla^m \Pi^L(\boldsymbol{A}\nabla^m F) = \nabla^m F \tag{2.17}$$

for all $F \in \dot{W}^2_m(\mathbb{R}^{n+1})$; this identity follows by uniqueness of the Newton potential as the solution operator in formula (2.14).

We remark that this Newton potential Π^L is somewhat different in smoothness from the traditional Newton potential. This potential, which we will denote N^L, is the unique solution to the equation $Lu = f$, $u \in \dot{W}^2_m(\mathbb{R}^{n+1})$, or, more precisely, to

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m N^L f \rangle_{\mathbb{R}^{n+1}} = \langle \varphi, f \rangle_{\mathbb{R}^{n+1}} \tag{2.18}$$

for all $\varphi \in \dot{W}^2_m(\mathbb{R}^{n+1})$. The input f should thus be taken in $\dot{W}^2_{-m}(\mathbb{R}^{n+1})$, the dual space to $\dot{W}^2_m(\mathbb{R}^{n+1})$. One can write $f = \operatorname{div}_m \dot{\boldsymbol{H}}$ for some $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$. The above formulas then become $Lu = \operatorname{div}_m \dot{\boldsymbol{H}}$, $u \in \dot{W}^2_m(\mathbb{R}^{n+1})$, or, more precisely,

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m N^L \operatorname{div}_m \dot{\boldsymbol{H}} \rangle_{\mathbb{R}^{n+1}} = (-1)^m \langle \nabla^m \varphi, \dot{\boldsymbol{H}} \rangle_{\mathbb{R}^{n+1}}.$$

In other words, $\Pi^L = (-1)^m N^L \operatorname{div}_m$. We shall be working exclusively with Π^L, but this analogy may be useful to keep in mind.

The main result of [45] may be stated as follows.

Theorem 2.1 ([45, Theorem 62 and Lemma 69]) *Let L be an operator of order $2m$ that satisfies the bounds* (2.4) *and* (2.5)*. Then there exists a function $E^L(X, Y)$ with the following properties.*

Let q and s be two integers that satisfy $q + s < n + 1$ and the bounds $0 \le q \le \min(m, (n+1)/2)$, $0 \le s \le \min(m, (n+1)/2)$.

Then we have the symmetry property

$$\partial^\zeta_X \partial^\xi_Y E^L(X, Y) = \overline{\partial^\zeta_X \partial^\xi_Y E^{L^*}(Y, X)} \tag{2.19}$$

as locally L^2 functions, for all multiindices ζ, ξ with $|\zeta| = m - q$ and $|\xi| = m - s$.

There is some $\varepsilon > 0$ such that if $X_0, Y_0 \in \mathbb{R}^{n+1}$, if $0 < 4r < R < |X_0 - Y_0|/3$, and if $q < (n+1)/2$ then

$$\int_{B(Y_0,r)} \int_{B(X_0,R)} |\nabla^{m-s}_X \nabla^{m-q}_Y E^L(X, Y)|^2 \, dX \, dY \le C r^{2q} R^{2s} \left(\frac{r}{R}\right)^\varepsilon. \tag{2.20}$$

If $n + 1$ is even and $q = (n+1)/2$ then we instead have the bound

$$\int_{B(Y_0,r)} \int_{B(X_0,R)} |\nabla^{m-s}_X \nabla^{m-q}_Y E^L(X, Y)|^2 \, dX \, dY \le C(\delta) \, r^{2q} R^{2s} \left(\frac{R}{r}\right)^\delta \tag{2.21}$$

for all $\delta > 0$ and some constant $C(\delta)$ depending on δ.

Furthermore, if $|\alpha| = m$ then

$$\partial^\alpha \Pi^L \dot{\boldsymbol{H}}(X) = \sum_{|\beta|=m} \int_{\mathbb{R}^{n+1}} \partial^\alpha_X \partial^\beta_Y E^L(X, Y) \, H_\beta(Y) \, dY \tag{2.22}$$

for almost every $X \notin \operatorname{supp} \dot{\boldsymbol{H}}$, and for all $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$ whose support is not all of $\mathbb{R}^{n+1}$.

Finally, if $\widetilde{E}^L$ is any other function that satisfies the bounds (2.20), (2.21) *and formula* (2.22)*, then*

$$\begin{aligned}\widetilde{E}^L(X, Y) = E^L(X, Y) &+ \sum_{|\zeta|<m-(n+1)/2} f_\zeta(X) \, Y^\zeta + \sum_{|\xi|<m-(n+1)/2} g_\xi(Y) \, X^\xi \\ &+ \sum_{|\zeta|=|\xi|=m-(n+1)/2} c_{\zeta,\xi} \, X^\zeta \, Y^\xi\end{aligned} \tag{2.23}$$

for some functions f_ζ and g_ζ and some constants $c_{\zeta,\xi}$. Thus, $\nabla^{m-q}_X \nabla^{m-s}_Y E^L(X, Y)$ is a well-defined, locally L^2 function provided q and s satisfy the conditions specified above.

Note that formula (2.22) and the definition of Π^L assures that E^L is indeed analogous to the traditional fundamental solution, which, roughly speaking, solves $LE^L = \delta$. That is, E^L is formally the kernel of the potential N defined above.

We record one further property of the fundamental solution for t-independent operators. By the uniqueness property for the fundamental solution, if $\boldsymbol{A}$ is t-independent, then we have that

$$\partial_{x,t}^{\zeta}\partial_{y,s}^{\xi}E^L(x,t,y,s) = \partial_{x,t}^{\zeta}\partial_{y,s}^{\xi}E^L(x,t+r,y,s+r)$$

for almost every $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and almost every t, s, $r \in \mathbb{R}$, and all multiindices ζ, ξ as in formula (2.19). In particular, for such ζ and ξ we have that

$$\partial_{x,t}^{\zeta}\partial_{y,s}^{\xi}\partial_t E^L(x,t,y,s) = -\partial_{x,t}^{\zeta}\partial_{y,s}^{\xi}\partial_s E^L(x,t,y,s). \tag{2.24}$$

Remark 2.2 We comment on the additional terms in formula (2.23). Notice that E^L is defined essentially by the relation (2.22). But this relation involves only derivatives of order $2m$; in other words, it is only $\nabla_X^m\nabla_Y^m E^L(X,Y)$ that is well-defined. The lower-order derivatives are defined only up to adding polynomials. (The ∂_X^α derivative is included in formula (2.22) because $\Pi^L\dot{\boldsymbol{H}} \in \dot{W}_m^2(\mathbb{R}^{n+1})$, and so $\Pi^L\dot{\boldsymbol{H}}$ is also defined only up to adding polynomials.)

If q and s are small enough, then there is a unique normalization of the derivatives $\nabla_X^{m-q}\nabla_Y^{m-s}E^L(X,Y)$ that satisfies the bound (2.20) or (2.21); in [45] this normalization was found using the Gagliardo-Nirenberg-Sobolev inequality. However, if $2m \geq n+1$ then E^L itself (and possibly some of its derivatives) are still not well-defined. The extra terms on the right-hand side of formula (2.23) are precisely the terms compatible with the requirement $\nabla_X^{m-q}\nabla_Y^{m-s}\widetilde{E}^L(X,Y) = \nabla_X^{m-q}\nabla_Y^{m-s}E^L(X,Y)$ for q, s small enough.

Consequently, throughout this paper we will be careful to use only derivatives of E^L of sufficiently high order; in fact, we will use only derivatives of the form $\partial_X^\zeta\partial_Y^\xi E^L(X,Y)$ for $|\xi| \geq m-1$, $|\zeta| \geq m-1$ and $|\xi|+|\zeta| \geq 2m-1$.

In some very special cases, there are natural normalization conditions for the fundamental solution even if $2m > n+1$; for example, if $L = (-\Delta)^m$ and $n+1 \leq 2m$ is even, then we may take $E^L(X,Y) = C_{m,n}|X-Y|^{2m-(n+1)}\log|X-Y|$. Notice the presence of logarithmic growth in the fundamental solution. However, if we take $2m-(n+1)+1$ derivatives (in either X or Y), then the logarithm vanishes; this is the lowest order of derivative that Theorem 2.1 guarantees is well-defined.

2.4 The double and single layer potentials

In this paper we seek to formulate a notion of layer potentials for higher-order elliptic operators of the form specified in Section 2.1. The goal of this paper is to produce bounds on layer potentials in the domain $\Omega = \mathbb{R}^{n+1}_\pm$; thus, we will define boundary values and layer potentials only for the half-spaces.

We begin by recalling the second-order Green's formula (1.9). To generalize this formula to higher order, notice that, for any function $u \in \dot{W}_m^2(\mathbb{R}^{n+1}_+)$,

$$\mathbf{1}_+ u = (\mathbf{1}_+ u - \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m u)) + \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m u)$$

as $\dot{W}_m^2(\mathbb{R}^{n+1}_\pm)$-functions. We claim that the quantity

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}} = -\mathbf{1}_+F + \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m F) \qquad \text{if } \dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}^+ F \tag{2.25}$$

is well-defined; that is, the right-hand side depends only on $\dot{\mathbf{Tr}}_{m-1}^+ F$. We will define $\mathcal{S}^L\dot{\boldsymbol{g}}$ in such a way that $\mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ u) = \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m u)$ as $\dot{W}_m^2(\mathbb{R}^{n+1})$-functions for any $u \in \dot{W}_m^2(\mathbb{R}^{n+1}_+)$ with $Lu=0$ in $\mathbb{R}^{n+1}_+$; this then yields the higher-order Green's formula

$$\mathbf{1}_{\mathbb{R}^{n+1}_+}\nabla^m u = -\nabla^m\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}^+ u) + \nabla^m\mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ u). \tag{2.26}$$

We will shortly find formulas (2.28) and (2.32) for the double and single layer potentials in terms of the fundamental solution; these formulas will parallel formulas (1.3) and (1.4).

The quantity $\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}$ given by formula (2.25), like the Neumann boundary values of formula (2.9), depends on the particular choice of coefficients $\boldsymbol{A}$ associated with L. However, we will see in formulas (2.31) and (2.32) below that the single layer potential depends only on the elliptic operator L and not on any particular choice of coefficients $\boldsymbol{A}$; in other words, the dependence of the quantity $\mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+ u) = \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m u)$ on the choice of $\boldsymbol{A}$ will turn out to lie entirely in the term $\dot{\mathbf{M}}_{\boldsymbol{A}}^+ u$ and not in the operator $\mathcal{S}^L$.

We now establish our claim that, if $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}^+ F$ for some $F \in W_m^2(\mathbb{R}^{n+1}_+)$, then $\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}$ is a well-defined element of $\dot{W}_m^2(\mathbb{R}^{n+1}_+)$ and $\dot{W}_m^2(\mathbb{R}^{n+1}_-)$. It suffices to show that, if $\dot{\mathbf{Tr}}_{m-1}^+ F = 0$, then the right-hand side of formula (2.25) is zero.

Suppose that $\dot{\mathbf{Tr}}_{m-1}^+ F = \operatorname{Tr}\nabla^{m-1}F = 0$. As in the argument preceding formula (2.9), $\nabla^{m-1}F$ lies in the completion in $\dot{W}_1^2(\mathbb{R}^{n+1}_+)$ of the set of smooth functions compactly supported in $\mathbb{R}^{n+1}_+$. Now, if φ is compactly supported in $\mathbb{R}^{n+1}_+$, then we may extend φ to a function in all of $\mathbb{R}^{n+1}$ by letting $\varphi = 0$ in $\mathbb{R}^{n+1}_-$. By density, we have that $\nabla^{m-1}F$ also extends by zero to a function in $\dot{W}_1^2(\mathbb{R}^{n+1})$; thus, we may extend F to a polynomial of degree $m-1$ in $\mathbb{R}^{n+1}_-$. Without loss of generality we may take this to be the zero polynomial. Thus, if $\dot{\mathbf{Tr}}_{m-1}^+ F = 0$, then $\mathbf{1}_+F \in \dot{W}_m^2(\mathbb{R}^{n+1})$. We may apply the identity (2.17) to $\mathbf{1}_+F$, and so the right-hand side of formula (2.25) is zero in $\dot{W}_m^2(\mathbb{R}^{n+1})$, that is, up to adding polynomials of degree $m-1$.

We will need two alternative formulations of $\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}$. Notice that we may extend F to a $\dot{W}_m^2(\mathbb{R}^{n+1})$ function even if $\dot{\mathbf{Tr}}_{m-1}^+ F = \dot{\boldsymbol{f}} \neq 0$; then $\dot{\mathbf{Tr}}_{m-1}^- F = \dot{\boldsymbol{f}}$ as well. Then by formula (2.17),

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}} = \mathbf{1}_-F - \Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m F) \qquad \text{if } \dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}^- F. \tag{2.27}$$

By formula (2.22), if $|\alpha| = m$, then for almost every $x \in \mathbb{R}^n$ and $t > 0$, we have that

$$\partial^\alpha \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}(x,t) = -\sum_{|\beta|=|\xi|=m} \int_{\mathbb{R}^{n+1}_-} \partial_{x,t}^\alpha \partial_{y,s}^\beta E^L(x,t,y,s)\, A_{\beta\xi}(y,s)\, \partial^\xi F(y,s)\, ds\, dy. \tag{2.28}$$

A corresponding formula, involving an integral over $\mathbb{R}^{n+1}_+$, is valid if $t < 0$.

Recall that the bound (1.7) is a bound on $\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}$ in terms of the L^2 norm of the tangential derivative $\nabla_\parallel \dot{\boldsymbol{f}}$ of $\dot{\boldsymbol{f}}$. In order to use existing theorems concerning L^2 boundedness, we will want to slightly modify the definition of the double layer potential, by defining

$$\widetilde{\mathcal{D}}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m,\|} F)(x,t) = \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1} F)(x,t) \tag{2.29}$$

for all sufficiently well-behaved functions F (e.g., for $F \in C_0^\infty(\mathbb{R}^{n+1})$). Then the bound (1.7) is equivalent to the bound

$$\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} |\nabla^m \partial_t \widetilde{\mathcal{D}}^{\boldsymbol{A}}\dot{\boldsymbol{f}}(x,t)|^2\, |t|\, dt\, dx \leq C\|\dot{\boldsymbol{f}}\|_{L^2(\mathbb{R}^n)}^2 \tag{2.30}$$

for all $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m,\|}\,\varphi$ for some $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Note that this bound has a L^2 norm of $\dot{\boldsymbol{f}}$, not of $\nabla_\parallel \dot{\boldsymbol{f}}$, on the right-hand side.

We now must define the single layer potential. Let $\dot{\boldsymbol{g}}$ be a bounded linear operator on the space

$$\dot{W}A_{m-1/2}^2(\mathbb{R}^n) = \{\dot{\mathbf{Tr}}_{m-1} F : F \in \dot{W}_m^2(\mathbb{R}^{n+1})\} = \{\dot{\mathbf{Tr}}_{m-1}^\pm F : F \in \dot{W}_m^2(\mathbb{R}^{n+1}_\pm)\}.$$

The operator $T_{\dot{\boldsymbol{g}}}F = \langle \dot{\boldsymbol{g}}, \dot{\mathbf{Tr}}_{m-1}^\pm F\rangle_{\mathbb{R}^n}$ is a bounded linear operator on $\dot{W}_m^2(\mathbb{R}^{n+1}_\pm)$. We may identify $\dot{W}_m^2(\mathbb{R}^{n+1}_\pm)$ with a closed subspace of $(L^2(\mathbb{R}^{n+1}_\pm))^q$, where q is the number of multiindices α of length m, via the map $F \mapsto \nabla^m F$. We may then extend $T_{\dot{\boldsymbol{g}}}$ to an operator on $(L^2(\mathbb{R}^{n+1}_\pm))^q$. Let $\dot{\boldsymbol{G}}^\pm \in (L^2(\mathbb{R}^{n+1}_\pm))^q$ be the kernel of $T_{\dot{\boldsymbol{g}}}$, so $T_{\dot{\boldsymbol{g}}}\dot{\boldsymbol{H}} = \langle \dot{\boldsymbol{G}}^\pm, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}_\pm}$ for all $\dot{\boldsymbol{H}} \in (L^2(\mathbb{R}^{n+1}_\pm))^q$. Let $\mathbf{1}_\pm\dot{\boldsymbol{G}}^\pm$ be the extension of $\dot{\boldsymbol{G}}^\pm$ to $\mathbb{R}^{n+1}$ by zero. In particular, $\langle \dot{\boldsymbol{G}}^\pm, \nabla^m\varphi\rangle_{\mathbb{R}^{n+1}_\pm} = \langle \dot{\boldsymbol{g}}, \dot{\mathbf{Tr}}_{m-1}\,\varphi\rangle_{\partial\mathbb{R}^{n+1}_\pm}$ for all $\varphi \in \dot{W}_m^2(\mathbb{R}^{n+1}_\pm)$. Let

$$\mathcal{S}^L\dot{\boldsymbol{g}} = \Pi^L(\mathbf{1}_\pm\dot{\boldsymbol{G}}^\pm) \quad \text{if} \quad \langle \dot{\boldsymbol{G}}^\pm, \nabla^m\varphi\rangle_{\mathbb{R}^{n+1}_\pm} = \langle \dot{\boldsymbol{g}}, \dot{\mathbf{Tr}}_{m-1}^\pm\,\varphi\rangle_{\partial\mathbb{R}^{n+1}_\pm} \text{ for all } \varphi \in \dot{W}_m^2(\mathbb{R}^{n+1}_\pm). \tag{2.31}$$

By boundedness of Π^L, if $\dot{\boldsymbol{g}}$ is a bounded linear operator on $\dot{W}A^2_{m-1/2}(\mathbb{R}^n)$ then $\mathcal{S}^L\dot{\boldsymbol{g}} \in \dot{W}^2_m(\mathbb{R}^{n+1})$. Furthermore, if $\varphi \in \dot{W}^2_m(\mathbb{R}^{n+1})$ then

$$\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \Pi^L(\mathbf{1}_\pm \dot{\boldsymbol{G}}^\pm)\rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \varphi, \dot{\boldsymbol{G}}^\pm\rangle_{\mathbb{R}^{n+1}_\pm} = \langle \dot{\mathbf{Tr}}_{m-1}\,\varphi, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^{n+1}}$$

by definition of Π^L and $\dot{\boldsymbol{G}}^\pm$, and so the value of $\Pi^L(\mathbf{1}_\pm \dot{\boldsymbol{G}}^\pm)$ as an element of $\dot{W}^2_m(\mathbb{R}^{n+1})$ depends only on $\dot{\boldsymbol{g}}$, not on our choice of $\dot{\boldsymbol{G}}^\pm$ (or indeed whether we characterize $\mathcal{S}^L\dot{\boldsymbol{g}}$ by $\dot{\boldsymbol{G}}^+$ or $\dot{\boldsymbol{G}}^-$).

In particular, the formula $\mathcal{S}^L(\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,u) = \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m u)$ follows immediately from the definitions (2.9) and (2.31) of Neumann boundary data and the single layer potential.

Again we will wish to formulate our layer potentials in terms of the fundamental solution. By formula (2.22), if $|\alpha| = m$ and $t > 0$ then

$$\partial^\alpha \mathcal{S}^L\dot{\boldsymbol{g}}(x,t) = \partial^\alpha \Pi^L(\mathbf{1}_-\dot{\boldsymbol{G}}^-)(x,t) = \sum_{|\beta|=m} \int_{\mathbb{R}^{n+1}_-} \partial^\alpha_{x,t}\partial^\beta_{y,s} E^L(x,t,y,s)\, G^-_\beta(y,s)\,ds\,dy.$$

But by the bound (2.20), $\varphi(y,s) = \partial^\alpha_{x,t}E^L(x,t,y,s)$ is a $\dot{W}^2_m(\mathbb{R}^{n+1}_-)$-function for almost every $x \in \mathbb{R}^n$ and $t > 0$; thus we may write

$$\partial^\alpha \mathcal{S}^L\dot{\boldsymbol{g}}(x,t) = \sum_{|\gamma|=m-1} \int_{\mathbb{R}^n} \partial^\alpha_{x,t}\partial^\gamma_{y,s} E^L(x,t,y,0)\, g_\gamma(y)\,dy. \tag{2.32}$$

2.5 Function spaces on the boundary

We have now defined $\mathcal{D}^{\boldsymbol{A}}$ and $\mathcal{S}^L$ as operators on $\dot{W}A^2_{m-1/2}(\mathbb{R}^n)$ and its dual space, respectively. We wish to extend $\mathcal{D}^{\boldsymbol{A}}$ and $\mathcal{S}^L$ to bounded operators on the space $L^2(\mathbb{R}^n)$. However, notice that $\mathcal{D}^{\boldsymbol{A}}$ acts naturally only on traces of gradients; that is, density arguments will only allow us to extend $\mathcal{D}^{\boldsymbol{A}}$ to a subspace of $L^2(\mathbb{R}^n)$. We will define this subspace as follows.

Definition 2.3 We let $\dot{W}A^2_{m-1}(\mathbb{R}^n)$ be the completion of the set

$$\{\dot{\mathbf{Tr}}_{m-1}\,\varphi : \varphi \text{ smooth and compactly supported in } \mathbb{R}^{n+1}\}$$

under the L^2 norm.

We let $\dot{W}A^2_{m,\|}(\mathbb{R}^n)$ be the completion of the set

$$\mathfrak{D} = \{\dot{\mathbf{Tr}}_{m,\|}\,\varphi : \varphi \text{ smooth and compactly supported in } \mathbb{R}^{n+1}\} \tag{2.33}$$

under the L^2 norm.

It is well known that that the space $\dot{W}A^2_{m-1/2}(\mathbb{R}^n)$ used above is the completion of $\mathfrak{D}$ under the norm in of Besov space $\dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$. This space is often called a Whitney-Besov space and has been used in the theory of higher-order boundary-value problems; see, for example, [7, 9–11, 60, 61]. The spaces $\dot{W}A^2_{m-1}(\mathbb{R}^n)$ or $\dot{W}A^2_{m,\|}(\mathbb{R}^n)$ are called Whitney-Sobolev spaces; they have also been used extensively in the theory, for example, in [1–6, 54]. The goal of this paper is to extend the double and single layer potentials to bounded operators on Whitney-Sobolev spaces by establishing boundedness results.

Remark 2.4 We remark that $\widetilde{\mathcal{D}}^{\boldsymbol{A}}$ is a well-defined operator on the space $\mathfrak{D}$ of formula (2.33). We will extend $\widetilde{\mathcal{D}}^{\boldsymbol{A}}$ to $\dot{W}A^2_{m,\|}(\mathbb{R}^n)$ by density.

We will also extend the single layer potential; in this case we wish to extend $\mathcal{S}^L$ to all arrays of functions $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$. It will be convenient to have a dense subspace $\mathfrak{N}$ at our disposal on which $\mathcal{S}^L$ is known to be well-defined. We claim that $\mathcal{S}^L\dot{\boldsymbol{g}}$ is well-defined for any $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ that is compactly supported and integrates to zero.

The argument is as follows. Recall that $\mathcal{S}^L\dot{\boldsymbol{g}}$ is well-defined whenever $\dot{\boldsymbol{g}}$ is a bounded linear operator on

$$\dot{W}A^2_{m-1/2}(\mathbb{R}^n) = \{\dot{\mathbf{Tr}}_{m-1}\,\Phi : \nabla^m\Phi \in L^2(\mathbb{R}^{n+1}_+)\}.$$

Now, suppose that $\int g_\gamma = 0$. Choose some function $\Phi \in \dot W^2_m(\mathbb{R}^{n+1}_+)$ that is smooth up to the boundary. Then

$$\int_{\mathbb{R}^n} g_\gamma\,\partial^\gamma \Phi = \int_{\mathbb{R}^n} g_\gamma\,(\partial^\gamma\Phi - c_\Phi)$$

for any constant c_Φ. Suppose that g_γ is supported in $\mathbb{R}^n \cap B((x_0,0),R)$ for some $x_0\in\mathbb{R}^n$ and some $R>0$. Let $\Omega = \mathbb{R}^{n+1}_+ \cap B((x_0,0),R)$. It is well known (see, for example, [56]) the trace map is bounded from $L^2(\Omega)\cap \dot W^2_1(\Omega)$ to $L^2(\partial\Omega)$. Thus,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} g_\gamma\,\partial^\gamma\Phi\right| &\leq \|g_\gamma\|_{L^2(\mathbb{R}^n)}\|\partial^\gamma\Phi - c_\Phi\|_{L^2(\partial\Omega)}\\ &\leq CR^{1/2}\|g_\gamma\|_{L^2(\mathbb{R}^n)}\|\partial^\gamma\Phi - c_\Phi\|_{L^2(\Omega)} + CR^{3/2}\|g_\gamma\|_{L^2(\mathbb{R}^n)}\|\nabla\partial^\gamma\Phi\|_{L^2(\Omega)}.\end{aligned}$$

By the Poincaré inequality, if we choose c_Φ correctly then we may control the quantity $R^{-1}\|\partial^\gamma\Phi - c_\Phi\|_{L^2(\Omega)}$ by $\|\nabla\partial^\gamma\Phi\|_{L^2(\Omega)} \leq \|\nabla^m\Phi\|_{L^2(\mathbb{R}^{n+1}_+)}$, and so we see that $\dot{\boldsymbol g}$ gives rise to a bounded operator on $\dot W\!A^2_{m-1/2}(\mathbb{R}^n)$. Thus, for such $\dot{\boldsymbol g}$, $\mathcal{S}^L\dot{\boldsymbol g}$ is a well-defined element of $\dot W^2_m(\mathbb{R}^{n+1})$.

3 Preliminary arguments

In this section we will establish some basic results that will be useful throughout the paper.

We begin with some bounds on solutions to elliptic equations. Specifically, we begin with the following higher-order generalization of the Caccioppoli inequality; in its full generality it was proven in [45], but the $j=m$ case was proven in [62] and an intriguing version appears in [63].

Lemma 3.1 (The Caccioppoli inequality) *Suppose that L is a divergence form elliptic operator associated to coefficients $\boldsymbol A$ satisfying the ellipticity conditions* (2.4) *and* (2.5). *Let $u\in \dot W^2_m(B(X_0,2r))\cap L^2(B(X_0,2r))$ with $Lu=0$ in $B(X_0,2r)$.*

Then we have the bound

$$\int_{B(X,r)} |\nabla^j u(x,s)|^2\,dx\,ds \leq \frac{C}{r^2}\int_{B(X,2r)}|\nabla^{j-1}u(x,s)|^2\,dx\,ds$$

for any j with $1\leq j\leq m$.

We remark that if $\boldsymbol A$ is t-independent and $Lu=0$, then $L(\partial_t^k u)=0$ for any integer $k>0$; thus, we also have that

$$\int_{B(X,r)} |\nabla^j \partial_s^{k+1} u(x,s)|^2\,dx\,ds \leq \frac{C}{r^2}\int_{B(X,2r)}|\nabla^{j}\partial_s^k u(x,s)|^2\,dx\,ds$$

for any j with $0\leq j\leq m$ and any $k\geq 0$.

We may use the following lemma to bound u not in balls of dimension $n+1$, but on horizontal slices of dimension n; the second-order case of this lemma is known and is Proposition 2.1 in [17].

Lemma 3.2 *Let $t\in\mathbb{R}$ be a constant, and let $Q\subset\mathbb{R}^n$ be a cube.*

Suppose that $\dot{\boldsymbol u}$ is an array of functions in $L^2(2Q\times(t-\ell(Q),t+\ell(Q)))$ whose weak vertical derivative $\partial_s\dot{\boldsymbol u}(x,s)$ is locally in L^2, and that satisfies the Caccioppoli-like inequality

$$\int_{B(X,r)} |\partial_s \dot{\boldsymbol u}(x,s)|^2\,dx\,ds \leq \frac{c_0}{r^2}\int_{B(X,2r)}|\dot{\boldsymbol u}(x,s)|^2\,dx\,ds$$

whenever $B(X,2r)\subset\{(x,s) : x\in 2Q, t-\ell(Q)<s<t+\ell(Q)\}$.

Then

$$\int_Q |\dot{\boldsymbol u}(x,t)|^2\,dx \leq \frac{C(c_0)}{\ell(Q)}\int_{2Q}\int_{t-\ell(Q)}^{t+\ell(Q)}|\dot{\boldsymbol u}(x,s)|^2\,ds\,dx.$$

In particular, if $Lu = 0$ in $2Q \times (t - \ell(Q), t + \ell(Q))$, and L is an operator of order $2m$ associated to t-independent coefficients A, then

$$\int_Q |\nabla^j \partial_t^k u(x,t)|^2 \, dx \leq \frac{C}{\ell(Q)} \int_{2Q} \int_{t-\ell(Q)}^{t+\ell(Q)} |\nabla^j \partial_s^k u(x,s)|^2 \, ds \, dx$$

for any $0 \leq j \leq m$ and any integer $k \geq 0$.

Proof. Begin by observing that

$$\left(\int_Q |\dot{\boldsymbol{u}}(x,t)|^2 \, dx\right)^{1/2} \leq \left(\int_Q \left|\dot{\boldsymbol{u}}(x,t) - \fint_t^{t+\ell(Q)/2} \dot{\boldsymbol{u}}(x,s) \, ds\right|^2 dx\right)^{1/2} + \left(\int_Q \fint_t^{t+\ell(Q)/2} |\dot{\boldsymbol{u}}(x,s)|^2 \, ds \, dx\right)^{1/2}.$$

But

$$\int_Q \left|\dot{\boldsymbol{u}}(x,t) - \fint_t^{t+\ell(Q)/2} \dot{\boldsymbol{u}}(x,s) \, ds\right|^2 dx \leq \int_Q \left|\fint_0^{\ell(Q)/2} \int_0^s \partial_r \dot{\boldsymbol{u}}(x,t+r) \, dr \, ds\right|^2 dx \leq \int_Q \int_0^{\ell(Q)/2} |\partial_r \dot{\boldsymbol{u}}(x,t+r)|^2 \, dr \, dx.$$

Applying the Caccioppoli inequality completes the proof. □

Throughout this paper we will frequently need to bound the fundamental solution of Theorem 2.1 on horizontal slices. The following estimate follows from Lemma 3.2, the Caccioppoli inequality and the bound (2.20); we will use it several times. Suppose that Q is a cube and that either $j \geq 1$ or $j = 0$ and $\ell(Q) \leq |s - t|$. Suppose further that q, s, i and k are nonnegative integers with $q \leq m$, $s \leq m$ and $q - k < (n+1)/2$, $s - i \leq (n+1)/2$. Then for some $\varepsilon > 0$,

$$\int_Q \int_{A_j(Q)} |\nabla_{x,t}^{m-q} \nabla_{y,s}^{m-s} \partial_t^k \partial_s^i E^L(x,t,y,s)|^2 \, dy \, dx \leq \frac{C}{\ell(Q)^{2r+2}} 2^{-j(2(i-s)+1+\varepsilon)} \tag{3.1}$$

where $r = k + i - q - s = (m - q) + (m - s) + k + i - 2m$. In applying this formula it is always useful to remember formula (2.24), that is, that we may take vertical derivatives in either the s variable or the t variable.

Now, recall that we seek to bound the double layer potential $\mathcal{D}^{\boldsymbol{A}}$. Furthermore, recall that if $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1} F$, then we may write $\mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}}$ in terms of F; see formulas (2.25) or (2.28). The following lemma provides extensions of $\dot{\boldsymbol{f}}$ with useful quantitative bounds.

Lemma 3.3 *Let $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1} F$ for some smooth, compactly supported function F. Then there is some function H defined in $\mathbb{R}^{n+1}_+$ such that $\dot{\mathbf{Tr}}_{m-1} H = \dot{\boldsymbol{f}}$ and such that*

$$\|\nabla^m H\|^2_{L^2(\mathbb{R}^{n+1})} \leq C \int_{\mathbb{R}^n} |\xi| \, |\widehat{\dot{\boldsymbol{f}}}(\xi)|^2 \, d\xi, \tag{3.2}$$

$$\sup_{t \neq 0} \|\nabla^{m-1} H(\,\cdot\,,t)\|^2_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\dot{\boldsymbol{f}}(x)|^2 \, dx, \tag{3.3}$$

$$\sup_{t \neq 0} \|\nabla^m H(\,\cdot\,,t)\|^2_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\nabla_\| \dot{\boldsymbol{f}}(x)|^2 \, dx, \tag{3.4}$$

$$\int_{\mathbb{R}^n} \int_0^\infty |\nabla^m H(x,t)|^2 \, t \, dt \, dx \leq \int_{\mathbb{R}^n} |\dot{\boldsymbol{f}}(x)|^2 \, dx. \tag{3.5}$$

Furthermore, if $\dot{\boldsymbol{f}} = 0$ in some cube Q, then $\nabla^{m-1} H = 0$ in $\{(x,t) : \operatorname{dist}(x, \mathbb{R}^n \setminus Q) > t\}$, and in particular in $(1/2)Q \times (0, \ell(Q)/4)$.

Proof. For each $0 \leq j \leq m-1$, let $f_j(x) = \partial_{n+1}^j F(x,0)$; observe that up to adding polynomials of appropriate degree, f_j is determined entirely by $\dot{\boldsymbol{f}} = \nabla^{m-1}F(x,0)$.

Let $\eta : \mathbb{R}^n \mapsto \mathbb{R}$ be smooth, nonnegative, supported in $B(0,1)$, and satisfy $\int_{\mathbb{R}^n} \eta = 1$ and $\int_{\mathbb{R}^n} x^\zeta\, \eta(x)\, dx = 0$ for all multiindices $\zeta \in \mathbb{N}^n$ with $1 \leq |\zeta| \leq m-1$. Let $\eta_t(x) = t^{-n}\eta(x/t)$. Let

$$H_j(x,t) = \frac{1}{j!} t^j\, f_j * \eta_t(x) = \frac{1}{j!} t^j \int_{\mathbb{R}^n} f_j(x-ty)\,\eta(y)\,dy, \quad H(x,t) = \sum_{j=0}^{m-1} H_j(x,t).$$

By inspection, $\lim_{t\to 0} \partial_t^j H_j(x,t) = f_j(x)$, and if $0 \leq k \leq m-1$ with $j \neq k$, then $\lim_{t\to 0} \partial_t^k H_j(x,t) = 0$. Thus $\dot{\mathbf{Tr}}_{m-1} H = \dot{\boldsymbol{f}}$, as desired.

We may bound H in terms the functions f_j using the Fourier transform in the x-variable and Plancherel's theorem, and we may bound appropriate derivatives of f_j using the array $\dot{\boldsymbol{f}}$. We omit the routine details.

If $\dot{\boldsymbol{f}} = 0$ in Q, then $\nabla^{m-1-j} f_j = 0$ in Q, and so f_j is a polynomial in Q. Thus, we may write $f_j(x-ty) = \sum_{|\gamma|<m-1-j} t^{|\gamma|} y^\gamma P_\gamma(x)$ for some polynomials $P_\gamma(x)$. Notice $P_0(x) = f_j(x)$. By our moment condition on η, if $\operatorname{dist}(x, \mathbb{R}^n \setminus Q) > t$, then

$$H_j(x,t) = \frac{1}{j!} t^j \int_{\mathbb{R}^n} f_j(x-ty)\,\eta(y)\,dy = \frac{1}{j!} t^j \int_{\mathbb{R}^n} P_0(x)\,\eta(y)\,dy = \frac{1}{j!} t^j f_j(x).$$

Thus, $H(x,t)$ is equal to a polynomial of degree at most $m-2$ in this region, as desired. □

4 Operators to be bounded

Recall that Theorem 1.1 involves bounding the quantities $\nabla^m \partial_{n+1} \mathcal{S}^L \dot{\boldsymbol{g}}$ and $\nabla^m \partial_{n+1} \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}}$ (or, substituting the bound (2.30) for the bound (1.7), the quantity $\nabla^m \partial_{n+1} \widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}$). In this section we will reduce to the case of the purely vertical derivatives; that is, we will show that bounding $\partial_{n+1}^{m+k} \mathcal{S}^L \dot{\boldsymbol{g}}$ and $\partial_{n+1}^{m+k} \widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}$, for any $k \geq 1$, suffices to bound $\nabla^m \partial_{n+1} \mathcal{S}^L \dot{\boldsymbol{g}}$ and $\nabla^m \partial_{n+1} \widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}$. We will also establish some notation for these operators.

Let $k \geq 1$ be an integer, to be chosen later. Let

$$\Theta_t^S \dot{\boldsymbol{g}}(x) = t^k \partial_t^{m+k} \mathcal{S}^L \dot{\boldsymbol{g}}(x,t). \tag{4.1}$$

Observe that by formula (2.32), if $t > 0$ then

$$\Theta_t^S \dot{\boldsymbol{g}}(x) = \sum_{|\gamma|=m-1} t^k \int_{\mathbb{R}^n} \partial_t^{m+k} \partial_{y,s}^\gamma E^L(x,t,y,0)\, g_\gamma(y)\, dy. \tag{4.2}$$

Notice that by the bound (3.1), if k is large enough and if $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^{n+1})$, then the integral converges for almost every $(x,t) \in \mathbb{R}_+^{n+1}$. We will elaborate on this point in Section 6.

If $\dot{\boldsymbol{f}}$ lies in the space $\mathfrak{D}$ of Remark 2.4, we let

$$\Theta_t^D \dot{\boldsymbol{f}}(x) = t^k \partial_t^{m+k} \widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}(x,t). \tag{4.3}$$

Establishing a bound on Θ_t^D in terms of the L^2 norm of $\dot{\boldsymbol{f}}$ will allow us to extend Θ_t^D to all of $\dot{W}A^2_{m,|}(\mathbb{R}^n)$.

Note that Θ_t^S, Θ_t^D implicitly depend on $k \geq 1$.

We begin by reducing the proof of Theorem 1.1 to establishing bounds on Θ_t^S and Θ_t^D; the remainder of this paper will be devoted to establishing these bounds.

Remark 4.1 The conclusion of Theorem 1.1 is a bound in the whole space $\mathbb{R}^{n+1}$; for notational convenience, we will establish a bound only in the upper half-space $\mathbb{R}_+^{n+1}$ and note that the corresponding bound in $\mathbb{R}_-^{n+1}$ follows by careful argument involving the change of variables $(x,t) \mapsto (x,-t)$.

Lemma 4.2 *Let $\dot{\boldsymbol{f}} \in \mathfrak{D}$ and $\dot{\boldsymbol{g}} \in \mathfrak{N}$, where $\mathfrak{D}$ and $\mathfrak{N}$ are as in Remark 2.4. If $k \geq 1$, then we have the bounds*

$$\int_{\mathbb{R}^{n+1}_+} |\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}}(x,t)|^2 \, t \, dx \, dt \leq C \int_{\mathbb{R}^{n+1}_+} |\Theta_t^S \dot{\boldsymbol{g}}(x)|^2 \, \frac{1}{t} \, dx \, dt,$$
$$\int_{\mathbb{R}^{n+1}_+} |\nabla^m \partial_t \widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}(x,t)|^2 \, t \, dx \, dt \leq C \int_{\mathbb{R}^{n+1}_+} |\Theta_t^D \dot{\boldsymbol{f}}(x)|^2 \, \frac{1}{t} \, dx \, dt.$$

P r o o f. We follow the proof of the similar formula (5.5) in [17]. Let $u = \mathcal{S}^L \dot{\boldsymbol{g}}$ or $u = \widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}$, and define

$$U_j(t) = \int_{\mathbb{R}^n} |\nabla^m \partial_t^j u(x,t)|^2 \, dx, \quad V_j(t) = \int_{\mathbb{R}^n} |\partial_t^{m+j} u(x,t)|^2 \, dx.$$

To prove the lemma we need only establish the bound

$$\int_0^\infty t \, U_1(t) \, dt \leq C \int_0^\infty t^{2k-1} V_k(t) \, dt.$$

By Lemma 3.2 and the Caccioppoli inequality, if $j \geq -m$ then

$$U_{m+j}(t) \leq \frac{C}{t^{2m}} \fint_{t/2}^{2t} V_j(s) \, ds$$

and thus we may easily show that

$$\int_0^\infty t^{2m+2k-1} \, U_{m+k}(t) \, dt \leq \int_0^\infty t^{2k-1} V_k(t) \, dt.$$

Thus, we need only show that

$$\int_0^\infty t \, U_1(t) \, dt \leq C \int_0^\infty t^{2m+2k-1} \, U_{m+k}(t) \, dt.$$

Observe that $\nabla^m u \in L^2(\mathbb{R}^{n+1}_+)$. By Lemma 3.2 and the Caccioppoli inequality, we have that if $j \geq 0$, then

$$U_j(t) \leq \frac{C}{t^{1+2j}} \|\nabla^m u\|^2_{L^2(\mathbb{R}^{n+1}_+)}.$$

Suppose $j > 0$. Then if $0 < \varepsilon < S < \infty$, we have that

$$\begin{aligned} \int_\varepsilon^S t^{2j-1} \, U_j(t) \, dt &= \int_\varepsilon^S t^{2j-1} \, U_j(S) \, dt - \int_\varepsilon^S t^{2j-1} \int_t^S U_j'(s) \, ds \, dt \\ &\leq \frac{C}{S} \|\nabla^m u\|^2_{L^2(\Omega)} + \frac{1}{2j} \int_\varepsilon^S s^{2j} \, |U_j'(s)| \, ds. \end{aligned}$$

Observe that $|U_j'(s)| \leq 2\sqrt{U_j(s) \, U_{j+1}(s)} \leq \frac{1}{s} U_j(s) + s \, U_{j+1}(s)$. Thus,

$$\int_\varepsilon^S t^{2j-1} \, U_j(t) \, dt \leq \frac{C}{S} \|\nabla^m u\|^2_{L^2(\Omega)} + \frac{1}{2j} \int_\varepsilon^S s^{2j-1} \, U_j(s) \, ds + \frac{1}{2j} \int_\varepsilon^S s^{2j+1} \, U_{j+1}(s) \, ds.$$

Rearranging terms, we have that if $j \geq 1$ then

$$\int_\varepsilon^S t^{2j-1} \, U_j(t) \, dt \leq \frac{C}{S} \|\nabla^m u\|^2_{L^2(\Omega)} + C \int_\varepsilon^S s^{2j+1} \, U_{j+1}(s) \, ds.$$

Taking the limit as $\varepsilon \to 0^+$ and $S \to \infty$, we have that if $j > 0$ then

$$\int_0^\infty t^{2j-1} \, U_j(t) \, dt \leq C \int_0^\infty s^{2j+1} \, U_{j+1}(s) \, ds.$$

Iterating, we see that

$$\int_0^\infty t\, U_1(t)\, dt \leq C(k) \int_0^\infty t^{2m+2k-1}\, U_{m+k}(t)\, dt$$

as desired. □

Remark 4.3 In Sections 5–11 we will bound the operators Θ_t^D and Θ_t^S for some values of k. In particular, we will make many arguments that are only valid for k large enough; we will not make any arguments that require k to be small, and thus there will be some k large enough that all our arguments are valid.

5 A vector-valued $T(b)$ theorem

Our goal now is to produce square-function estimates for the operators Θ_t^S and Θ_t^D. In this section, we will review some known theorems that may be used to establish square-function estimates on singular integral operators.

We begin with one of the first such results, the Christ-Journé $T1$ theorem from Section 2 of [55], which is a square function analogue of the well-known result of David and Journé [64].

Theorem 5.1 *Suppose that the family of linear operators* $\{\Theta_t\}_{t>0}$ *are given by*

$$\Theta_t f(x) = \int_{\mathbb{R}^n} \psi_t(x,y)\, f(y)\, dy$$

for some kernels $\psi_t : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ *that satisfy*

$$|\psi_t(x,y)| \leq C_0 \frac{t^\varepsilon}{(t+|x-y|)^{n+\varepsilon}},$$

$$|\psi_t(x,y) - \psi_t(x,z)| \leq C_1 \frac{t^\varepsilon |y-z|^\varepsilon}{(t+|x-y|)^{n+2\varepsilon}} \qquad \text{for all } |y-z| \leq \frac{1}{2}(t+|x-y|)$$

for some constants C_0*,* C_1 *and some* $\varepsilon > 0$*. If*

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Theta_t 1(x)|^2 \frac{dx\,dt}{t} \leq C_2$$

then we have the bound

$$\int_0^\infty \int_{\mathbb{R}^n} |\Theta_t f(x)|^2\, \frac{dx\,dt}{t} \leq C \|f\|^2_{L^2(\mathbb{R}^n)}$$

where C *depends only on the constants* ε*,* C_0*,* C_1*,* C_2 *and the dimension* $n+1$*.*

We will use this theorem directly in Section 10.2 below. However, this theorem is too restrictive to apply to the operators Θ_t^D and Θ_t^S. (In particular, Θ_t^D and Θ_t^S lack smooth kernels.) There are many generalizations of this theorem; we will need the following $T1$ and Tb theorems from [43]. (We will define a CLP family in Section 7.)

Theorem 5.2 ([43, Theorem 4.5]) *Consider a family of operators* $\{\Theta_t\}_{t>0}$ *taking values in* $\mathbb{C}^{p+1}$*,* $p \geq 0$*, so that* $\Theta_t = (\Theta_t^1, \Theta_t^2, \ldots, \Theta_t^{p+1})$*, where each* Θ_t^j *acts on scalar-valued* $L^2(\mathbb{R}^n)$*, and where for* $\vec{g} = (g_1, g_2, \ldots, g_{p+1}) \in L^2(\mathbb{R}^n \mapsto \mathbb{C}^{p+1})$*, we set*

$$\Theta_t \vec{g} = \sum_{j=1}^{p+1} \Theta_t^j g_j.$$

Suppose that there is some $\theta > 0$ *and some* $C > 0$ *such that, for all dyadic cubes* Q*, all integers* $j \geq 0$*, and all functions* $\vec{g}_j \in L^2(A_j(Q))$*, where* $A_j(Q)$ *is as in formula* (2.2)*, we have the estimate*

$$\|\Theta_t(\mathbf{1}_{A_j(Q)} \vec{g}^j)\|_{L^2(Q)} \leq C 2^{-j(n+2+\theta)/2} \|\vec{g}^j\|_{L^2(A_j(Q))} \quad \text{if } \ell(Q) \leq t \leq 2\ell(Q). \tag{5.1}$$

Suppose further that for some $\theta > 0$, some CLP family of operators Q_s, and some subspace H of $L^2(\mathbb{R}^n)$, we have that

$$\|\Theta_t Q_s \vec{h}\|_{L^2(\mathbb{R}^n)} \leq C\left(\frac{s}{t}\right)^\theta \|\vec{h}\|_{L^2(\mathbb{R}^n)} \quad \textit{for all } \vec{h} \in H \textit{ and all } s \leq t. \tag{5.2}$$

Finally, suppose that

$$\int_0^{\ell(Q)} \int_Q |\Theta_t \mathbf{1}(x)|^2 \frac{dx\,dt}{t} \leq C_0 |Q| \tag{5.3}$$

where $\mathbf{1}$ denotes the $(p+1) \times (p+1)$ identity matrix. Equivalently, we may require that

$$\int_0^{\ell(Q)} \int_Q |\Theta_t^j 1(x)|^2 \frac{dx\,dt}{t} \leq C_0 |Q|$$

for each $1 \leq j \leq p+1$, where 1 denotes the function that is one everywhere.

Then for all $\vec{h} \in H$, we have that

$$\int_0^\infty \int_{\mathbb{R}^n} |\Theta_t \vec{h}(x)|^2 \frac{dx\,dt}{t} \leq C\|\vec{h}\|^2_{L^2(\mathbb{R}^n)}. \tag{5.4}$$

Remark 5.3 The uniform L^2 bound

$$\sup_{t>0} \|\Theta_t \vec{g}\|_{L^2(\mathbb{R}^n)} \leq C\|\vec{g}\|_{L^2(\mathbb{R}^n)} \tag{5.5}$$

follows from the bound (5.1) by summing over dyadic cubes Q of side-length 2^j, $2^j \leq t < 2^{j+1}$. In particular, establishing the bound (5.1) suffices to show that Θ_t is a well-defined operator on $L^2(\mathbb{R}^n)$.

The major advantage of Theorem 5.2 over Theorem 5.1, from our perspective, is that we need not have point-wise estimates on the kernels of our operators Θ_t. Rough kernels appear in the theory of second-order equations (see [43, Section 3]) and are an essential part of our treatment of layer potentials for higher-order equations. On the other hand, we note that the proof of Theorem 5.2 is an easy modification of that of Theorem 5.1; see [43] for details.

We now outline the bounds that we will prove using Theorem 5.2.

In Section 6, we will show that Θ_t^D and Θ_t^S satisfy the estimate (5.1). Notice that Θ_t^D is required to satisfy this estimate for all $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$, not only all $\dot{\boldsymbol{g}} \in \dot{W}A^2_{m,|}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n \mapsto \mathbb{C}^q)$; to deal with this technical requirement, in Section 6.1 we will extend Θ_t^D to an operator defined on all of $L^2(\mathbb{R}^n)$. This extension is used only for this technical requirement and will take a strange form; a similar extension will be used for another purpose in Section 10.2.

In Section 7 we will show that Θ_t^S satisfies the estimate (5.2) for all $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, and that Θ_t^D satisfies the estimate (5.2) for all $\dot{\boldsymbol{h}} \in \dot{W}A^2_{m,|}(\mathbb{R}^n)$. (We do not need to extend this estimate to all $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$.)

Finally, in Section 8, we will show that if

$$\Theta_t^\perp f(x) = \Theta_t^S(f\dot{\boldsymbol{e}}_\perp)(x), \tag{5.6}$$

$$\Theta_t^{S'} \dot{\boldsymbol{f}}(x) = \sum_{\gamma_{n+1}<|\gamma|=m-1} \Theta_t^S(f_\gamma \dot{\boldsymbol{e}}_\gamma)(x), \tag{5.7}$$

then $\Theta_t^{S'} \dot{\mathbf{1}} = 0$ for almost every x and t. Thus, the estimate (5.4) is valid for $\Theta_t = \Theta_t^{S'}$. Indeed, one can see that the splitting (5.6)–(5.7) corresponds to the case when $\gamma = (0, \ldots, m-1)$ in formula (4.2) (that is, all derivatives under the integral are in t, s) and the case when each term of the integrand has at least one y-derivative, respectively.

We will not be able to show directly that $\Theta_t^\perp 1$ or $\Theta_t^D \dot{\mathbf{1}}$ satisfy the bound (5.3). In Section 9, we will show that if $2m > n$, then

$$\Theta_t^D \dot{\mathbf{1}}(x) = \Upsilon_t(x) + \Theta_t^S \dot{\boldsymbol{\alpha}}(x)$$

where Υ_t is a Carleson measure (that is, satisfies the estimate (5.3)) and where $\dot{\boldsymbol{\alpha}}(x)$ is a uniformly bounded function. Standard techniques will allow us to control $\Theta_t^S \dot{\boldsymbol{\alpha}}$ by $\Theta_t^S \dot{\mathbf{1}}$, using only the fact that $\dot{\boldsymbol{\alpha}}$ is bounded (that is, without using any special cancellation properties); see Lemma 9.1 below. Thus, a bound on $\Theta_t^\perp 1$ together with the equation $\Theta_t^{S'} \dot{\mathbf{1}} = 0$ will give us a bound on $\Theta_t^D \dot{\mathbf{1}}$ and thus allow us to use Theorem 5.2.

However, this argument does require control on $\Theta_t^\perp$, and Theorem 5.2 will not suffice to bound $\Theta_t^\perp$.

We will bound $\Theta_t^\perp$ (giving us a bound on Θ_t^D) using the following theorem, with $\Theta_t^{p+1} = \Theta_t^\perp$ and $\Theta_t' = (\Theta_t^D, \Theta_t^{S'})$.

Theorem 5.4 ([43, Theorem 2.13]) *Consider a family $\{\Theta_t\}_{t>0}$ of operators taking values in $\mathbb{C}^{p+1}$, $p \geq 0$, so that $\Theta_t = (\Theta_t', \Theta_t^{p+1}) = (\Theta_t^1, \Theta_t^2, \ldots, \Theta_t^{p+1})$, where each Θ_t^j acts on scalar-valued $L^2(\mathbb{R}^n)$, and where for $\vec{g} = (g', g_{p+1}) = (g_1, g_2, \ldots, g_{p+1}) \in L^2(\mathbb{R}^n \mapsto \mathbb{C}^{p+1})$, we set*

$$\Theta_t \vec{g} = \sum_{j=1}^{p+1} \Theta_t^j g_j, \qquad \Theta_t' g' = \sum_{j=1}^{p} \Theta_t^j g_j.$$

Suppose that Θ_t satisfies the bound (5.1)*, that Θ_t^{p+1} satisfies the bound* (5.2) *for all $h \in L^2(\mathbb{R}^n)$, and that there is some subspace $H' \subset L^2(\mathbb{R}^n \mapsto \mathbb{C}^p)$ such that Θ_t' satisfies the bound* (5.2) *for all $\vec{h} \in H'$.*

We define the $\mathcal{C}, \delta$-norm as

$$\|\Upsilon_t\|_{\mathcal{C},\delta}^2 = \sup_{\ell(Q)>\delta} \frac{1}{|Q|} \int_\delta^{\min(\ell(Q),1/\delta)} \int_Q |\Upsilon_t(x)|^2 \frac{dx\,dt}{t}.$$

Suppose that for each $1 \leq j \leq p$,

$$\|\Theta_t^j 1\|_{\mathcal{C},\delta} \leq C_1 + C_1 \|\Theta_t^{p+1} 1\|_{\mathcal{C},\delta} \qquad \text{for all } \delta > 0 \text{ small enough.} \tag{5.8}$$

Suppose that for each dyadic cube $Q \subset \mathbb{R}^n$, we have a measure μ_Q such that

$$d\mu_Q = \phi_Q\,dx, \qquad \|\nabla \phi_Q\|_{L^\infty(\mathbb{R}^n)} \leq C_0 \ell(Q)^{-1}, \qquad \frac{1}{C_0} \leq \phi_Q \text{ on } Q. \tag{5.9}$$

Suppose further that for each such Q there exists a vector-valued function $\vec{b}_Q = (\vec{b}_Q', b_Q^{p+1}) \in H' \times L^2(\mathbb{R}^n)$ such that

$$\int_0^{\ell(Q)} \int_Q |\Theta_t \vec{b}_Q(x)|^2 \frac{dx\,dt}{t} \leq C_0 |Q|, \tag{5.10}$$

$$\int_{\mathbb{R}^n} |\vec{b}_Q(x)|^2\,dx \leq C_0 |Q|, \tag{5.11}$$

$$\operatorname{Re} \fint_Q b_Q^{p+1}\,d\mu_Q \geq \sigma, \tag{5.12}$$

$$\left| \fint_Q \vec{b}_Q'\,d\mu_Q \right| \leq \eta\sigma, \qquad \eta \leq 1/(2C_1 + 4). \tag{5.13}$$

Then for all $\vec{f} \in H' \times L^2(\mathbb{R}^n)$,

$$\int_0^\infty \int_{\mathbb{R}^n} |\Theta_t \vec{f}(x)|^2 \frac{dx\,dt}{t} \leq C \|\vec{f}\|_{L^2(\mathbb{R}^n)}^2. \tag{5.14}$$

This theorem is a local Tb theorem; that is, we may test $\Theta_t \vec{b}_Q$ near Q, for some $\vec{b}_Q$ adapted to our particular cube Q, rather than testing $\Theta_t \mathbf{1}$ in an arbitrary cube. There is an extensive body of work devoted to generalizing $T1$ theorems to Tb theorems and local Tb theorems; see, for example, the survey paper [65], and in particular [66–69] for a few of the important milestones of the theory.

As mentioned above, we will establish the bound (5.8) in Section 9, for $\Theta_t^{p+1} = \Theta_t^\perp$ and $\Theta_t' = (\Theta_t^D, \Theta_t^{S'})$, provided $2m > n$. We will construct the measure μ_Q and test functions $\dot{\boldsymbol{b}}_Q = (\dot{\boldsymbol{b}}_Q^S, \dot{\boldsymbol{b}}_Q^D)$ at the beginning of

Section 10, and therein will establish the estimates (5.10); we will establish the bounds (5.11), (5.12) and (5.13) in Sections 10.1 and 10.2. The assumption $2m > n$ will be useful in Section 10 as well as Section 9. This will allow us to bound $\Theta_t^\perp$, and so together with Lemma 4.2 will complete the proof of Theorem 1.1 in the case $2m > n$. We will extend to the case $2m \leq n$ in Section 11.

6 The decay estimate (5.1)

In this section, we will show that the operators Θ_t^S and Θ_t^D satisfy the bound (5.1) for all $\dot{\boldsymbol{g}}^j$ in $L^2(A_j(Q))$.

By formula (4.2) for Θ_t^S,

$$\int_Q |\Theta_t^S \dot{\boldsymbol{g}}^j|^2 = \int_Q t^{2k} \bigg| \sum_{|\gamma|=m-1} \int_{A_j(Q)} \partial_t^{m+k} \partial_{y,s}^\gamma E^L(x,t,y,0)\, g_\gamma(y)\, dy \bigg|^2 dx.$$

By Hölder's inequality

$$\int_Q |\Theta_t^S \dot{\boldsymbol{g}}^j|^2 \leq C \|\dot{\boldsymbol{g}}\|_{L^2(A_j(Q))} \int_Q t^{2k} \int_{A_j(Q)} |\partial_t^{m+k} \nabla_{y,s}^{m-1} E^L(x,t,y,0)|^2\, dy\, dx.$$

Finally, by the bound (3.1) on the fundamental solution,

$$\int_Q |\Theta_t^S \dot{\boldsymbol{g}}^j|^2 \leq C 2^{-j(2k-1+\varepsilon)} \|\dot{\boldsymbol{g}}\|_{L^2(A_j(Q))}^2.$$

Thus, if k is large enough then the operator Θ_t^S satisfies bound (5.1).

Remark 6.1 Suppose that $\dot{\boldsymbol{g}}^j(z) = \partial_{z_i} \dot{\boldsymbol{h}}^j(z)$ for some $1 \leq i \leq n$ and some $\dot{\boldsymbol{h}}^j$ supported in $A_{j,1}(Q)$. Then

$$\int_Q |\Theta_t^S (\partial_i \dot{\boldsymbol{h}}^j)|^2 = \int_Q t^{2k} \bigg| \sum_{|\gamma|=m-1} \int_{A_{j,1}(Q)} \partial_t^{m+k} \partial_{y,s}^\gamma E^L(x,t,y,0)\, \partial_{y_i} h_\gamma^j(y)\, dy \bigg|^2 dx.$$

Integrating by parts in y_i, and applying the bound (3.1), we see that

$$\int_Q |\Theta_t^S (\partial_i \dot{\boldsymbol{h}}^j)|^2 \leq \frac{C}{t^2} 2^{-j(2k+1+\varepsilon)} \|\dot{\boldsymbol{h}}^j\|_{L^2(A_{j,1}(Q))}^2.$$

In particular, for any $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, we have the uniform L^2 estimate

$$\|\Theta_t^S (\partial_i \dot{\boldsymbol{h}})\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{t} \|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}, \qquad 1 \leq i \leq n. \tag{6.1}$$

This formula will be useful in Section 7.

We now wish to show that Θ_t^D satisfies the decay estimate (5.1), that is, that

$$\|\Theta_t^D \dot{\boldsymbol{f}}^j\|_{L^2(Q)} \leq C 2^{-j(n+2+\theta)/2} \|\dot{\boldsymbol{f}}^j\|_{L^2(A_j(Q))} \qquad \text{for all } \ell(Q) \leq t \leq 2\ell(Q)$$

for all $\dot{\boldsymbol{f}}^j \in L^2(\mathbb{R}^n)$ and supported in $A_j(Q)$.

Choose some dyadic cube Q and some t with $\ell(Q) \leq t < 2\ell(Q)$. Suppose first that $\dot{\boldsymbol{f}}^j \in \mathfrak{D}$, where $\mathfrak{D}$ is as in Remark 2.4, and is supported in $A_{j,1}(Q)$. Recall that $\widetilde{\mathcal{D}}^{\boldsymbol{A}} \dot{\boldsymbol{f}}^j$ (and thus $\Theta_t^D \dot{\boldsymbol{f}}^j$) is defined in terms of extensions F^j of $\dot{\boldsymbol{f}}^j$. Thus, we begin by choosing an appropriate extension. Let H^j be the function given by Lemma 3.3; we then have that

$$\dot{\mathbf{Tr}}_{m,|}^- H^j = \dot{\boldsymbol{f}}^j, \qquad \sup_{t<0} \|\nabla^m H^j(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \leq C \|\dot{\boldsymbol{f}}^j\|_{L^2(\mathbb{R}^n)}.$$

We may assume without loss of generality that $\operatorname{Tr} H^j \equiv 0$ outside of $2^{j+2}Q$. Let $\eta_j(x,t)$ be smooth and satisfy the bound $|\nabla^i \eta_j(x,t)| \leq C_i(2^j\ell(Q))^{-i}$, with

$$\eta_j(x,t) = 1 \text{ if } x \in 2^{j+2}Q \text{ and } -2^j\ell(Q) < t < 2^j\ell(Q),$$
$$\eta_j(x,t) = 0 \text{ if } x \notin 2^{j+3}Q \text{ or } |t| > 2^{j+1}\ell(Q).$$

Then $\dot{\mathbf{Tr}}_{m,\|}^-(\eta_j H^j) = \dot{\mathbf{Tr}}_{m,\|}^- H^j = \dot{\boldsymbol{f}}^j$. We take $F^j = \eta_j H^j$. Observe that if $j \geq 2$, then $\nabla^m F^j(x,t) = \nabla^m H^j(x,t) = 0$ if $|t| < \operatorname{dist}(x, \mathbb{R}^n \setminus 2^{j-1}Q)$. Furthermore, we still have the bound

$$\sup_{t<0} \|\nabla^m F^j(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{f}}^j\|_{L^2(\mathbb{R}^n)}.$$

Now, by the definition (4.3) of Θ_t^D and by formulas (2.29), and (2.28)

$$\int_Q |\Theta_t^D \dot{\boldsymbol{f}}^j|^2 = t^{2k} \int_Q \left| \sum_{|\beta|=m} \int_{\mathbb{R}^{n+1}_-} \partial_t^{m+k} \partial_{y,s}^\beta E^L(x,t,y,s)\, (\boldsymbol{A}\nabla^m F^j)_\beta(y,s)\, ds\, dy \right|^2 dx.$$

Applying the bound (3.1), we see that

$$\int_Q |\Theta_t^D \dot{\boldsymbol{f}}^j|^2 \leq C2^{-j(2k+\varepsilon)} \|\dot{\boldsymbol{f}}^j\|^2_{L^2(\mathbb{R}^n)}$$

for all $\dot{\boldsymbol{f}}^j \in \mathfrak{D}$ supported in $A_{j,1}(Q)$; by density we may extend to all $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m,\|}(\mathbb{R}^n)$ supported in $A_{j,1}(Q)$.

6.1 Extending Θ_t^D to all of $L^2(\mathbb{R}^n)$

We now must extend Θ_t^D to an operator defined on all of $L^2(\mathbb{R}^n)$ that still satisfies the estimate (5.1). Essentially, this argument consists of defining a projection operator from $L^2(\mathbb{R}^n)$ to $\dot{W}A^2_{m,\|}(\mathbb{R}^n)$, the space on which Θ_t^D naturally acts.

Because $L^2(\mathbb{R}^n)$ is a Hilbert space, there is an orthogonal projection operator $O_W : L^2(\mathbb{R}^n) \mapsto \dot{W}A^2_{m,\|}(\mathbb{R}^n)$. For example, if $m = 1$ then $O_W \vec{f} = \nabla_\| u$, where $\Delta_\| u = \nabla_\| \cdot \vec{f}$. This is the most natural mapping from $L^2(\mathbb{R}^n)$ to $\dot{W}A^2_{m,\|}(\mathbb{R}^n)$; however, this mapping does not satisfy adequate decay estimates. Thus, we must refine this mapping by applying cutoffs before and after projecting.

Let W_j be the closure in $L^2(\mathbb{R}^n)$ of

$$\{\mathbf{1}_{2^{j+2}Q}\, \dot{\mathbf{Tr}}_{m-1}\, \varphi + (1 - \mathbf{1}_{2^{j+2}Q})\dot{\boldsymbol{f}} : \varphi \in C_0^\infty,\ \dot{\boldsymbol{f}} \in L^2(\mathbb{R}^n)\}.$$

Loosely, elements of W_j are higher-order traces in the cube $2^{j+2}Q$ and are merely arbitrary L^2 arrays outside of that cube. Let O_j denote orthogonal projection from $L^2(\mathbb{R}^n)$ onto the subspace W_j; observe that $O_j\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}$ outside of $2^{j+2}Q$. Furthermore, if φ is a nice function then $O_j(\dot{\mathbf{Tr}}_{m-1}\,\varphi) = \dot{\mathbf{Tr}}_{m-1}\,\varphi$.

Let η_j be a smooth partition of unity; that is, $\sum_j \eta_j(x,t) = 1$ for t near zero, with η_j supported in $A_{j,1}(Q) \times (-2^j\ell(Q), 2^j\ell(Q))$ and satisfying $|\nabla^i \eta_j(x,t)| \leq C2^{-ij}\ell(Q)^{-i}$ for all $(x,t) \in \mathbb{R}^{n+1}$.

Define $\pi_j : W_j \mapsto \dot{W}A^2_{m,\|}(\mathbb{R}^n)$ as follows. Suppose that $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m,\|}\,\varphi$ in $2^{j+2}Q$ for some smooth function φ. We may renormalize φ so that $\int_Q \operatorname{Tr} \partial^\zeta \varphi = 0$ for all $|\zeta| \leq m-1$. Let $\pi_j\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m,\|}(\eta_j\varphi)$. We remark that $\pi_j\dot{\boldsymbol{f}}$ is well-defined, that is, $\dot{\mathbf{Tr}}_{m,\|}(\eta_j\varphi)$ depends only on $\dot{\mathbf{Tr}}_{m,\|}\,\varphi$. Furthermore, observe that $\pi_j\dot{\boldsymbol{f}}$ is supported in $A_{j,1}(Q)$. Finally, by the Poincaré inequality

$$\|\pi_j\dot{\boldsymbol{f}}\|_{L^2(A_{j,1}(Q))} \leq C2^{jn/2}\|\dot{\mathbf{Tr}}_{m,\|}\,\varphi\|_{L^2(2^{j+2}Q)} = C2^{jn/2}\|\dot{\boldsymbol{f}}\|_{L^2(2^{j+2}Q)}.$$

We will extend to an operator on all of $L^2(\mathbb{R}^n)$ using the orthogonal projection operators O_j. Observe, first, that $\pi_jO_j\dot{\boldsymbol{f}} = 0$ outside of $A_{j,1}(Q)$, and second, that if $\dot{\boldsymbol{f}} = 0$ in $2^{j+2}Q$ then $O_j\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}$ and so $\pi_jO_j\dot{\boldsymbol{f}} = 0$.

If φ is smooth and compactly supported in $\mathbb{R}^{n+1}$, and renormalized as above, then

$$\dot{\mathbf{Tr}}_{m,|}\,\varphi = \sum_{j=0}^{\infty} \dot{\mathbf{Tr}}_{m,|}(\eta_j\varphi) = \sum_{j=0}^{\infty} \pi_j(\dot{\mathbf{Tr}}_{m-1}\,\varphi) = \sum_{j=0}^{\infty} \pi_j(O_j(\dot{\mathbf{Tr}}_{m-1}\,\varphi)).$$

We define $\Theta_t^D \dot{\boldsymbol{f}} = \sum_{j=0}^{\infty} \Theta_t^D(\pi_j O_j \dot{\boldsymbol{f}})$. Now, if $\dot{\boldsymbol{f}}^j$ is supported in $A_j(Q)$, observe that $\pi_i O_i \dot{\boldsymbol{f}} = 0$ for all $i \leq j-2$, and furthermore that $\|\pi_i O_i \dot{\boldsymbol{f}}^j\|_{L^2(\mathbb{R}^n)} \leq C 2^{in/2} \|\dot{\boldsymbol{f}}^j\|_{L^2(A_j(Q))}$ for all $i \geq j-1$. Then

$$\begin{aligned}\|\Theta_t^D \dot{\boldsymbol{f}}^j\|_{L^2(Q)} &\leq \sum_{i=j-1}^{\infty} \|\Theta_t^D(\pi_i O_i \dot{\boldsymbol{f}}^j)\|_{L^2(Q)} \leq C \sum_{i=j-1}^{\infty} 2^{-i(k+\varepsilon/2)} \|\pi_i O_i \dot{\boldsymbol{f}}^j\|_{L^2(A_{i,1}(Q))} \\ &\leq C \sum_{i=j-1}^{\infty} 2^{-i(k+\varepsilon/2)} 2^{in/2} \|\dot{\boldsymbol{f}}^j\|_{L^2(A_j(Q))}\end{aligned}$$

and so Θ_t^D satisfies the decay estimate (5.1) provided k is large enough.

7 The quasi-orthogonality estimate (5.2)

In [43], a family of operators $\{Q_s\}_{s>0}$ was defined to be a Calderón-Littlewood-Paley family, or CLP family, if

$$Q_s f(x) = \int_{\mathbb{R}^n} s^{-n}\,\varphi(y/s)\,f(x-y)\,dy$$

for some $\varphi \in L^1(\mathbb{R}^n)$ that satisfies the conditions

$$|\widehat{\varphi}(\xi)| \leq C \min(|\xi|^\sigma, |\xi|^{-\sigma}), \qquad |\varphi(x)| \leq C(1+|x|)^{-n-\sigma} \tag{7.1}$$

for some $C > 0$ and $\sigma > 0$, where $\widehat{\varphi}$ denotes the Fourier transform of φ, and such that Q satisfies the conditions

$$\|Q_s f\|_{L^2(\mathbb{R}^n)} + \|s\nabla Q_s f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)} \quad \text{for all } s > 0, \tag{7.2}$$

$$\int_{\mathbb{R}^n}\int_0^\infty |Q_s f(x)|^2 \frac{ds\,dx}{s} \leq C\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{7.3}$$

$$\int_0^\infty Q_s^2 \frac{ds}{s} = I \tag{7.4}$$

where convergence to the identity in the last formula is in the strong operator topology on $\mathcal{B}(L^2(\mathbb{R}^n))$.

We now provide some conditions on φ that ensure validity of the bounds (7.2)–(7.4).

Lemma 7.1 *Suppose that $\psi \in L^1(\mathbb{R}^n)$ satisfies the bounds* (7.1) *for some $C > 0$ and $\sigma > 0$. Suppose further that $|\widehat{\psi}(\xi)| \leq C|\xi|^{-1}$. Finally, suppose that ψ is radial, real-valued, and not identically zero.*

Then there is some constant c such that if $\varphi = c\psi$, then $Q_s f(x) = \int_{\mathbb{R}^n} s^{-n}\varphi(y/s)\,f(x-y)\,dy$ is a CLP family.

P r o o f. Observe that

$$\widehat{Q_s f}(\xi) = c\,\widehat{\psi}(s\,\xi)\,\widehat{f}(\xi).$$

Given this relation, the estimates (7.2) and (7.3) follow from the estimate $|\widehat{\psi}(\xi)| \leq C\min(|\xi|^\sigma, |\xi|^{-1})$ by Plancherel's theorem. To establish the identity (7.4), we normalize ψ as follows. Because ψ is radial, so is $\widehat{\psi}$. Thus, the integral

$$\int_0^\infty c^2 \widehat{\psi}(s\,\xi)^2\,\frac{1}{s}\,ds$$

is independent of ξ provided $\xi \neq 0$. If ψ is both radial and real-valued, then $\widehat{\psi}$ is real-valued, and so this integral is positive; furthermore, our bounds on $\widehat{\psi}$ imply finiteness of this integral. We may choose c so that this integral equals 1. It is then straightforward to establish the condition (7.4). □

In this section, let $\widehat{\varphi}$ be bounded, radial and supported in $B(0,2)\setminus B(0,1/2)$, and such that formula (7.4) is valid. (In Section 9.2 we will use a CLP family again; in that section it will be more convenient to take φ, rather than $\widehat{\varphi}$, compactly supported.) We wish to establish the bound (5.2), for the operators $\Theta_t=\Theta_t^S$ or $\Theta_t=\Theta_t^D$. We proceed as in [43].

We begin with Θ_t^S. Fix some $\dot{\boldsymbol{h}}\in L^2(\mathbb{R}^n)$; we wish to bound $\Theta_t^S Q_s\dot{\boldsymbol{h}}$. For each $1\leq j\leq n$, let f_γ^j satisfy

$$\widehat{f_\gamma^j}(\xi)=\frac{\xi_j}{2\pi i|\xi|^2}\widehat{Q_s h_\gamma}(\xi)=\frac{\xi_j}{2\pi i|\xi|^2}\,\widehat{\varphi}(s\,\xi)\,\widehat{h}_\gamma(\xi).$$

Then $Q_s h_\gamma=\sum_{j=1}^n \partial_j f_\gamma^j$, and so

$$\Theta_t^S Q_s\dot{\boldsymbol{h}}(x)=\sum_{j=1}^n \Theta_t^S(\partial_j \dot{\boldsymbol{f}}^j)(x).$$

By the bound (6.1),

$$\|\Theta_t^S Q_s\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}\leq\sum_{j=1}^n\|\Theta_t^S(\partial_j\dot{\boldsymbol{f}}^j)\|_{L^2(\mathbb{R}^n)}\leq C\frac{1}{t}\sum_{j=1}^n\|\dot{\boldsymbol{f}}^j\|_{L^2(\mathbb{R}^n)}.$$

Notice that $|\widehat{\varphi}(\xi)|\leq C|\xi|$, and so $\|\dot{\boldsymbol{f}}\|_{L^2(\mathbb{R}^n)}\leq Cs\|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}$. Thus,

$$\|\Theta_t^S Q_s\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}\leq C\frac{s}{t}\|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}.$$

Therefore, Θ_t^S satisfies the bound (5.2) for $\theta=1$ (and thus for any $\theta\leq 1$).

We now consider Θ_t^D. Let the subspace H be $\dot{W}A^2_{m,|}(\mathbb{R}^n)$; recall that this is the natural space upon which $\widetilde{\mathcal{D}}^{\boldsymbol{A}}$ and Θ_t^D act. It suffices to show that

$$\|\Theta_t^D Q_s\,\dot{\mathbf{Tr}}_{m,|}\,\varphi\|_{L^2(\mathbb{R}^n)}\leq C\left(\frac{s}{t}\right)^\theta\|\dot{\mathbf{Tr}}_{m,|}\,\varphi\|_{L^2(\mathbb{R}^n)}$$

for some $\theta>0$ and for all smooth, compactly supported functions φ. To establish this bound, we begin with the following lemma.

Lemma 7.2 *If φ is smooth and compactly supported in $\mathbb{R}^{n+1}$, then*

$$Q_s(\dot{\mathbf{Tr}}_{m,|}\,\varphi)=\dot{\mathbf{Tr}}_{m,|}\,\Phi_s$$

for some function Φ_s that satisfies

$$\|\nabla^m\Phi_s\|_{L^2(\mathbb{R}^{n+1}_-)}\leq C\sqrt{s}\|\dot{\mathbf{Tr}}_{m,|}\,\varphi\|_{L^2(\mathbb{R}^n)}.$$

Proof. For each j with $0\leq j\leq m-1$, let $f^j(x)=\partial_{n+1}^j\varphi(x,0)$. Observe that if β is a multiindex and $\beta_{n+1}<|\beta|=m$, then

$$(\dot{\mathbf{Tr}}_{m,|}\,\varphi)_\beta=\partial_x^{\beta_\parallel}f^{\beta_{n+1}}(x,0).$$

Q_s is a convolution operator and so commutes with horizontal derivatives, and thus

$$(Q_s\,\dot{\mathbf{Tr}}_{m,|}\,\varphi)_\beta=\partial_x^{\beta_\parallel}Q_s f^{\beta_{n+1}}(x,0).$$

Let $g_s^j=Q_s f^j$. Then $\widehat{g}_s^j(\xi)=\widehat{\varphi}(s\,\xi)\,\widehat{f^j}(\xi)$ and so $|\xi|^{-1}\,|\widehat{g}_s^j(\xi)|^2\leq Cs|\widehat{f^j}(\xi)|^2$. Thus,

$$\int_{\mathbb{R}^n}|\widehat{g}_s^j(\xi)|^2|\xi|^{2m-2j-1}\,d\xi\leq Cs\int_{\mathbb{R}^n}|\widehat{f^j}(\xi)|^2|\xi|^{2m-2j}\,d\xi=Cs\int_{\mathbb{R}^n}|\nabla_\parallel^{m-j}f^j(x)|^2\,dx.$$

Because $|\nabla_\|^{m-j} f^j| \leq |\dot{\mathbf{Tr}}_{m,|}\,\varphi|$, we have that

$$\int_{\mathbb{R}^n} |\widehat{g}_s^j(\xi)|^2 |\xi|^{2m-2j-1}\,d\xi \leq Cs\|\dot{\mathbf{Tr}}_{m,|}\,\varphi\|_{L^2(\mathbb{R}^n)}^2.$$

Let $\dot{\boldsymbol{g}}_s = Q_s(\dot{\mathbf{Tr}}_{m-1}\,\varphi)$; then $\dot{\boldsymbol{g}}_s$ satisfies $(\dot{\boldsymbol{g}}_s)_\gamma = \partial^{\gamma_\|} g_s^{\gamma_{n+1}}$ for each $|\gamma| = m-1$. We have established that

$$\int_{\mathbb{R}^n} |\widehat{\dot{\boldsymbol{g}}}_s(\xi)|^2 |\xi|\,d\xi \leq Cs\|\dot{\mathbf{Tr}}_{m,|}\,\varphi\|_{L^2(\mathbb{R}^n)}^2.$$

Extending $\dot{\boldsymbol{g}}_s$ using Lemma 3.3 completes the proof. Notice that more general extension theorems, also appropriate in this case, are well known; see, for example, [70, Theorem 2.9.3] or [71, Theorem 2.7.2]. □

The estimate (5.2) for Θ_t^D follows quickly from Lemma 7.2. By the definition (4.3),

$$\Theta_t^D(Q_s\,\dot{\mathbf{Tr}}_{m,|}\,\varphi)(x) = t^k \partial_t^{m+k} \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\,\Phi_s)(x,t).$$

But by the definition (2.27),

$$\Theta_t^D(Q_s\,\dot{\mathbf{Tr}}_{m,|}\,\varphi)(x) = -t^k \partial_t^{m+k} \Pi^L(\mathbf{1}_- A\nabla^m \Phi_s)(x,t).$$

But $u = \Pi^L(\mathbf{1}_- A\nabla^m\Phi_s)$ satisfies $Lu = 0$ in $\mathbb{R}^{n+1}_+$; furthermore, by the bound (2.15), $\|\nabla^m u\|_{L^2(\mathbb{R}^{n+1}_+)} \leq C\|\nabla^m\Phi_s\|_{L^2(\mathbb{R}^{n+1}_-)} \leq C\sqrt{s}\|\dot{\mathbf{Tr}}_{m,|}\,\varphi\|_{L^2(\mathbb{R}^n)}$. Applying the Caccioppoli inequality and Lemma 3.2 in small cubes of sidelength t/C suffices to establish that Θ_t^D satisfies the bound (5.2) for $\theta = 1/2$.

8 The semi-horizontal single layer potential

In this section we will prove the following theorem.

Theorem 8.1 *Suppose that $\gamma_{n+1} < |\gamma| = m-1$. Then we have the square-function estimate*

$$\int_{\mathbb{R}^{n+1}_+} |\Theta_t^S(g\dot{\boldsymbol{e}}_\gamma)(x)|^2 \,\frac{1}{t}\,dx\,dt \leq C\|g\|_{L^2(\mathbb{R}^n)}^2.$$

Proof. Let $\Theta_t g = \Theta_t^S(g\dot{\boldsymbol{e}}_\gamma)$. We want to apply Theorem 5.2. As shown in Sections 6 and 7, the bounds (5.1) and (5.2) are valid for this choice of Θ_t. We are left with the estimate (5.3).

Recall that by formula (4.2),

$$\Theta_t g(x) = t^k \int_{\mathbb{R}^n} \partial_t^{m+k} \partial_{y,s}^\gamma E^L(x,t,y,0)\,g(y)\,dy.$$

In particular,

$$\Theta_t 1(x) = t^k \int_{\mathbb{R}^n} \partial_t^{m+k} \partial_{y,s}^\gamma E^L(x,t,y,0)\,dy.$$

Let j satisfy $1 \leq j \leq n$ and $\gamma_j > 0$; by assumption on γ such a j exists. Let $\zeta = \gamma - \vec{e}_j + \vec{e}_{n+1}$. By formula (2.24), we have that

$$\Theta_t 1(x) = -t^k \int_{\mathbb{R}^n} \partial_{y_j}\big(\partial_t^{m+k-1} \partial_{y,s}^\zeta E^L(x,t,y,0)\big)\,dy.$$

By the bound (3.1), for almost every $(x,t) \in \mathbb{R}^{n+1}$, if k is large enough then $v(y) = \partial_t^{m+k-1}\partial_{y,s}^\zeta E^L(x,t,y,0)$ lies in both $L^1(\mathbb{R}^n)$ and in $\dot{W}_1^1(\mathbb{R}^n)$. Thus,

$$\int_{\mathbb{R}^n} \partial_{y_j}\big(\partial_t^{m+k-1} \partial_{y,s}^\zeta E^L(x,t,y,0)\big)\,dy = 0$$

for almost every $(x,t) \in \mathbb{R}^{n+1}$. Thus, $\Theta_t 1 = 0$, and in particular the bound (5.3) is valid. □

We are now left with the double layer potential Θ_t^D and the vertical single layer potential $\Theta_t^\perp f = \Theta_t^S(f\dot{\boldsymbol{e}}_\perp)$. In the following sections we will use the full force of Theorem 5.4 to bound $\Theta_t^\perp$; we will bound Θ_t^D as a component of the auxiliary operator Θ_t' in Theorem 5.4.

9 The Carleson estimate (5.8)

We will let $\Theta_t = (\Theta_t^D, \Theta_t^S)$, with $\Theta_t^\perp$ denoting the purely vertical component of Θ_t^S (that is, $\Theta_t^\perp f = \Theta_t^S(f\dot{\boldsymbol{e}}_\perp)$). The bounds (5.1) and (5.2) are valid. We wish to show that the bound (5.8) is also valid. Observe that we may state this bound as

$$\|\Theta_t' \vec{e}_j\|_{\mathcal{C},\delta} \leq C_1 + C_1\|\Theta_t^\perp 1\|_{\mathcal{C},\delta} \qquad \text{for all } 1 \leq j \leq q, \text{ if } \delta > 0 \text{ small enough.}$$

In Section 8, we showed that $\Theta_t^S \dot{\boldsymbol{e}}_\gamma = 0$ whenever $\gamma_{n+1} < |\gamma| = m-1$; the bound (5.8), with $\Theta_t' = \Theta_t^{S'}$, follows immediately. Thus, we need only bound $\Theta_t^D \dot{\boldsymbol{e}}_\beta$ by $\Theta_t^\perp 1$ and a constant.

Recall that by formulas (4.3), (2.29) and (2.28), if $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m,|} F$, then

$$\Theta_t^D \dot{\boldsymbol{f}}(x) = -\sum_{|\alpha|=|\beta|=m} t^k \int_{\mathbb{R}^{n+1}_-} \partial_t^{m+k} \partial_{y,s}^\alpha E^L(x,t,y,s)\, A_{\alpha\beta}(y,s)\, \partial^\beta F(y,s)\, ds\, dy.$$

Using the bound (3.1), we see that if $\nabla^m F$ is bounded then the integral converges absolutely for almost every $(x,t) \in \mathbb{R}^{n+1}_+$.

Recall that Θ_t^D acts on arrays of functions of the form $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m,|}\, \varphi$; these arrays $\dot{\boldsymbol{\varphi}}$ are indexed by multiindices β with $\beta_{n+1} < |\beta| = m$. Fix some such β. By choosing

$$F(x,t) = \frac{1}{\beta!}(x,t)^\beta$$

we see that

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = -\sum_{|\alpha|=m} t^k \int_{\mathbb{R}^{n+1}_-} \partial_t^{m+k} \partial_{y,s}^\alpha E^L(x,t,y,s)\, A_{\alpha\beta}(y)\, ds\, dy. \tag{9.1}$$

We use formula (2.24) to convert one of the derivatives in t into a derivative in s; we evaluate the integral ds to see that

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = \sum_{|\alpha|=m} t^k \int_{\mathbb{R}^n} \partial_t^{m+k-1} \partial_{y,s}^\alpha E^L(x,t,y,0)\, A_{\alpha\beta}(y)\, dy.$$

Observe that this is a sum of terms depending on α and β. In Section 9.1 we will bound the terms for which $\alpha_{n+1} > 0$. In Sections 9.2–9.5, we will bound the terms for which $\alpha_{n+1} = \beta_{n+1} = 0$; this case is the most involved, and will closely parallel the argument in [43, Section 3.1]. Finally, in Section 9.6 we will bound the remaining terms, that is, the terms for which $\alpha_{n+1} = 0$ and $\beta_{n+1} > 0$; this bound will rely on the bound in the case $\alpha_{n+1} = \beta_{n+1} = 0$.

9.1 Terms with $\alpha_{n+1} > 0$

Observe that if $\alpha_{n+1} > 0$, then $\alpha = \gamma + \vec{e}_{n+1}$ for a unique γ with $|\gamma| = m-1$. For any such γ, let $\widetilde{\gamma} = \gamma + \vec{e}_{n+1}$. Then by formula (2.24),

$$\sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}} \partial_{y,s}^\alpha \partial_t^{m+k-1} E^L(x,t,y,0)\, A_{\alpha\beta}(y) = -\sum_{|\gamma|=m-1} \partial_{y,s}^\gamma \partial_t^{m+k} E^L(x,t,y,0)\, A_{\widetilde{\gamma}\beta}(y).$$

Thus,

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = \sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}} t^k \int_{\mathbb{R}^n} \partial_t^{m+k-1} \partial_{y,s}^\alpha E^L(x,t,y,0)\, A_{\alpha\beta}(y)\, dy$$
$$- \sum_{|\gamma|=m-1} t^k \int_{\mathbb{R}^n} \partial_t^{m+k} \partial_{y,s}^\gamma E^L(x,t,y,0)\, A_{\widetilde{\gamma}\beta}(y)\, dy.$$

In this section we will bound the second sum; we will consider the first sum in Sections 9.2–9.6.

By formula (4.2), the second sum is equal to $\Theta_t^S \dot{\boldsymbol{a}}_\beta$, where $(a_\beta)_\gamma = A_{\tilde{\gamma},\beta}$. Notice that $\dot{\boldsymbol{a}}_\beta$ is bounded. Our goal is to show that $\|\Theta_t^D \dot{\boldsymbol{e}}_\beta\|_{\mathcal{C},\delta} \leq C_1 + C_1\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}$. In this section we will show that $\|\Theta_t^S \dot{\boldsymbol{a}}_\beta\|_{\mathcal{C},0} \leq C_1$; because $\|\Theta_t^S \dot{\boldsymbol{a}}_\beta\|_{\mathcal{C},\delta} \leq \|\Theta_t^S \dot{\boldsymbol{a}}_\beta\|_{\mathcal{C},0}$ this will reduce matters to the terms for which $\alpha_{n+1} = 0$.

We may control $\Theta_t^S \dot{\boldsymbol{a}}_\beta$ using a standard technique in the study of $T1$ theorems. Let

$$P_t f(x) = \int_{\mathbb{R}^n} \frac{1}{t^n}\psi\left(\frac{x-y}{t}\right) f(y)\,dy$$

where ψ is smooth, nonnegative and satisfies $\int_{\mathbb{R}^n} \psi = 1$. We do not require that ψ be compactly supported. Fix some γ with $|\gamma| = m-1$, and let

$$\Psi_t a(x) = \Theta_t^S(a\,\dot{\boldsymbol{e}}_\gamma)(x) - P_t a(x)\,\Theta_t^S \dot{\boldsymbol{e}}_\gamma(x).$$

Then

$$\|P_t(a_\beta)_\gamma\,\Theta_t^S \dot{\boldsymbol{e}}_\gamma\|_{\mathcal{C},\delta} \leq \|\dot{\boldsymbol{a}}\|_{L^\infty}\|\Theta_t^S \dot{\boldsymbol{e}}_\gamma\|_{\mathcal{C},\delta}.$$

Either $\gamma = \gamma_\perp$ and so $\Theta_t^S \dot{\boldsymbol{e}}_\gamma = \Theta_t^\perp 1$, or $\gamma_{n+1} < |\gamma|$ and so $\Theta_t^S \dot{\boldsymbol{e}}_\gamma = 0$. (See the proof of Theorem 8.1.) So to bound $\|\Theta_t^S \dot{\boldsymbol{a}}_\beta\|_{\mathcal{C},\delta}$, we need only control $\Psi_t a(x)$ for arbitrary bounded functions a.

We begin our analysis of $\Psi_t a$ by applying Theorem 5.2 to Ψ_t. Observe that $\Psi_t 1(x) = 0$, and so the bound (5.3) is valid. We need only verify the bounds (5.1) and (5.2) for the operator Ψ_t; because these bounds have been verified for Θ_t^S, we need only consider $\Upsilon_t a(x) = P_t a(x)\,\Theta_t^S \dot{\boldsymbol{e}}_\gamma(x)$.

We begin with the bound (5.2). Observe that

$$\widehat{P_t Q_s h}(\xi) = \widehat{\psi}(t\,\xi)\,\widehat{\varphi}(s\,\xi)\,\widehat{h}(\xi)$$

where Q_s is the operator defined in Section 7. Recall that $\widehat{\varphi}$ is supported in $B(0,2)\setminus B(0,1/2)$. If we require that $\widehat{\psi}$ be smooth and supported in $B(0,1/2)$, then $P_t Q_s h = 0$ whenever $s \leq t$; thus Υ_t satisfies the bound (5.2).

We now establish the bound (5.1). By Section 6, we know that the operator Θ_t^S satisfies the decay estimate (5.1). From this we may verify that, if $Q \subset \mathbb{R}^n$ is a cube with $\ell(Q) \leq t < 2\ell(Q)$, then

$$\|\Theta_t^S \dot{\boldsymbol{e}}_\gamma\|_{L^2(Q)} \leq C|Q|^{1/2}.$$

If $\widehat{\psi}$ is smooth as well as being supported in $B(0,1/2)$, then ψ is a Schwartz function and satisfies the estimate $|\psi(y)| \leq C_N(1+|y|)^{-N}$ for any $N > 0$. If g_j is supported in $A_j(Q)$, and $\ell(Q) \leq t < 2\ell(Q)$, then

$$\sup_{x\in Q}|P_t g_j(x)| \leq C_N t^{-n} 2^{-jN}\|g_j\|_{L^1(A_j(Q))} \leq C_N t^{-n/2} 2^{-j(N-n/2)}\|g_j\|_{L^2(A_j(Q))}$$

and so if we choose N large enough, then the operator Υ_t satisfies the bound (5.1).

Thus, by Theorem 5.2, we have that the operators Ψ_t satisfy the square-function estimate (5.4). To show that $\|\Psi_t a\|_{\mathcal{C}} \leq C\|a\|_{L^\infty(\mathbb{R}^n)}$, we need only show that the estimate (5.4) for L^2 test functions implies an estimate for L^∞ test functions.

Lemma 9.1 *Suppose that the operators Ψ_t satisfy the square-function estimate*

$$\int_0^\infty \int_{\mathbb{R}^n} |\Psi_t f(x)|^2 \frac{dx\,dt}{t} \leq C_0\|f\|^2_{L^2(\mathbb{R}^n)}$$

for all $f \in L^2(\mathbb{R}^n \mapsto \mathbb{C})$, and that for some $\theta > -2$, Ψ_t satisfies the off-diagonal decay estimate

$$\|\Psi_t g_j\|_{L^2(Q)} \leq C_1 2^{-j(n+2+\theta)/2}\|g_j\|_{L^2} \qquad \text{if } \ell(Q) \leq t \leq 2\ell(Q)$$

for all $j \geq 1$ and all g_j supported in $A_j(Q) = 2^{j+1}Q \setminus 2^j Q$.

Then there is some C depending only on C_0, C_1 and θ such that Ψ_t satisfies the Carleson condition

$$\|\Psi_t b\|^2_{\mathcal{C},0} = \sup_Q \frac{1}{|Q|}\int_Q \int_0^{\ell(Q)} |\Psi_t b(x)|^2 \frac{dt\,dx}{t} \leq C\|b\|^2_{L^\infty(\mathbb{R}^n)}$$

for all bounded functions b.

P r o o f. Choose some cube Q and some bounded function b. Let $b_j = b\mathbf{1}_{A_j(Q)}$; recall that $b_0 = b\mathbf{1}_{A_0(Q)} = b\mathbf{1}_{2Q}$. Then by the square-function estimate,

$$\left(\int_Q \int_0^{\ell(Q)} |\Psi_t b_0(x)|^2 \frac{dt\,dx}{t}\right)^{1/2} \leq C\|b_0\|_{L^2(2Q)} \leq C\|b\|_{L^\infty(\mathbb{R}^n)}|Q|^{1/2}.$$

Furthermore, if $j \geq 1$, then by the decay estimate applied in cubes $R \subset Q$ with side-length $t \leq \ell(R) < 2t$,

$$\left(\int_0^{\ell(Q)} \int_Q |\Psi_t b_j(x)|^2 \,dx\, \frac{dt}{t}\right)^{1/2} \leq C\|b\|_{L^\infty(\mathbb{R}^n)}|Q|^{1/2}2^{-j(2+\theta)/2}.$$

Summing in j, we see that

$$\left(\int_Q \int_0^{\ell(Q)} |\Psi_t b(x)|^2 \frac{dt\,dx}{t}\right)^{1/2} \leq C\|b\|_{L^\infty(\mathbb{R}^n)}|Q|^{1/2}$$

as desired. □

9.2 Terms with $\alpha_{n+1} = \beta_{n+1} = 0$: preliminaries

We have now established that

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = \sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}} t^k \int_{\mathbb{R}^n} \partial_t^{m+k-1}\partial_{y,s}^\alpha E^L(x,t,y,0)\, A_{\alpha\beta}(y)\,dy - \Theta_t^S \dot{\boldsymbol{a}}_\beta(x)$$

and that $\|\Theta_t^S \dot{\boldsymbol{a}}_\beta\|_{\mathcal{C},\delta} \leq C_1$. In order to show that Θ_t^D satisfies the estimate (5.8), we need only show that the operator Θ_t^β, defined as

$$\Theta_t^\beta f(x) = \sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}} t^k \int_{\mathbb{R}^n} \partial_t^{m+k-1}\partial_y^\alpha E^L(x,t,y,0)\, A_{\alpha\beta}(y)\, f(y)\,dy, \tag{9.2}$$

satisfies the Carleson-measure estimate

$$\|\Theta_t^\beta 1\|_{\mathcal{C},\delta} \leq C_1 + C_1\|\Theta_t^\perp 1\|_{\mathcal{C},\delta} \tag{9.3}$$

for any multiindex β with $\beta_{n+1} < |\beta| = m$.

We will make use of the horizontal operator $L_\|$, defined as follows. Recall that L is an operator acting on $\dot{W}^2_{m,loc}(\mathbb{R}^{n+1})$-functions. We may (formally) define the operator $L_\|$, acting on $\dot{W}^2_{m,loc}(\mathbb{R}^n)$-functions, as

$$L_\| f = (-1)^m \sum_{\substack{|\alpha|=|\beta|=m\\ \alpha_{n+1}=\beta_{n+1}=0}} \partial^\alpha(A_{\alpha\beta}\partial^\beta f) \tag{9.4}$$

where ∂^α, ∂^β are understood to be derivatives in the n horizontal directions. (The operator $L_\|$ has a weak formulation, as in formula (2.6).)

To establish the bound (9.3) for $\beta_{n+1} = 0$, we will follow the argument of [43, Section 3.1]. We remark that the argument we will make in this section is valid only in the case where the order $2m$ of L (and thus $L_\|$) satisfies the inequality $2m > n$. Thus, the argument of Sections 9 and 10 will only establish boundedness of Θ_t^D and $\Theta_t^\perp$ in the case of operators of very high order. In Section 11 we will show that bounds on Θ_t^D and Θ_t^S, for operators of high order, imply the corresponding bounds for operators of lower order, completing the proof of Theorem 1.1.

We will use some tools from the proof of the Kato conjecture, in particular from the paper [44]. The following lemma was established therein.

Lemma 9.2 *Suppose that $2m \geq n$. There is some W depending only on the standard constants such that, for each cube $Q \subset \mathbb{R}^n$, there exist W functions $f_{Q,w}$ that satisfy the estimates*

$$\int_R |\nabla_\|^m f_{Q,w}|^2 \leq C|Q| \qquad \text{for any cube } R \text{ with } \ell(R) = \ell(Q), \tag{9.5}$$

$$|L_\| f_{Q,w}(x)| \leq \frac{C}{\ell(Q)^m}, \tag{9.6}$$

and such that, for any array $\dot{\boldsymbol{\gamma}}_t$,

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\dot{\boldsymbol{\gamma}}_t(x)|^2 \frac{dx\,dt}{t} \leq C\sum_{w=1}^W \sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\langle \dot{\boldsymbol{\gamma}}_t(x), A_t^Q \nabla_\|^m f_{Q,w}(x)\rangle|^2 \frac{dx\,dt}{t} \tag{9.7}$$

where $A_t^Q f(x) = \fint_{Q'} f(y)\,dy$, for $Q' \subset Q$ the unique dyadic subcube that satisfies $x \in Q'$ and $t \leq \ell(Q') < 2t$.

Specifically, the bound (9.6) is the bound (2.19) in [44]. The bound (9.5) follows from the bound (2.18) in [44] (if $R = Q$) and the observation that, by Lemma 3.1 in [44] and the definition of $f_{Q,w}$ therein, $\nabla_\|^m f_{Q,w} = \nabla_\|^m f_{R,w}$ whenever $\ell(Q) = \ell(R)$. Finally, the bound (9.7) is simply Lemma 2.2 of [44]. The requirement that $2m \geq n$ is a sufficient condition (see [44, Propositon 2.5] or [72, 73]) for $L_\|$ to satisfy a pointwise upper bound; this condition is assumed in the proofs of the above results.

Let $(\gamma_t)_\beta = \mathbf{1}_{\delta<t<1/\delta}\Theta_t^\beta 1$; notice that $\dot{\boldsymbol{\gamma}}_t$ includes $\Theta_t^\beta 1$ provided $\beta_{n+1} = 0$. Thus, the estimates (9.3) and thus (5.8), for $\beta_{n+1} = 0$, follow from the estimate

$$\sup_Q \frac{1}{|Q|}\int_\delta^{\min(1/\delta,\ell(Q))}\int_Q \Big|\sum_{\substack{|\beta|=m\\ \beta_{n+1}=0}} \Theta_t^\beta 1(x)\, A_t^Q \partial_\|^\beta f_{Q,w}(x)\Big|^2 \frac{dx\,dt}{t} \leq C_1 + C_1\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}.$$

for $\delta < \ell(Q)$ and $\delta < 1$.

We will divide this quantity into a sum of controllable terms as follows. Let $P_t f(x) = f * \psi_t(x)$, where $\psi_t(x) = t^{-n}\psi(x/t)$. We require that ψ be smooth and nonnegative, that $\int \psi = 1$, and that $\psi(x) = 0$ whenever $|x| > 1/2$. (We will later impose some additional constraints on ψ. Notice that it is convenient to use a different approximate identity P_t in this section from that used in Section 9.1.) Let

$$\begin{aligned}
R_t^{1,\beta} F(x) &= \Theta_t^\beta 1(x)\,(A_t^Q F(x) - P_t F(x)),\\
R_t^{2,\beta} F(x) &= \Theta_t^\beta 1(x)\, P_t F(x) - \Theta_t^\beta (P_t F)(x),\\
R_t^{3,\beta} F(x) &= \Theta_t^\beta (P_t F - F)(x),\\
R_t^{4,\beta} F(x) &= \Theta_t^\beta F(x)
\end{aligned}$$

so that we seek to establish the estimate

$$\frac{1}{|Q|}\int_\delta^{\min(1/\delta,\ell(Q))}\int_Q \Big|\sum_{\substack{|\beta|=m\\ \beta_{n+1}=0}} R_t^{j,\beta}\partial^\beta f_{Q,w}(x)\Big|^2 \frac{dx\,dt}{t} \leq C_1 + C_1\|\Theta_t^\perp 1\|_{\mathcal{C},\delta} \tag{9.8}$$

for $j = 1, 2, 3, 4$ and for all cubes $Q \subset \mathbb{R}^n$.

We begin with $R_t^{4,\beta}$. Observe that by the definition (9.2) of Θ_t^β,

$$\sum_{\substack{|\beta|=m\\ \beta_{n+1}=0}} R_t^{4,\beta}\partial^\beta f_{Q,w}(x) = \sum_{\substack{|\alpha|=|\beta|=m\\ \alpha_{n+1}=\beta_{n+1}=0}} t^k \int_{\mathbb{R}^n} \partial_y^\alpha \partial_t^{m+k-1} E^L(x,t,y,0)\, A_{\alpha\beta}(y)\,\partial^\beta f_{Q,w}(y)\,dy.$$

Using formula (2.24) and then integrating by parts in y, we see that this quantity is equal to

$$\sum_{\substack{|\beta|=m\\ \beta_{n+1}=0}} R_t^{4,\beta}\partial^\beta f_{Q,w}(x) = (-1)^{k-1} t^k \int_{\mathbb{R}^n} \partial_t^m \partial_s^{k-1} E^L(x,t,y,0)\, L_\| f_{Q,w}(y)\,dy$$

and by the bounds (9.6) and (3.1), we see that if k is large enough then

$$\frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q \Big|\sum_{\substack{|\beta|=m\\ \beta_{n+1}=0}} R_t^{4,\beta}\partial^\beta f_{Q,w}(x)\Big|^2\frac{dx\,dt}{t}\leq C.$$

9.3 The term $R_t^{1,\beta}\partial^\beta f_{Q,w}$

Next, we bound $R_t^{1,\beta}\partial^\beta f_{Q,w}$. We begin by establishing a L^∞ bound on $\Theta_t^\beta 1$. (Recall that our goal is a Carleson estimate on $\Theta_t^\beta 1$; the L^∞ estimate is weaker but suffices to establish a Carleson estimate on $R_t^{1,\beta}\partial^\beta f_{Q,w}$.)

Recall the following special case of Morrey's inequality (see, for example, [56, Section 5.6.3]): if $x\in Q$ and $Q\subset\mathbb{R}^n$ is a cube, then for every $v\in \dot W^2_m(Q)\cap L^2(Q)$, there is a representative of v that satisfies

$$|v(x)|\leq \sum_{i=0}^m C\ell(Q)^i\left(\fint_Q |\nabla_\parallel^i v|^2\right)^{1/2}\quad\text{provided } 2m>n.$$

We apply this bound to the function $v(x)=\partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)$, a locally Sobolev function for almost all y and t. Then by the bound (3.1), we have that if $|\alpha|=m$ and either $|t|=\ell(R)$ or $|t|<\ell(R)$ and $j\geq 1$, then

$$\int_{A_j(R)}|\partial_{y,s}^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)|^2\,dy\leq C\ell(R)^{-n-2k}2^{-j(n+2k)}. \tag{9.9}$$

Observe that this bound is valid for any $k>1/2-n/2$; in this section we will need this bound only for k large, but in Section 10 we will need this bound for $k=0$ and $k=1$ as well. Also, by formula (2.19) it is valid with the roles of y and x reversed.

Using Hölder's inequality and summing over j, we see that if $k>0$ then

$$|\Theta_t^\beta 1(x)|=\Big|\sum_{\substack{|\alpha|=m\\ \alpha_{n+1}=0}} t^k\int_{\mathbb{R}^n}\partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)\,A_{\alpha\beta}(y)\,dy\Big|\leq C$$

and so

$$|R_t^{1,\beta}\partial^\beta f_{Q,w}(x)|\leq C|A_t^Q\partial^\beta f_{Q,w}(x)-P_t\partial^\beta f_{Q,w}(x)|.$$

Thus, we need only bound $A_t^Q\partial^\beta f_{Q,w}(x)-P_t\partial^\beta f_{Q,w}(x)$. We will do this using a standard orthogonality argument.

Let $\widetilde R_t^1=A_t^Q-P_t$. Recall that the kernel ψ of P_t is supported in $B(0,1/2)$; thus, if $x\in Q$ and $t<\ell(Q)$, then $\widetilde R_t^1F(x)=\widetilde R_t^1(\mathbf{1}_{2Q}F)(x)$, and so

$$\int_0^{\ell(Q)}\int_Q|R_t^{1,\beta}\partial^\beta f_{Q,w}(x)|^2\frac{dx\,dt}{t}\leq C\int_0^\infty\int_{\mathbb{R}^n}|\widetilde R_t^1(\mathbf{1}_{2Q}\partial^\beta f_{Q,w})(x)|^2\frac{dx\,dt}{t}.$$

Let $\{Q_s\}$ be a CLP family, as in Section 7, but with the kernel φ (and not its Fourier transform $\widehat\varphi$) supported in $B(0,1/2)$. By the identity (7.4),

$$\int_0^{\ell(Q)}\int_Q|R_t^{1,\beta}\partial^\beta f_{Q,w}(x)|^2\frac{dx\,dt}{t}\leq C\int_0^\infty\int_{\mathbb{R}^n}\left|\int_0^\infty \widetilde R_t^1Q_s^2(\mathbf{1}_{2Q}\partial^\beta f_{Q,w})(x)\,\frac{ds}{s}\right|^2\frac{dx\,dt}{t}.$$

By Hölder's inequality, for any number $\varepsilon>0$,

$$\int_0^{\ell(Q)}\int_Q|R_t^{1,\beta}\partial^\beta f_{Q,w}(x)|^2\frac{dx\,dt}{t}$$
$$\leq\frac{C}{\varepsilon}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}|\widetilde R_t^1Q_s^2(\mathbf{1}_{2Q}\partial^\beta f_{Q,w})(x)|^2\,dx\,\max\left(\frac{s}{t},\frac{t}{s}\right)^\varepsilon\frac{ds}{s}\frac{dt}{t}.$$

We claim that

$$\|\widetilde{R}_t^1 Q_s g\|_{L^2(\mathbb{R}^n)} \leq C \min\left(\frac{s}{t}, \frac{t}{s}\right)^{1/6} \|g\|_{L^2(\mathbb{R}^n)}. \tag{9.10}$$

Choose $\varepsilon = 1/6$. Assuming validity of the bound (9.10), we have that

$$\int_0^{\ell(Q)} \int_Q |R_t^{1,\beta} \partial^\beta f_{Q,w}(x)|^2 \frac{dx\,dt}{t} \\ \leq C \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} |Q_s(\mathbf{1}_{2Q} \partial^\beta f_{Q,w})(x)|^2 \,dx\, \min\left(\frac{s}{t}, \frac{t}{s}\right)^{1/6} \frac{ds}{s}\frac{dt}{t}.$$

Interchanging the order of integration and evaluating the integral in t, we see that

$$\int_0^{\ell(Q)} \int_Q |R_t^{1,\beta} \partial^\beta f_{Q,w}(x)|^2 \frac{dx\,dt}{t} \leq C \int_0^\infty \int_{\mathbb{R}^n} |Q_s(\mathbf{1}_{2Q} \partial^\beta f_{Q,w})(x)|^2 \,dx\, \frac{ds}{s}.$$

By the bounds (7.3) and (9.5), we have that

$$\int_0^{\ell(Q)} \int_Q |R_t^{1,\beta} \partial^\beta f_{Q,w}(x)|^2 \frac{dx\,dt}{t} \leq C \|\partial^\beta f_{Q,w}\|^2_{L^2(2Q)} \leq C|Q|$$

and so the bound (9.8) is valid for $j = 1$. Thus, to complete our bound on $R_t^{1,\beta} \partial^\beta f_{Q,w}$, we need only establish the estimate (9.10).

Suppose first that $t \leq s$ and so $\min(s/t, t/s) = t/s$. By definition of $\widetilde{R}_t^Q$ and Q_s,

$$\widetilde{R}_t^1 Q_s g(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\frac{1}{|Q'|}\mathbf{1}_{Q'}(y) - \psi_t(x-y)\right) \varphi_s(y-z)\, g(z)\,dy\,dz.$$

Notice that $\int_{\mathbb{R}^n} \frac{1}{|Q'|}\mathbf{1}_{Q'}(y) - \psi_t(x-y)\,dy = 0$, and that the integrand is zero unless $|x-y| < Ct$. Thus

$$\widetilde{R}_t^1 Q_s g(x) \leq \int_{\mathbb{R}^n} \int_{B(x,Ct)} \left(\frac{1}{|Q'|}\mathbf{1}_{Q'}(y) - \psi_t(x-y)\right) \big(\varphi_s(y-z) - \varphi_s(x-z)\big)\,dy\, g(z)\,dz.$$

Suppose that $y \in B(x, Ct)$. Because φ is supported in $B(0, 1/2)$ and $s \geq t$, if $|x - z| > 2Cs$, then $\varphi_s(y-z) - \varphi_s(x-z) = 0$. Otherwise,

$$|\varphi_s(y-z) - \varphi_s(x-z)| \leq Cs^{-n-1}|y-x|.$$

Thus,

$$\begin{aligned}\widetilde{R}_t^1 Q_s g(x) &= C \int_{B(x,2Cs)} \frac{t}{s^{n+1}} \int_{B(x,Ct)} \left|\frac{1}{|Q'|}\mathbf{1}_{Q'}(y) - \psi_t(x-y)\right| dy\,|g(z)|\,dz \\ &\leq C\frac{t}{s} \fint_{B(x,Cs)} |g(z)|\,dz \leq C\frac{t}{s}\mathcal{M}g(x)\end{aligned}$$

where $\mathcal{M}g$ denotes the Hardy-Littlewood maximal function of g. It is well known that $\mathcal{M}$ is bounded $L^p(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$ for any $1 < p \leq \infty$, and so the estimate (9.10) is valid whenever $t < s$.

Recall that the kernel φ_s of Q_s also integrates to zero and that the kernel ψ_t of P_t is also smooth. Thus, by a similar argument, if $s \leq t$ then $\|P_t Q_s g\|_{L^2(\mathbb{R}^n)} \leq C(s/t)\|g\|_{L^2(\mathbb{R}^n)}$. Bounding $A_t^Q Q_s g$ is somewhat more involved, because the kernel $\frac{1}{|Q'|}\mathbf{1}_{Q'}$ of A_t^Q is not smooth.

Suppose $s \leq t$. Let $\eta = \eta_{t,s,x}$ be a smooth cutoff function that is identically 1 in Q' and is supported in $(1 + \sqrt{s/t})Q'$. We may require that $|\nabla\eta| \leq C/\sqrt{st}$. Let

$$B_{t,s}^Q G(x) = \frac{1}{|Q'|} \int \eta(y)\, G(y)\,dy.$$

By the same argument as above, we may show that if $s \le t$ then $|B_{t,s}^Q Q_s g(x)| \le C(s/t)^{1/2}\mathcal{M}g(x)$. To conclude the argument, notice that

$$|A_t^Q Q_s g(x) - B_{t,s}^Q Q_s g(x)| \le \frac{C}{t^n}\int_{\operatorname{supp}\eta\setminus Q'} |Q_s g|.$$

Notice that $|\operatorname{supp}\eta \setminus Q'| \le Ct^n\sqrt{s/t}$. We apply Hölder's inequality to see that

$$\begin{aligned}|A_t^Q Q_s g(x) - B_{t,s}^Q Q_s g(x)| &\le C\left(\frac{1}{t^n}\int_{\operatorname{supp}\eta} |Q_s g|^{3/2}\right)^{2/3}\left(\frac{s}{t}\right)^{1/6} \\ &\le C\left(\frac{s}{t}\right)^{1/6}\mathcal{M}(|Q_s g|^{3/2})(x)^{2/3}.\end{aligned}$$

Because $3/2 < 2$, the estimate (9.10) is valid for $s \le t$ as well as $t \le s$. This establishes our desired bound on $R_t^{1,\beta}\partial^\beta f_{Q,w}$.

9.4 The term $R_t^{2,\beta}\partial^\beta f_{Q,w}$

Next, we bound $R_t^{2,\beta}\partial^\beta f_{Q,w}$. We will use the following lemma from [17]; this is a square-function $T1$ theorem that is somewhat simpler than Theorem 5.2 but has more stringent requirements.

Lemma 9.3 (Lemna 3.5(ii) in [17]) *Suppose that $\{R_t\}_{t>0}$ is a family of operators defined on $L^2(\mathbb{R}^n)$ and satisfying*

$$\|R_t(F\mathbf{1}_{A_j(Q)})\|_{L^2(Q)}^2 \le C2^{-nj}\left(\frac{t}{2^j\ell(Q)}\right)^4 \|F\|_{L^2(A_j(Q))}^2 \tag{9.11}$$

for all $0 < t < \ell(Q)$ and all $j \ge 1$. Suppose further that for all $t > 0$, all $F \in L^2(\mathbb{R}^n)$, and all smooth $\vec F \in L^2(\mathbb{R}^n)$, we have the bounds

$$\|R_t F\|_{L^2(\mathbb{R}^n)} \le C\|F\|_{L^2(\mathbb{R}^n)}, \qquad \|R_t \operatorname{div}_\| \vec F\|_{L^2(\mathbb{R}^n)} \le \frac{C}{t}\|\vec F\|_{L^2(\mathbb{R}^n)}. \tag{9.12}$$

Finally, suppose that $R_t 1 = 0$ for all $t > 0$.
Then

$$\int_{\mathbb{R}^n}\int_0^\infty |R_t F(x)|^2 \frac{dt\,dx}{t} \le C\|F\|_{L^2(\mathbb{R}^n)}^2 \tag{9.13}$$

for all $F \in L^2(\mathbb{R}^n)$.

By the definitions of Θ_t^β and $R_t^{2,\beta}$, we have that

$$\begin{aligned}R_t^{2,\beta}F(x) &= \Theta_t^\beta 1(x)\,P_tF(x) - \Theta_t^\beta(P_tF)(x) \\ &= \sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}} t^k\int_{\mathbb{R}^n} \partial_t^{m+k-1}\partial_y^\alpha E^L(x,t,y,0)\,A_{\alpha\beta}(y)\big(P_tF(x) - P_tF(y)\big)\,dy.\end{aligned}$$

Let

$$\widetilde R_t^{2,\alpha,\beta}F(x) = t^k\int_{\mathbb{R}^n} \partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)\,A_{\alpha\beta}(y)\,(P_tF(x) - P_tF(y))\,dy. \tag{9.14}$$

In this section we need only bound $\widetilde R_t^{2,\alpha,\beta}$ for $\alpha_{n+1} = 0$; in Section 9.5 we will need an estimate on $\widetilde R_t^{2,\alpha,\beta}$ in the case where $\alpha_{n+1} > 0$.

Observe that $P_t 1(x) = P_t 1(y) = 1$ and so $\widetilde R_t^{2,\alpha,\beta} 1 = 0$.

Now, recall that $P_tF(x) = \int t^{-n}\psi((x-y)/t)\,F(y)\,dy$ for some smooth, compactly supported function ψ; then

$$\|P_tF\|_{L^2(\mathbb{R}^n)} \leq C\|F\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|P_t(\operatorname{div}_\| \vec F)\|_{L^2(\mathbb{R}^n)} \leq \frac{C}{t}\|\vec F\|_{L^2(\mathbb{R}^n)}.$$

We will use this fact to establish the bounds (9.11) and (9.12).

Let Q be a cube and let $0 < t < \ell(Q)$. If $j \geq 1$ and F is supported in $A_j(Q)$, observe that $P_tF(x)$ is supported in $A_{j,1}(Q)$. Thus, by the bound (3.1),

$$\|\widetilde R_t^{2,\alpha,\beta}(F\mathbf{1}_{A_j(Q)})\|_{L^2(Q)} \leq C\frac{t^k}{\ell(Q)^k}2^{-j(k-1/2+\varepsilon/2)}\|P_t(F\mathbf{1}_{A_j(Q)})\|_{L^2(\mathbb{R}^n)}.$$

(In the case $j=1$ some extra care must be taken to establish this estimate; however, it may be done by considering the cases $t > \ell(Q)/2$ and $t \leq \ell(Q)/2$ separately.)

This implies the bound (9.11). We are left with the uniform L^2 bounds (9.12).

Suppose that F is supported in $8Q$ and that $\ell(Q)/2 < t \leq \ell(Q)$. Then $P_tF(y) = 0$ for all $y \notin 16Q$ and so

$$\begin{aligned}|\widetilde R_t^{2,\alpha,\beta}F(x)| &\leq Ct^k\int_{\mathbb{R}^n}|\partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)||P_tF(x)-P_tF(y)|\,dy\\ &\leq Ct^k\int_{16Q}|\partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)||P_tF(x)-P_tF(y)|\,dy\\ &\quad + Ct^k\sum_{j=1}^\infty|P_tF(x)|\int_{A_j(Q)}|\partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)|\,dy.\end{aligned}$$

Applying the bound (9.9), we see that

$$|\widetilde R_t^{2,\alpha,\beta}F(x)| \leq C|P_tF(x)| + Ct^{-n/2}\|P_tF\|_{L^2(16Q)}.$$

Thus,

$$\|R_t(F\mathbf{1}_{8Q})\|_{L^2(Q)} \leq C\|P_t(F\mathbf{1}_{8Q})\|_{L^2(\mathbb{R}^n)}.$$

We sum over cubes of side-length t; this yields the bound

$$\|R_tF\|_{L^2(Q)} \leq C\|P_tF\|_{L^2(\mathbb{R}^n)}$$

and, combined with the existing bounds on P_tF and $P_t\operatorname{div}_\| F$, yields the desired estimates (9.12).

Thus, Lemma 9.3 applies and the operator $\widetilde R_t^{2,\alpha,\beta}$ satisfies a square-function estimate (9.13). In particular, using the bound (9.5) on $\nabla_\|^m f_{Q,w}$ and arguing as in the proof of Lemma 9.1, we have the desired Carleson bound (9.8) for $j=2$.

9.5 The term $R_t^{3,\beta}\partial^\beta f_{Q,w}$

Finally, we consider the term $R_t^{3,\beta}\partial^\beta f_{Q,w}$. As in the case of $R_t^{4,\beta}$, but unlike $R_t^{1,\beta}$ and $R_t^{2,\beta}$, we will not be able to bound the individual terms $R_t^{3,\beta}\partial^\beta f_{Q,w}$; we will only be able to bound

$$\begin{aligned}\widetilde R_t^3 f_{Q,w}(x) &= \sum_{\substack{|\beta|=m\\ \beta_{n+1}=0}} R_t^{3,\beta}\partial^\beta f_{Q,w}(x)\\ &= \sum_{\substack{|\alpha|=|\beta|=m\\ \alpha_{n+1}=\beta_{n+1}=0}} t^k\int_{\mathbb{R}^n}\partial_y^\alpha\partial_t^{m+k-1}E^L(x,t,y,0)\,A_{\alpha\beta}(y)\,\partial^\beta(P_tf_{Q,w}-f_{Q,w})(y)\,dy.\end{aligned}$$

Another complication of this section is that we will need to use the term $\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}$ on the right-hand side of the bound (9.8).

Let $f = (f_{Q,w} - p_{Q,w})\eta_Q$, where η_Q is a smooth cutoff function that is identically 1 in $2Q$ and is supported in $4Q$, and where $p_{Q,w}$ is a polynomial of degree $m-1$. By the Poincaré inequality, we may choose $p_{Q,w}$ so that $\|\nabla_\|^m f\|_{L^2(\mathbb{R}^n)} \leq \|\nabla_\|^m f_{Q,w}\|_{L^2(4Q)}$, and by the bound (9.5),

$$\|\nabla_\|^m f\|_{L^2(\mathbb{R}^n)} \leq C|Q|^{1/2}.$$

Furthermore, $\partial^\beta f = \partial^\beta f_{Q,w}$ in $2Q$ whenever $|\beta| = m$. Using the bound (9.9) on E^L and the bound (9.5) on $\nabla_\|^m f_{Q,w}$, we may show that

$$\frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |\widetilde{R}_t^3 f_{Q,w}(x) - \widetilde{R}_t^3 f(x)|^2 \frac{dt\,dx}{t} \leq C$$

and so to establish the bound (9.8) for $j = 3$, we need only establish the bound

$$\frac{1}{|Q|}\int_Q\int_\delta^{\min(1/\delta),\ell(Q)} |\widetilde{R}_t^3 f(x)|^2 \frac{dt\,dx}{t} \leq C + C\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}.$$

By the definition (9.4) of $L_\|$, and by formula (2.19),

$$\widetilde{R}_t^3 f(x) = t^k \int_{\mathbb{R}^n} \partial_t^{m+k-1}\overline{L_\|^* E^{L^*}(y,0,x,t)}\,(P_t f - f)(y)\,dy$$

where $L_\|^*$ is taken in the y variable. Recalling that $L_{y,s}^*(E^{L^*}(y,s,x,t)) = 0$ away from (x,t), we see that

$$\overline{L_\|^* E^{L^*}(y,0,x,t)} = (-1)^{m+1} \sum_{\substack{|\xi|=|\zeta|=m \\ \xi_{n+1}+\zeta_{n+1}\geq 1}} \partial_{y,s}^\zeta(A_{\xi\zeta}(y)\,\partial_{y,s}^\xi E^L(x,t,y,0)).$$

Thus, we need only establish the bound

$$\frac{1}{|Q|}\int_Q\int_\delta^{\min(1/\delta),\ell(Q)} |\widetilde{R}_t^{3,\xi,\zeta} f(x)|^2 \frac{dt\,dx}{t} \leq C + C\|\Theta_t^\perp 1\|_{\mathcal{C},\delta} \tag{9.15}$$

where

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) = (-1)^m t^k \int_{\mathbb{R}^n} \partial_{y,s}^\zeta(A_{\xi\zeta}(y)\,\partial_{y,s}^\xi \partial_t^{m+k-1} E^L(x,t,y,0))\,(P_t f - f)(y)\,dy$$

and where at least one of ζ_{n+1} and ξ_{n+1} is positive.

For each multiindex ζ, we write $\zeta = \zeta_\| + \zeta_\perp \vec{e}_\perp$, where $\zeta_\perp = \zeta_{n+1}$ and where $\zeta_\|$ is a multiindex with $(\zeta_\|)_{n+1} = 0$. Integrating by parts and applying formula (2.24), we see that

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) = t^k \int_{\mathbb{R}^n} A_{\xi\zeta}(y)\,\partial_{y,s}^\xi \partial_t^{m+k-1+\zeta_\perp} E^L(x,t,y,0)\,\partial^{\zeta_\|}(P_t f - f)(y)\,dy.$$

To establish the bound (9.15), we will want to bound $\int_Q |\widetilde{R}_t^{3,\xi,\zeta} f|^2$ for $0 < t < \ell(Q)$. Let $S \subset Q$ be a dyadic subcube with $t/2 < \ell(S) \leq t$. Let $\sum_{j=0}^\infty \eta_j$ be a smooth partition of unity with η_j supported in $A_{j,1}(S)$ and with $|\nabla^i \eta_j| \leq C 2^{-ji}\ell(S)^{-i}$. Let $f_j = \eta_j\, f$. Then

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) = \sum_{j=0}^\infty t^k \int_{\mathbb{R}^n} A_{\xi\zeta}(y)\,\partial_{y,s}^\xi \partial_t^{m+k-1+\zeta_\perp} E^L(x,t,y,0)\,\partial^{\zeta_\|}(P_t f_j - f_j)(y)\,dy.$$

By the bound (9.9), if $x \in S$ then

$$\left| t^k \int_{\mathbb{R}^n} A_{\xi\zeta}(y)\,\partial_{y,s}^\xi \partial_t^{m+k-1+\zeta_\perp} E^L(x,t,y,0)\,\partial^{\zeta_\|}(P_t f_j - f_j)(y)\,dy \right|^2$$
$$\leq C t^{-n-2\zeta_\perp} 2^{-j(n+2k+2\zeta_\perp)} \int_{A_{j,2}(S)} |\partial^{\zeta_\|}(P_t f_j - f_j)(y)|^2\,dy.$$

Thus,

$$\int_Q |\widetilde{R}_t^{3,\xi,\zeta} f(x)|^2\,dx \leq \sum_{\substack{S\subset Q \text{ dyadic}\\ t/2<\ell(S)\leq t}} \sum_{j=0}^\infty C t^{-2\zeta_\perp} 2^{-j(n+2k+2\zeta_\perp)} \int_{A_{j,2}(S)} |\partial^{\zeta_\parallel}(P_t f_j - f_j)(y)|^2\,dy.$$

Summing carefully, we see that

$$\int_Q |\widetilde{R}_t^{3,\xi,\zeta} f(x)|^2\,dx \leq C t^{-2\zeta_\perp} \int_{\mathbb{R}^n} |\partial^{\zeta_\parallel}(P_t f - f)(y)|^2\,dy.$$

Now, by Plancherel's theorem,

$$\int_0^\infty t^{-2\zeta_\perp} \int_{\mathbb{R}^n} |\partial^{\zeta_\parallel}(P_t f - f)(y)|^2\,dy\,\frac{dt}{t} \leq \int_{\mathbb{R}^n} |\omega|^{2|\zeta_\parallel|}\,|\widehat{f}(\omega)|^2 \int_0^\infty t^{-2\zeta_\perp}(1-\widehat{\psi}(t\omega))^2\,\frac{dt}{t}\,d\omega$$

where $\psi_t(x) = t^{-n}\psi(x/t)$ is the convolution kernel of P_t. We require that ψ be radial and make the change of variables $s = t|\omega|$. Then

$$\int_0^\infty t^{-2\zeta_\perp} \int_{\mathbb{R}^n} |\partial^{\zeta_\parallel}(P_t f - f)(y)|^2\,dy\,\frac{dt}{t} \leq \int_{\mathbb{R}^n} |\omega|^{2|\zeta|}\,|\widehat{f}(\omega)|^2 \int_0^\infty s^{-2\zeta_\perp}\Big(1-\widehat{\psi}\Big(s\frac{\omega}{|\omega|}\Big)\Big)^2\,\frac{ds}{s}\,d\omega.$$

We require that $\int \psi = 1$, and that the higher moments are zero, that is, that $\int x^\theta\,\psi(x)\,dx = 0$ for all $|\theta| \leq 2m$. This implies that $|1-\widehat{\psi}(s\omega/|\omega|)| \leq C s^{2m+1}$ and so $s^{-2\zeta_\perp - 1}(1-\widehat{\psi}(s\omega/|\omega|))^2$ is integrable near zero. Because ψ is smooth and compactly supported, we have that $\widehat{\psi}(s)$ is bounded. If $\zeta_\perp > 0$ then the integral in s converges. (We have that $\widehat{\psi}(s\omega/|\omega|) \to 0$ as $s\to\infty$, and so the integral must diverge if $\zeta_\perp = 0$.) Because $\|\nabla_\parallel^m f\|_{L^2(\mathbb{R}^n)} \leq C\sqrt{|Q|}$, we have that

$$\int_0^{\ell(Q)} \int_Q |\widetilde{R}_t^{3,\xi,\zeta} f(x)|^2\,\frac{dx\,dt}{t} \leq C|Q|$$

whenever $\zeta_\perp > 0$.

We are left with the terms $\widetilde{R}_t^{3,\xi,\zeta} f$ for $\zeta_\perp = 0$; recall that we need only consider $\zeta_\perp + \xi_\perp \geq 1$ and so we may assume $\xi_\perp \geq 1$. Because $\zeta_\perp = 0$, we have that

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) = t^k \int_{\mathbb{R}^n} A_{\xi\zeta}(y)\,\partial_{y,s}^\xi \partial_t^{m+k-1} E^L(x,t,y,0)\,(\partial^\zeta P_t f(y) - \partial^\zeta f(y))\,dy.$$

Recall from Section 9.4 that the operator $\widetilde{R}_t^{2,\xi,\zeta}$, given by formula (9.14), satisfies Carleson measure estimates. Thus, we need only bound

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) + \widetilde{R}_t^{2,\xi,\zeta} f(x) = t^k \int_{\mathbb{R}^n} A_{\xi\zeta}(y)\,\partial_{y,s}^\xi \partial_t^{m+k-1} E^L(x,t,y,0)\,(\partial^\zeta P_t f(x) - \partial^\zeta f(y))\,dy.$$

Let $\gamma = \xi - \vec{e}_\perp$. We use formula (2.24); we then see that

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) + \widetilde{R}_t^{2,\xi,\zeta} f(x) = -t^k \int_{\mathbb{R}^n} A_{\xi\zeta}(y)\,\partial_{y,s}^\gamma \partial_t^{m+k} E^L(x,t,y,0)\,(\partial^\zeta P_t f(x) - \partial^\zeta f(y))\,dy.$$

We recognize the integrand as being much like the kernel of the single layer potential. By formula (4.2) for Θ_t^S, we have that

$$\widetilde{R}_t^{3,\xi,\zeta} f(x) + \widetilde{R}_t^{2,\xi,\zeta} f(x) = -P_t(\partial^\zeta f)(x)\,\Theta_t^S(A_{\xi\zeta}\dot{\boldsymbol{e}}_\gamma)(x) + \Theta_t^S(A_{\xi\zeta}\,\partial^\zeta f\,\dot{\boldsymbol{e}}_\gamma)(x).$$

If $\gamma \neq \gamma_\perp$, then the operator Θ_t given by $\Theta_t f = \Theta_t^S(f\dot{\boldsymbol{e}}_\gamma)$ satisfies the bound (5.4) (see Section 8). If $\gamma = \gamma_\perp$, then

$$\Theta_t^S(A_{\xi\zeta}\,\partial^\zeta f\,\dot{\boldsymbol{e}}_\gamma) = \Theta_t^\perp(A_{\xi\zeta}\,\partial^\zeta f).$$

The operator $\Theta_t^\delta = 1_{\delta<t<1/\delta}\Theta_t^\perp$ satisfies the conditions of Theorem 5.2, albeit with constants depending on $\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}$, and so also satisfies the bound (5.4). Thus, in either case, we have the bound

$$\int_{\mathbb{R}^n}\int_\delta^{1/\delta} |\Theta_t^S(A_{\xi\zeta}\,\partial^\zeta f\,\dot{\boldsymbol{e}}_\gamma)(x)|^2\frac{dx\,dt}{t} \leq (C + C\|\Theta_t^\perp\|_{\mathcal{C},\delta})|Q|.$$

To bound $P_t(\partial^\zeta f)(x)\,\Theta_t^S(A_{\xi\zeta}\dot{\boldsymbol{e}}_\gamma)(x)$, recall Carleson's lemma (see, for example, [74, Chapter II, Section 2.2]).

Lemma 9.4 *Let $F(x,t)$ be a function and $d\mu$ be a measure defined on $\mathbb{R}^{n+1}_+$. Then*

$$\left|\int_{\mathbb{R}^{n+1}_+} F(x,t)\,d\mu(x,t)\right| \leq C\biggl(\sup_{R\subset\mathbb{R}^n}\frac{1}{|R|}\int_R\int_0^{\ell(R)}|d\mu|\biggr)\biggl(\int_{\mathbb{R}^n}\sup_{|x-y|<t}|F(y,t)|\,dx\biggr)$$

provided the right-hand side is finite, where the supremum is taken over cubes $R\subset\mathbb{R}^n$.

We wish to bound

$$\frac{1}{|Q|}\int_Q\int_\delta^{\min(1/\delta,\ell(Q))} |P_t(\partial^\zeta f)(x)|^2\,|\Theta_t^S(A_{\xi\zeta}\dot{\boldsymbol{e}}_\gamma)(x)|^2\,\frac{1}{t}\,dt\,dx$$

for δ small enough. Let $F(x,t) = |P_t(\partial^\zeta f)(x)|^2$; because P_t is a smooth identity with a convolution kernel it is elementary to show that $\sup_{|x-y|<t}|F(y,t)| \leq C\mathcal{M}(\partial^\zeta f)(x)$. Let

$$d\mu(x,t) = \mathbf{1}_{\delta<t<1/\delta}|\Theta_t^S(A_{\xi\zeta}\dot{\boldsymbol{e}}_\gamma)(x)|^2\,\frac{1}{t}\,dt\,dx.$$

By Lemma 9.1 and the preceding remarks, we have that

$$\sup_R\frac{1}{|R|}\int_R\int_0^{\ell(R)}|d\mu| \leq C + C\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}^2.$$

This establishes the desired bound on $\widetilde{R}_t^{3,\xi,\zeta} f$.

9.6 Terms with $\alpha_{n+1} = 0$ and $\beta_{n+1} > 0$

We conclude this section by bounding $\Theta_t^\beta 1$ for multiindices β with $\beta_{n+1} > 0$. Recall that

$$\Theta_t^\beta f(x) = \sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}} t^k\int_{\mathbb{R}^n}\partial_t^{m+k-1}\partial_y^\alpha E^L(x,t,y,0)\,A_{\alpha\beta}(y)\,f(y)\,dy.$$

By a well known argument of Fefferman and Stein [75], using decay of the kernel of Θ_t^β (that is, the bound (3.1)), we find that if k is large enough then

$$\|\Theta_t^\beta 1\|_{\mathcal{C}} \leq C + \sup_Q\frac{C}{|Q|}\int_0^{\ell(Q)}\int_Q|\Theta_t^\beta(\mathbf{1}_{4Q})(x)|^2\frac{dx\,dt}{t}.$$

Let $(F_\beta)_\alpha := A_{\alpha\beta}\,\mathbf{1}_{4Q}$; then $\dot{\boldsymbol{F}}_\beta$ is an L^2 array-valued function. More precisely, let q be the number of multiindices $\zeta\in\mathbb{N}^n$ of length m; alternatively, q is the number of multiindices $\zeta\in\mathbb{N}^{n+1}$ of length m with $\zeta_{n+1}=0$. We will think of $\mathbb{C}^q$ as the vector space of arrays of numbers indexed by such multiindices. Then for each β, $\dot{\boldsymbol{F}}_\beta$ is a function in $L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)$.

Now, observe that

$$\boldsymbol{A}_\parallel\nabla_\parallel^m L_\parallel^{-1}\operatorname{div}_{m,\parallel} : L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)\mapsto L^2(\mathbb{R}^n\mapsto\mathbb{C}^q) \tag{9.16}$$

is a bounded operator, where $\nabla^m_\|$ is defined in Section 2, and where formally $\mathrm{div}_{m,\|}\dot{\boldsymbol{F}}=\sum_{|\zeta|=m,\,\zeta\in\mathbb{N}^n}\partial^\zeta F_\zeta$; the weak definition is precisely analogous to the definition (2.1) of $\mathrm{div}_m\dot{\boldsymbol{F}}$.

Thus, we have a Hodge decomposition of $L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)$. Specifically, if $\dot{\boldsymbol{F}}\in L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)$, then

$$\dot{\boldsymbol{F}}=\dot{\boldsymbol{H}}+\boldsymbol{A}_\|\nabla^m\Phi$$

for some $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)$ and some $\Phi\in\dot{W}^2_m(\mathbb{R}^n)$, with $\mathrm{div}_{m,\|}\dot{\boldsymbol{H}}=0$ and with

$$\|\dot{\boldsymbol{H}}\|_{L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)}+\|\Phi\|_{\dot{W}^{m,2}(\mathbb{R}^n\mapsto\mathbb{C})}\leq C\|\dot{\boldsymbol{F}}\|_{L^2(\mathbb{R}^n\mapsto\mathbb{C}^q)}.$$

Applying the Hodge decomposition to $\dot{\boldsymbol{F}}_\beta$, we see that

$$\Theta^\beta_t\mathbf{1}_{4Q}(x)=\sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}}t^k\int_{\mathbb{R}^n}\partial_t^{m+k-1}\partial_y^\alpha E^L(x,t,y,0)\left(\dot{\boldsymbol{H}}_\beta+\boldsymbol{A}_\|\nabla^m_\|\Phi_\beta\right)_\alpha dy.$$

But because $\mathrm{div}_{m,\|}\dot{\boldsymbol{H}}_\beta=0$, we have that

$$\Theta^\beta_t\mathbf{1}_{4Q}(x)=\sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}}t^k\int_{\mathbb{R}^n}\partial_t^{m+k-1}\partial_y^\alpha E^L(x,t,y,0)\left(\boldsymbol{A}_\|\nabla^m_\|\Phi_\beta\right)_\alpha dy.$$

We may extend Φ_β to a function defined on $\mathbb{R}^{n+1}$ by letting $\Phi_\beta(y,s)=\Phi_\beta(y)$. Observe that $\partial^\zeta\Phi_\beta=0$ unless $\zeta_{n+1}=0$. Also, if $\alpha_{n+1}=0$, then $(\boldsymbol{A}_\|\nabla^m_\|\Phi_\beta)_\alpha=(\boldsymbol{A}\nabla^m\Phi_\beta)_\alpha$. Thus,

$$\Theta^\beta_t\mathbf{1}_{4Q}(x)=\sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}}\sum_{|\zeta|=m}t^k\int_{\mathbb{R}^n}\partial_t^{m+k-1}\partial_y^\alpha E^L(x,t,y,0)\,A_{\alpha\zeta}(y)\,\partial^\zeta\Phi_\beta(y)\,dy.$$

Recall from formulas (4.3) and (2.28) that

$$\Theta^D_t(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x)=-\sum_{|\alpha|=|\zeta|=m}t^k\int_{\mathbb{R}^{n+1}_-}\partial_t^{m+k}\partial_{y,s}^\alpha E^L(x,t,y,s)\,A_{\alpha\zeta}(y)\,\partial^\zeta\Phi_\beta(y)\,dy\,ds.$$

Using the identity (2.24) and integrating in s, we see that

$$\Theta^D_t(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x)=\sum_{|\alpha|=|\zeta|=m}t^k\int_{\mathbb{R}^n}\partial_t^{m+k-1}\partial_{y,s}^\alpha E^L(x,t,y,0)\,A_{\alpha\zeta}(y)\,\partial^\zeta\Phi_\beta(y)\,dy.$$

Thus,

$$\begin{aligned}\Theta^\beta_t\mathbf{1}_{4Q}(x)=-&\sum_{\substack{|\alpha|=m\\ \alpha_{n+1}>0}}\sum_{|\zeta|=m}t^k\int_{\mathbb{R}^n}\partial_t^{m+k-1}\partial_{y,s}^\alpha E^L(x,t,y,0)\,A_{\alpha\zeta}(y)\,\partial^\zeta\Phi_\beta(y)\,dy\\&+\Theta^D_t(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x).\end{aligned}$$

If $\alpha_{n+1}>0$, then $\alpha=\gamma+\vec{e}_{n+1}$ for some multiindex γ with $|\gamma|=m-1$. Conversely, if $|\gamma|=m-1$, let $\widetilde{\gamma}=\gamma+\vec{e}_{n+1}$. We may write

$$\begin{aligned}\Theta^\beta_t\mathbf{1}_{4Q}(x)=&\sum_{|\gamma|=m-1}\sum_{|\zeta|=m}t^k\int_{\mathbb{R}^n}\partial_t^{m+k}\partial_{y,s}^\gamma E^L(x,t,y,0)\,A_{\widetilde{\gamma}\zeta}(y)\,\partial^\zeta\Phi_\beta(y)\,dy\\&+\Theta^D_t(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x).\end{aligned}$$

By formula (4.2) for Θ_t^S, we see that

$$\Theta_t^\beta \mathbf{1}_{4Q}(x) = \Theta_t^S \dot{\boldsymbol{G}}_\beta(x) + \Theta_t^D(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x)$$

where $(G_\beta)_\gamma = \sum_\zeta A_{\tilde{\gamma}\zeta}\partial^\zeta\Phi_\beta$.

Observe that $\dot{\boldsymbol{G}}_\beta \in L^2(\mathbb{R}^n)$, and so by Theorem 5.2,

$$\int_{\mathbb{R}^n}\int_\delta^{1/\delta} |\Theta_t^S \dot{\boldsymbol{G}}_\beta(x)|^2 \frac{dt\,dx}{t} \leq (C + C\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}^2)\|\dot{\boldsymbol{G}}_\beta\|_{L^2(\mathbb{R}^n)}^2 \leq (C + C\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}^2)|Q|.$$

Recall that Θ_t^D acts on the space $\dot{W}A^2_{m,|}(\mathbb{R}^n)$, the completion of $\{\dot{\mathbf{Tr}}_{m,|}\,\varphi : \varphi \in C_0^\infty(\mathbb{R}^{n+1})\}$ under the L^2 norm. Consider the subspace W, the completion of

$$\{\dot{\mathbf{Tr}}_{m,|}\,\varphi : \varphi \in C_0^\infty(\mathbb{R}^{n+1}),\ \partial_{n+1}^j\varphi(x,0) = 0 \text{ for all } x\in\mathbb{R}^n \text{ and all } j\geq 1\}$$

under the L^2 norm. We may let $\Theta_t^{D\|}$ denote the restriction of Θ_t^D to the space W. Notice that $\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta = \nabla_\|^m\Phi_\beta\big|_{\mathbb{R}^n}$, and so $\Theta_t^D(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x) = \Theta_t^{D,\|}(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x)$. As we established in Sections 9.2–9.5,

$$\|\Theta_t^{D\|}1\|_{\mathcal{C},\delta} \leq C + C\|\Theta_t^\perp 1\|_{\mathcal{C},\delta}$$

and so we may control $\Theta_t^{D,\|}(\dot{\mathbf{Tr}}_{m,|}\,\Phi_\beta)(x)$. This completes the argument that $\Theta_t^D 1$ satisfies a Carleson estimate.

10 Test functions $\dot{\boldsymbol{b}}_Q$

In this section we will choose test functions $\dot{\boldsymbol{b}}_Q$ such that we may apply Theorem 5.4 to bound $\Theta_t^\perp$ and Θ_t^D. (The remaining components of Θ_t^S were bounded in Section 8.) We will follow the example of [43], which considers the case $m = 1$.

As in Section 9, we will make the assumption $2m > n$. Again, by Morrey's inequality, this implies that functions locally in $\dot{W}^2_m(\mathbb{R}^n)$ are locally Hölder continuous. By Lemma 3.2, if $2m > n$ then solutions to elliptic equations are locally in $L^2(\mathbb{R}^n\times\{t\})$ for constants t, and thus are also locally Hölder continuous. (See also [17, Appendix B], in which a similar argument is made.)

Fix some dyadic cube Q. Let y_Q be its midpoint. Let

$$F_s(x,t) = \partial_s^{m-1}E^L(x,t,y_Q,s) \tag{10.1}$$

and let $F_\pm = F_{\pm\kappa\ell(Q)}$ for some small positive number κ to be chosen later. By the bound (9.9) and the symmetry relation (2.19), we may see that if $n+1\geq 3$ then $F_\pm(x,t)\in\dot{W}^2_m(\mathbb{R}^{n+1}_\mp)$; furthermore, by Theorem 2.1 we see that $LF_\pm = 0$ in $\mathbb{R}^{n+1}_\mp$. Thus, by the higher-order Green's formula (2.26) and the analogous formula in $\mathbb{R}^{n+1}_-$, if $t > 0$ then

$$\begin{aligned}\partial_t^m F_-(x,t) &= -\partial_t^m\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}^+\,F_-)(x,t) + \partial_t^m\mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+\,F_-)(x,t),\\ 0 &= \partial_t^m\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}^-\,F_+)(x,t) + \partial_t^m\mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^-\,F_+)(x,t).\end{aligned}$$

Adding and applying the definition (2.29) of $\widetilde{\mathcal{D}}$, we see that

$$\partial_t^m F_-(x,t) = \partial_t^m\widetilde{\mathcal{D}}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m,|}\,F_+ - \dot{\mathbf{Tr}}_{m,|}\,F_-)(x,t) + \partial_t^m\mathcal{S}^L(\dot{\mathbf{M}}_{\boldsymbol{A}}^+\,F_- + \dot{\mathbf{M}}_{\boldsymbol{A}}^-\,F_+)(x,t).$$

Thus, by the definitions (4.1) and (4.3) of Θ_t^S and Θ_t^D,

$$t^k\partial_t^{m+k}F_-(x,t) = \Theta_t^D(\dot{\mathbf{Tr}}_{m,|}\,F_+ - \dot{\mathbf{Tr}}_{m,|}\,F_-)(x) + \Theta_t^S(\dot{\mathbf{M}}_{\boldsymbol{A}}^+\,F_- + \dot{\mathbf{M}}_{\boldsymbol{A}}^-\,F_+)(x). \tag{10.2}$$

Let

$$\dot{\boldsymbol{b}}_Q^D = |Q|(\dot{\mathbf{Tr}}_{m,|}\,F_+ - \dot{\mathbf{Tr}}_{m,|}\,F_-). \tag{10.3}$$

Recall that $\dot{\mathbf{M}}_{\boldsymbol{A}}^{+}\, u$ is only defined as a linear functional on $\dot{W}A^2_{m-1/2}(\mathbb{R}^n)$, that is, as an operator acting on $m-1$th-order traces of $\dot{W}^2_m$-functions. Let $\dot{\boldsymbol{b}}_Q^S$ be a representative of the operator $|Q|(\dot{\mathbf{M}}_{\boldsymbol{A}}^{+}\, F_- + \dot{\mathbf{M}}_{\boldsymbol{A}}^{-}\, F_+)$; that is, $\dot{\boldsymbol{b}}_Q^S$ is an array of functions that satisfies

$$\begin{aligned}\langle \dot{\mathbf{Tr}}_{m-1}\,\varphi, \dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n} &= |Q|\langle \dot{\mathbf{Tr}}_{m-1}\,\varphi, \dot{\mathbf{M}}_{\boldsymbol{A}}^{+}\, F_- + \dot{\mathbf{M}}_{\boldsymbol{A}}^{-}\, F_+\rangle_{\mathbb{R}^n} \\ &= |Q|\langle \nabla^m\varphi, A\nabla^m F_-\rangle_{\mathbb{R}^{n+1}_+} + |Q|\langle \nabla^m\varphi, A\nabla^m F_+\rangle_{\mathbb{R}^{n+1}_-}\end{aligned} \tag{10.4}$$

for all φ smooth and compactly supported in $\mathbb{R}^{n+1}$. In Section 10.2 we will show that there is some such array of functions that in addition lies in $L^2(\mathbb{R}^n)$.

Now, by formula (2.24) and by definition of $\dot{\boldsymbol{b}}_Q^D$, $\dot{\boldsymbol{b}}_Q^S$ and F_-,

$$\begin{aligned}\Theta_t(\dot{\boldsymbol{b}}_Q^D, \dot{\boldsymbol{b}}_Q^S) = \Theta_t^D \dot{\boldsymbol{b}}_Q^D(x) + \Theta_t^S \dot{\boldsymbol{b}}_Q^S(x) &= |Q| t^k \partial_t^{m+k} F_-(x,t) \\ &= |Q| t^k \partial_t^m \partial_s^{m+k-1} E^L(x,t,y_Q,-\kappa\ell(Q)).\end{aligned}$$

An application of the bound (9.9), with the roles of x and y reversed, reveals that the bound (5.10) is valid for this choice of $\dot{\boldsymbol{b}}_Q = (\dot{\boldsymbol{b}}_Q^S, \dot{\boldsymbol{b}}_Q^D)$, albeit with constant C_0 that depends on our choice of κ.

We thus need only show that this choice of $\dot{\boldsymbol{b}}_Q$ satisfies the bounds (5.11), (5.12) and (5.13), with the distinguished component $b_Q^{p+1} = b_Q^{\perp}$ in (5.12) the $\dot{\boldsymbol{e}}_\perp$-component of $\dot{\boldsymbol{b}}_Q^S$.

Remark 10.1 Although we will not make use of this fact, we observe that by the definition (10.1) of $F_\pm$, the symmetry property (2.19), and formula (2.28) for the double layer potential, we have that

$$\begin{aligned}\langle \dot{\boldsymbol{b}}_Q^S, \dot{\mathbf{Tr}}_{m-1}\,\varphi\rangle_{\mathbb{R}^n} &= |Q|\partial_{n+1}^{m-1}\mathcal{D}^{\boldsymbol{A}^*}(\dot{\mathbf{Tr}}_{m-1}\,\varphi)(y_Q,-\kappa\ell(Q)) \\ &\quad - |Q|\partial_{n+1}^{m-1}\mathcal{D}^{\boldsymbol{A}^*}(\dot{\mathbf{Tr}}_{m-1}\,\varphi)(y_Q,\kappa\ell(Q)).\end{aligned}$$

Thus, $\dot{\boldsymbol{b}}_Q^S$ may be viewed as the kernel of the double layer potential. If $m=1$, then the classic jump relation $\mathrm{Tr}^-\,\mathcal{D}^{\boldsymbol{A}^*} f - \mathrm{Tr}^+\,\mathcal{D}^{\boldsymbol{A}^*} f = f$ is well known. In the higher-order case, the analogous jump relation (see [11]) is

$$\dot{\mathbf{Tr}}_{m-1}^-\,\mathcal{D}^{\boldsymbol{A}^*}\dot{\boldsymbol{f}} - \dot{\mathbf{Tr}}_{m-1}^+\,\mathcal{D}^{\boldsymbol{A}^*}\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}.$$

Thus, if κ is small enough and φ is smooth, then $\langle \dot{\boldsymbol{b}}_Q^S, \dot{\mathbf{Tr}}_{m-1}\,\varphi\rangle_{\mathbb{R}^n}$ is approximately $|Q|\partial_{n+1}^{m-1}\varphi(y_Q,0)$. Thus, we expect $\dot{\boldsymbol{b}}_Q^S$ to be approximately equal to $\dot{\boldsymbol{e}}_\perp$ near Q, and so it is reasonable to expect the bounds (5.12) and (5.13) to be valid for $\dot{\boldsymbol{b}}_Q^S$.

Remark 10.2 Recall from formula (2.23) that if $2m \geq n+1$, precisely the case considered in Sections 9–10, then the fundamental solution E^L in the definition (10.1) of F_s is only determined up to adding polynomials. However, note the presence of the vertical derivative ∂_s^{m-1} in the definition of F_s; this vertical derivative suffices to remove the terms of the form $f_\zeta(x,t)\,(y,s)^\zeta$ in formula (2.23), leaving $F_s(x,t)$ well-defined up to adding polynomials in x and t. The function F_s is a tool used to define $\dot{\boldsymbol{b}}_Q^S$ and $\dot{\boldsymbol{b}}_Q^D$; notice from formulas (10.3) and (10.4) that these quantities depend only on the higher-order derivatives of F_s, and so the lower-order terms in formula (2.23) do not affect our results.

Remark 10.3 The conclusions of this section are also valid if $n+1=2$; the analysis is somewhat more complicated because $F_\pm$ is no longer in $\dot{W}^2_m(\mathbb{R}^{n+1}_\pm)$.

By Morrey's inequality, Lemma 3.2, and the bound (2.21), we have that if $n+1=2$ and R is a cube of side-length $|s-t|$ then

$$\int_{A_j(R)} |\nabla_{x,t}^m \partial_s^{m-1} E^L(x,t,y,s)|^2\,dy \leq \frac{C(\delta)}{|s-t|} 2^{-j(1-\delta)} \tag{10.5}$$

for any $\delta>0$. In particular,

$$\int_{\mathbb{R}^1} |\nabla^m F_s(x,t)|^2\,dx \leq \frac{C}{|t-s|}.$$

The argument is similar to the proof of the bounds (3.1) and (9.9), but we must use the bound (2.21) instead of the bound (2.20) in order to take $m-1$ derivatives in the variable s rather than the variables (x,t).

We may use the bound (9.9) with the roles of x and y reversed to show that if $t<s<\sigma$ or $t>s>\sigma$, then

$$\int_{\mathbb{R}^1} |\nabla^m F_s(x,t)-\nabla^m F_\sigma(x,t)|^2\,dx \leq C\frac{|\sigma-s|^2}{|t-s|^3}.$$

Thus, if $0<s<\sigma$, then $F_{\pm s}-F_{\pm\sigma}\in \dot W^2_m(\mathbb{R}^{n+1}_\mp)$, and so we may apply the Green's formula (2.26) and the equivalent in $\mathbb{R}^{n+1}_-$ to see that

$$\begin{aligned}t^k\partial_t^{m+k}F_{-s}(x,t)-t^k\partial_t^{m+k}F_{-\sigma}(x,t)&\\ &=\Theta_t^D(\dot{\mathbf{Tr}}_{m,|}\,F_s-\dot{\mathbf{Tr}}_{m,|}\,F_{-s}-\dot{\mathbf{Tr}}_{m,|}\,F_\sigma+\dot{\mathbf{Tr}}_{m,|}\,F_{-\sigma})(x,t)\\ &\quad+\Theta_t^S(\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,F_{-s}+\dot{\mathbf{M}}^-_{\boldsymbol{A}}\,F_s-\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,F_{-\sigma}-\dot{\mathbf{M}}^-_{\boldsymbol{A}}\,F_\sigma)(x,t).\end{aligned}$$

Fix some $t>0$ and let $s=\kappa\,\ell(Q)$. Observe that if we take the limit as $\sigma\to\infty$, then the left-hand side approaches $t^k\partial_t^{m+k}F_{-s}(\,\cdot\,,t)$ in $L^2(\mathbb{R}^n)$. Furthermore, $\dot{\mathbf{Tr}}_{m,|}\,F_{\pm\sigma}\to 0$ in $L^2(\mathbb{R}^n)$. In Lemma 10.4 below, we will see that $\dot{\boldsymbol{b}}_Q^S\to 0$ as the implied constant $\kappa\to\infty$; by definition of $\dot{\boldsymbol{b}}_Q^S$, this implies that $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,F_{-\sigma}+\dot{\mathbf{M}}^-_{\boldsymbol{A}}\,F_{-\sigma}\to 0$ in $L^2(\mathbb{R}^n)$ as $\sigma\to\infty$. Thus, by the bound (5.5), we have that

$$t^k\partial_t^{m+k}F_{-s}(\,\cdot\,,t)=\Theta_t^D(\dot{\mathbf{Tr}}_{m,|}\,F_s-\dot{\mathbf{Tr}}_{m,|}\,F_{-s})(\,\cdot\,,t)+\Theta_t^S(\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,F_{-s}+\dot{\mathbf{M}}^-_{\boldsymbol{A}}\,F_s)(\,\cdot\,,t)$$

as $L^2(\mathbb{R}^n)$-functions. Applying the Caccioppoli inequality, Lemma 3.2 and Morrey's inequality, we see that this equality must be true pointwise as well. Thus, formula (10.2) is still valid if $n+1=2$ and we may proceed as above.

10.1 Bounds on $\dot{\boldsymbol{b}}_Q^D$

By the bounds (9.9) or (10.5), we have that

$$\int_{\mathbb{R}^n}|\dot{\boldsymbol{b}}_Q^D|^2\leq\frac{C}{\kappa^n}|Q|.$$

Thus, $\dot{\boldsymbol{b}}_Q^D$ satisfies the bound (5.11) with constant $C_0=C\kappa^{-n}$.

We now show that $\dot{\boldsymbol{b}}_Q^D$ satisfies the bound (5.13). Following [43, Section 3], we fix a small positive constant ω. Let ϕ_Q be supported on $(1+\omega)Q$ with $\phi_Q=1$ on $(1/2)Q$. We may choose ϕ_Q such that $|\nabla\phi_Q(x)|<2/\ell(Q)$ for all x, and such that $\phi_Q>\omega$ on Q. We then set $d\mu_Q=\phi_Q\,dx$. Observe that the conditions (5.9) are valid for $C_0=\max(2,1/\omega)$.

Then by definition of $\dot{\boldsymbol{b}}_Q^D$, if we let $F_Q=F_+-F_-$, then

$$\begin{aligned}\fint_Q\dot{\boldsymbol{b}}_Q^D\,d\mu_Q&=\int_Q\dot{\mathbf{Tr}}_{m,|}\,F_Q\,\phi_Q\\ &=\int_{\mathbb{R}^n}\dot{\mathbf{Tr}}_{m,|}\,F_Q\,\phi_Q-\int_{\mathbb{R}^n\setminus Q}\dot{\mathbf{Tr}}_{m,|}\,F_Q\,\phi_Q.\end{aligned}$$

Recall that each component of $\dot{\mathbf{Tr}}_{m,|}\,F_Q$ may be written as $\partial^\beta F_Q(x,0)$ for some β with $\beta_{n+1}<|\beta|=m$. In particular, $\beta=\vec e_j+\gamma$ for some $1\leq j\leq n$ and some multiindex γ. Integrating by parts, we see that

$$\biggl|\int_{\mathbb{R}^n}\dot{\mathbf{Tr}}_{m,|}\,F_Q\,\phi_Q\biggr|\leq\int_{\mathbb{R}^n}|\dot{\mathbf{Tr}}_{m-1}\,F_Q|\,|\nabla\phi_Q|.$$

Recalling the regions on which ϕ_Q and $\nabla\phi_Q$ are supported, we see that

$$\biggl|\fint_Q\dot{\boldsymbol{b}}_Q^D\,d\mu_Q\biggr|\leq\frac{C}{\ell(Q)}\int_{(1+\omega)Q\setminus(1/2)Q}|\nabla^{m-1}F_Q|+\int_{(1+\omega)Q\setminus Q}|\nabla^m F_Q|.$$

Now, observe that $LF_Q = 0$ away from $(y_Q, \pm\kappa\ell(Q))$. Thus, we may apply Hölder's inequality and Lemma 3.2 to see that

$$\left|\fint_Q \dot{\boldsymbol{b}}_Q^D\,d\mu_Q\right| \leq \frac{C|Q|^{1/2}}{\ell(Q)^{3/2}}\biggl(\int_{2Q\setminus(1/4)Q}\int_{-\ell(Q)}^{\ell(Q)}|\nabla^{m-1}F_Q|^2\biggr)^{1/2} + C\frac{|Q|^{1/2}}{\ell(Q)^{1/2}}\biggl(\int_{(3/2)Q\setminus(1/2)Q}\int_{-\ell(Q)/2}^{\ell(Q)/2}|\nabla^m F_Q|^2\biggr)^{1/2}.$$

We may use the Caccioppoli inequality (Lemma 3.1) to control the second term by the first term. For ease of notation let $S = (2Q\setminus(1/4)Q)\times(-\ell(Q),\ell(Q))$. Recalling the definition of F_Q, we see that

$$\left|\fint_Q \dot{\boldsymbol{b}}_Q^D\,d\mu_Q\right|^2 \leq \frac{C|Q|}{\ell(Q)^3}\int_S\left|\int_{-\kappa\ell(Q)}^{\kappa\ell(Q)}\nabla_{x,t}^{m-1}\partial_s^m E^L(x,t,y_Q,s)\,ds\right|^2dx\,dt.$$

Applying Hölder's inequality again, we see that

$$\left|\fint_Q \dot{\boldsymbol{b}}_Q^D\,d\mu_Q\right|^2 \leq \frac{C|Q|\kappa}{\ell(Q)^2}\int_S\int_{-\kappa\ell(Q)}^{\kappa\ell(Q)}|\nabla_{x,t}^{m-1}\partial_s^m E^L(x,t,y_Q,s)|^2ds\,dx\,dt$$

Using formula (2.24), (2.19) and the bound (9.9), we see that

$$\left|\fint_Q \dot{\boldsymbol{b}}_Q^D\,d\mu_Q\right|^2 \leq C\kappa^2$$

Thus, if we choose κ small enough, then $\dot{\boldsymbol{b}}_Q^D$ satisfies the bound (5.13). Notice that κ may be chosen depending only on the constant C_1 in the bound (5.8), that is, on the numbers determined in Section 9. It is acceptable for the numbers C_0 in the bounds (5.9), (5.10) and (5.11) to grow as $\kappa\to 0$. In particular, recall that $\dot{\boldsymbol{b}}_Q^D$ satisfies the bound (5.11) with a constant $C_0(\kappa) = C\kappa^{-n}$; this growth is acceptable.

Thus, $\dot{\boldsymbol{b}}_Q^D$ satisfies all the conditions of Theorem 5.4.

10.2 Bounds on $\dot{\boldsymbol{b}}_Q^S$

To conclude the proof of Theorem 1.1, at least in the case $2m > n$, we need only show that $\dot{\boldsymbol{b}}_Q^S$ satisfies the bounds (5.11), (5.12) and (5.13).

The most involved argument of this section will be the proof of the following lemma.

Lemma 10.4 *Suppose that $2m > n$. If $\dot{\boldsymbol{g}} \in \dot{W}A_{m-1}^2(\mathbb{R}^n)$, then*

$$|\langle\dot{\boldsymbol{g}},\dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n}| \leq \frac{C}{\kappa^{n/2}}\sqrt{|Q|}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}. \tag{10.6}$$

Furthermore, if $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$ and $\kappa < 1/16$, then we have a better estimate:

$$|\langle\dot{\boldsymbol{g}},\dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n}| \leq C\,\kappa\sqrt{|Q|}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}. \tag{10.7}$$

The bound (10.6) is valid even for κ large; recall that this bound was used in Remark 10.3 to show that the bound (5.10) is valid even in dimension $n+1=2$.

Notice that this implies that $\dot{\boldsymbol{b}}_Q^S$ is a bounded linear functional on $\dot{W}A_{m-1}^2(\mathbb{R}^n)$; if $m\geq 2$ then this is a proper subspace of $L^2(\mathbb{R}^n)$. Thus, $\dot{\boldsymbol{b}}_Q^S$ lies in a quotient space of $L^2(\mathbb{R}^n)$. Once this lemma is proven we may extend $\dot{\boldsymbol{b}}_Q^S$ to a bounded linear functional on $L^2(\mathbb{R}^n)$ (establishing the bound (5.11)); we will need to select our extension carefully to ensure that $\dot{\boldsymbol{b}}_Q^S$, after projection, satisfies the bound (5.13).

Proof of Lemma 10.4. It suffices to prove this lemma for all $\dot{\boldsymbol{g}}$ such that $\dot{\boldsymbol{g}} = \dot{\mathbf{Tr}}_{m-1}\,\eta$ for some smooth, compactly supported function η. Recall that

$$\begin{aligned}\langle \dot{\boldsymbol{g}}, \dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n} &= |Q|\langle \dot{\boldsymbol{g}}, \dot{\mathbf{M}}_{\boldsymbol{A}}^+ F_-\rangle_{\mathbb{R}^n} + |Q|\langle \dot{\boldsymbol{g}}, \dot{\mathbf{M}}_{\boldsymbol{A}}^- F_+\rangle_{\mathbb{R}^n}\\ &= |Q|\langle \nabla^m G, \boldsymbol{A}\nabla^m F_-\rangle_{\mathbb{R}^{n+1}_+} + |Q|\langle \nabla^m G, \boldsymbol{A}\nabla^m F_+\rangle_{\mathbb{R}^{n+1}_-}\end{aligned}$$

for any extension G of $\dot{\boldsymbol{g}}$.

We will need to construct our extension G of $\dot{\boldsymbol{g}}$ carefully. Let H be the extension of $\dot{\boldsymbol{g}}$ given by Lemma 3.3. Recall that H satisfies the estimates (3.3) and (3.5), and that if $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$ then $\nabla^{m-1}H(x,t) = 0$ whenever $|t| < \operatorname{dist}(x, \mathbb{R}^n \setminus (1/4)Q)$. Let φ be smooth, supported in $B(0,1/2) \subset \mathbb{R}^n$ and integrate to 1. Suppose further that the higher moments are zero, that is, $\int_{\mathbb{R}^n} x^\zeta\,\varphi(x)\,dx = 0$ for all $1 \leq |\zeta| \leq m$. Let

$$G(x,t) = \int_{\mathbb{R}^n} \frac{1}{t^n}\varphi\biggl(\frac{x-z}{t}\biggr) H(z,t)\,dz = \int_{\mathbb{R}^n} \varphi(z)\,H(x-zt,t)\,dz.$$

To study the derivatives of G, observe that if ζ is a multiindex, then for some constants $C_{\zeta,\xi}$,

$$\partial^\zeta G(x,t) = \sum_{|\xi|=|\zeta|,\,\xi_\parallel\geq\zeta_\parallel} C_{\zeta,\xi}\int_{\mathbb{R}^n} z^{\xi_\parallel-\zeta_\parallel}\varphi(z)\,\partial^\xi H(x-zt,t)\,dz$$

where $\zeta_\parallel$ denotes the horizontal part of ζ, that is, $\zeta_\parallel = (\zeta_1,\dots,\zeta_n)$. Let $J_{\zeta,\xi}(z) = C_{\zeta,\xi} z^{\xi_\parallel-\zeta_\parallel}\varphi(z)$, so that

$$\partial^\zeta G(x,t) = \sum_{|\xi|=|\zeta|}\int_{\mathbb{R}^n}\frac{1}{t^n}J_{\zeta,\xi}\biggl(\frac{x-z}{t}\biggr)\partial^\xi H(z,t)\,dz. \tag{10.8}$$

By our moment condition on φ, we have that

$$\int_{\mathbb{R}^n} J_{\zeta,\xi}(z,t)\,dz = 1 \quad \text{if } \zeta=\xi, \qquad \int_{\mathbb{R}^n} J_{\zeta,\xi}(z,t)\,dz = 0 \quad \text{otherwise.}$$

Furthermore, $J_{\zeta,\xi}$ is a smooth cutoff function, and so $\dot{\mathbf{Tr}}_{m-1}\,G = \dot{\boldsymbol{g}}$. Thus,

$$\langle \dot{\boldsymbol{g}}, \dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n} = |Q|\langle \nabla^m G, \boldsymbol{A}\nabla^m F_-\rangle_{\mathbb{R}^{n+1}_+} + |Q|\langle \nabla^m G, \boldsymbol{A}\nabla^m F_+\rangle_{\mathbb{R}^{n+1}_-}$$

for this choice of G.

We will need some special arguments to establish the bound (10.7). Arguing as in the proof of Lemma 3.3, we see that if $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$ then $\nabla^{m-1}G(x,t) = 0$ whenever $2|t| < \operatorname{dist}(x, \mathbb{R}^n \setminus (1/4)Q)$. In particular, if $\kappa < 1/8$ then $\nabla^{m-1}G = 0$ near $(y_Q, \pm\kappa\ell(Q))$. (We require $\kappa < 1/16$ so that $\nabla^{m-1}G = 0$ everywhere within a fixed radius of $(y_Q, \pm\kappa\ell(Q))$.) Observe that $LF_+ = 0$ away from these points, and so

$$0 = \langle \nabla^m G, \boldsymbol{A}\nabla^m F_+\rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m G, \boldsymbol{A}\nabla^m F_+\rangle_{\mathbb{R}^{n+1}_+} + \langle \nabla^m G, \boldsymbol{A}\nabla^m F_+\rangle_{\mathbb{R}^{n+1}_-}.$$

Thus, if $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$, then

$$\langle \dot{\boldsymbol{g}}, \dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n} = |Q|\langle \nabla^m G, \boldsymbol{A}\nabla^m (F_- - F_+)\rangle_{\mathbb{R}^{n+1}_+}.$$

We now introduce some notation. Let $\mathcal{G} = \{(x,t) : t > (1/2)\operatorname{dist}(x, \mathbb{R}^n \setminus (1/4)Q)\}$, so if $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$ then $\operatorname{supp}\nabla^{m-1}G \cap \mathbb{R}^{n+1}_+ \subset \mathcal{G}$. If $|\alpha| = m$, let

$$w_\alpha^s(x,t) = (\boldsymbol{A}(x)\nabla^m F_s(x,t))_\alpha = \sum_{|\beta|=m} A_{\alpha\beta}(x)\,\partial^\beta F_s(x,t)$$

with $w_\alpha^\pm = w_\alpha^{\pm\kappa\ell(Q)}$. Let $\widetilde{w}_\alpha = w_\alpha^+ - w_\alpha^-$. If $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$, then

$$\langle \dot{\boldsymbol{g}}, \dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n} = |Q|\sum_{|\alpha|=m}\int_{\mathcal{G}} \partial^\alpha G(x,t)\,\widetilde{w}_\alpha(x,t)\,dx\,dt.$$

In general,

$$\langle \dot{\boldsymbol{g}}, \dot{\boldsymbol{b}}_Q^S \rangle_{\mathbb{R}^n} = |Q| \sum_{|\alpha|=m} \int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, w_\alpha^-(x,t)\, dx\, dt + \int_{\mathbb{R}^{n+1}_-} \partial^\alpha G(x,t)\, w_\alpha^+(x,t)\, dx\, dt.$$

The second integral is similar to the first integral; thus, we will present the argument only for the first integral. In other words, we will work only in $\mathbb{R}^{n+1}_+$, not $\mathbb{R}^{n+1}_-$, whether our goal is to establish the bound (10.6) or (10.7).

We will need to bound w_α^s; it will also help to bound vertical derivatives of w_α^s. Let $j \geq 0$ be an integer. Observe that by formula (2.24) and the definition of F_s,

$$\int_{\mathbb{R}^n} |\partial_t^j w_\alpha^s(x,t)|^2\, dx = \int_{\mathbb{R}^n} |\partial_s^{m+j-1} \nabla_{x,t}^m E^L(x,t,y_Q,s)|^2\, dx.$$

By the bounds (9.9) or (10.5), if $2m > n$ and $j \geq 0$ then

$$\int_{\mathbb{R}^n} |\partial_t^j w_\alpha^s(x,t)|^2\, dx \leq C|t-s|^{-n-2j}.$$

Thus, we have the bounds

$$\sup_{t>0} \int_{\mathbb{R}^n} |w_\alpha^-(x,t)|^2\, dx \leq \frac{C}{\kappa^n |Q|}, \tag{10.9}$$

$$\int_{\mathbb{R}^n} \int_0^\infty |\partial_t w_\alpha^-(x,t)|^2\, t\, dt\, dx \leq \frac{C}{\kappa^n |Q|}, \tag{10.10}$$

$$\int_0^\infty \left(\int_{\mathbb{R}^n} |\partial_t w_\alpha^-(x,t)|^2\, dx \right)^{1/2} dt \leq \frac{C}{\sqrt{|Q|}\kappa^{n/2}}. \tag{10.11}$$

Now, by formula (2.24), $\partial_t^j w_\alpha^s(x,t) = (-1)^j \partial_s^j w_\alpha^s(x,t)$. Furthermore, if $t > 0$ $|s| < \kappa\ell(Q)$, then by the bound (9.9)

$$\int_{\mathbb{R}^n} \mathbf{1}_{\mathcal{G}}(x,t)\, |\partial_s w_\alpha^s(x,t)|^2\, dx \leq C(\ell(Q)+t)^{-n-2}.$$

Thus, recalling that $\widetilde{w}_\alpha(x,t) = w_\alpha^+(x,t) - w_\alpha^-(x,t)$, we have that

$$\sup_t \int_{\mathbb{R}^n} \mathbf{1}_{\mathcal{G}}(x,t)\, |\widetilde{w}_\alpha(x,t)|^2\, dx \leq \frac{C\kappa^2}{|Q|}, \tag{10.12}$$

$$\int_{\mathbb{R}^n} \int_0^\infty \mathbf{1}_{\mathcal{G}}(x,t)\, |\partial_t \widetilde{w}_\alpha(x,t)|^2\, t\, dt\, dx \leq \frac{C\kappa^2}{|Q|}, \tag{10.13}$$

$$\int_0^\infty \left(\int_{\mathbb{R}^n} \mathbf{1}_{\mathcal{G}}(x,t)\, |\partial_t \widetilde{w}_\alpha(x,t)|^2\, dx \right)^{1/2} dt \leq \frac{C\kappa}{\sqrt{|Q|}}. \tag{10.14}$$

Recall that we wish to bound

$$\int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, \widetilde{w}_\alpha(x,t)\, dx\, dt \qquad \text{or} \qquad \int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, w_\alpha^-(x,t)\, dx\, dt.$$

We will essentially proceed by integrating by parts to move one derivative from G to w_α; we will need separate arguments in the case where we integrate by parts in t (possible only if $\alpha_{n+1} > 0$) and in the case where we integrate by parts in a horizontal variable x_j (possible only if $\alpha_{n+1} < m$).

First, if $\alpha_{n+1} > 0$, then $\alpha = \gamma + \vec{e}_{n+1}$ for some multiindex γ with $|\gamma| = m-1$. So

$$\int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, w_\alpha^-(x,t)\, dx\, dt = \int_{\mathbb{R}^{n+1}_+} \partial_t \partial^\gamma G(x,t)\, w_\alpha^-(x,t)\, dx\, dt.$$

Integrating by parts in t, we see that

$$\int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, w_\alpha^-(x,t)\,dx\,dt = -\lim_{t\to 0^+}\int_{\mathbb{R}^n} \partial^\gamma G(x,t)\, w_\alpha^-(x,t)\,dx - \int_{\mathbb{R}^{n+1}_+} \partial^\gamma G(x,t)\,\partial_t w_\alpha^-(x,t)\,dx\,dt.$$

Recall that H satisfies the uniform L^2 bound (3.3); by formula (10.8), the same is true of G. We may control the first term using the estimate (10.9) and the second term using the estimate (10.11). This yields the bound

$$\left|\int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, w_\alpha^-(x,t)\,dx\,dt\right| \leq \frac{C}{\kappa^{n/2}\sqrt{|Q|}}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}.$$

Similarly,

$$\int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, \widetilde{w}_\alpha(x,t)\,dx\,dt = -\lim_{t\to 0^+}\int_{\mathbb{R}^n} \partial^\gamma G(x,t)\, \widetilde{w}_\alpha(x,t)\,dx - \int_{\mathbb{R}^{n+1}_+} \partial^\gamma G(x,t)\,\partial_t \widetilde{w}_\alpha(x,t)\,dx\,dt.$$

If $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$ then we may integrate over $\mathcal{G}$ rather than $\mathbb{R}^{n+1}_+$. By the bounds (10.14) and (10.12) on $\widetilde{w}_\alpha$ and the uniform L^2 estimate on $\nabla^{m-1}G$, we have that

$$\int_{\mathbb{R}^{n+1}_+} \partial^\alpha G(x,t)\, \widetilde{w}_\alpha(x,t)\,dx\,dt \leq \frac{C\kappa}{\sqrt{|Q|}}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}.$$

Now, we turn to the case where $\alpha_{n+1} < |\alpha| = m$. We still integrate by parts in t. We see that, if $w_\alpha = \widetilde{w}_\alpha$ or $w_\alpha = w_\alpha^-$, then

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} &\partial^\alpha G(x,t)\, w_\alpha(x,t)\,dx\,dt \\ &= -\int_{\mathbb{R}^{n+1}_+} t\,\partial_t\big(\partial^\alpha G(x,t)\, w_\alpha(x,t)\big)\,dx\,dt \\ &= -\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\, w_\alpha(x,t)\,dx\,dt - \int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha G(x,t)\,\partial_t w_\alpha(x,t)\,dx\,dt.\end{aligned}$$

Recall that G as well as H satisfies the estimate (3.5), and so by the bounds (10.10) and (10.13),

$$\left|\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha G(x,t)\,\partial_t w_\alpha^-(x,t)\,dx\,dt\right| \leq \frac{C}{\kappa^{n/2}\sqrt{|Q|}}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)},$$
$$\left|\int_{\mathcal{G}} t\,\partial^\alpha G(x,t)\,\partial_t \widetilde{w}_\alpha(x,t)\,dx\,dt\right| \leq \frac{C\kappa}{\sqrt{|Q|}}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}.$$

We are left with the term

$$\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\, w_\alpha(x,t)\,dx\,dt, \qquad \alpha_{n+1} < |\alpha| = m.$$

We have square-function estimates on $\partial_{n+1}w_\alpha$ rather than w_α; thus, we write

$$\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\, w_\alpha(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\,\int_t^\infty \partial_r w_\alpha(x,r)\,dr\,dx\,dt.$$

Observe that if $r > t > 0$ and $\dot{\boldsymbol{g}} = 0$ in $(1/4)Q$, then $\mathbf{1}_{\mathcal{G}}(x,r)\,\partial^\alpha\partial_t G(x,t) = \partial^\alpha\partial_t G(x,t)$; this is true because, if $\mathbf{1}_{\mathcal{G}}(x,r) \neq 1$, then $\nabla^m G(x,t) = 0$. Let $v_\alpha(x,r) = \partial_r w_\alpha^-(x,r)$ or $\mathbf{1}_{\mathcal{G}}(x,r)\,\partial_r \widetilde{w}_\alpha(x,r)$, depending on whether we seek to establish the bound (10.6) or (10.7). Then

$$\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\,w_\alpha(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\int_t^\infty v_\alpha(x,r)\,dr\,dx\,dt.$$

If $\alpha_{n+1} < |\alpha| = m$, let $j = j_\alpha$ be any integer such that $j \leq n$ and $\alpha_j > 0$, and let $\zeta = \zeta_\alpha = \alpha - \vec{e}_j + \vec{e}_{n+1}$. (For the sake of definiteness we may let j_α be the smallest such integer.) Then

$$\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\,w_\alpha(x,t)\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} t\,\partial_{j_\alpha}\partial^{\zeta_\alpha} G(x,t)\int_t^\infty v_\alpha(x,r)\,dr\,dx\,dt.$$

For each pair of multiindices α and β with $|\alpha| = |\beta| = m$, define the linear operator $T_{\alpha,\beta}$ by the relation

$$T_{\alpha,\beta}F(x,t) = \int_{\mathbb{R}^n} \frac{1}{t^n} J_{\alpha,\beta}\left(\frac{x-z}{t}\right) F(z,t)\,dz$$

where $J_{\alpha,\beta}$ is as defined above in the discussion of $\nabla^m G$ and $\nabla^m H$. Then

$$\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\,w_\alpha(x,t)\,dx\,dt = \sum_{|\beta|=m}\int_{\mathbb{R}^{n+1}_+} t\,\partial_{j_\alpha} T_{\zeta_\alpha,\beta}\partial^\beta H(x,t)\int_t^\infty v_\alpha(x,r)\,dr\,dx\,dt.$$

We may rearrange the terms of the integral to see that

$$\int_{\mathbb{R}^{n+1}_+} t\,\partial_{j_\alpha} T_{\zeta_\alpha,\beta}\partial^\beta H(x,t)\int_t^\infty v_\alpha(x,r)\,dr\,dx\,dt \\ = \int_0^\infty\int_{\mathbb{R}^n}\int_t^\infty\int_{\mathbb{R}^n} \frac{1}{t^n}(\partial_{j_\alpha} J_{\zeta_\alpha,\beta})\left(\frac{x-z}{t}\right) v_\alpha(x,r)\,dx\,dr\,\partial^\beta H(z,t)\,dz\,dt.$$

We will use the Christ-Journé $T1$ theorem (Theorem 5.1 above) to bound

$$W_{\alpha,r}(z,t) = \int_{\mathbb{R}^n}\frac{1}{t^n}(\partial_{j_\alpha} J_{\zeta_\alpha,\beta})\left(\frac{x-z}{t}\right) v_\alpha(x,r)\,dx.$$

Let $\psi_t(z,x) = \frac{1}{t^n}(\partial_{j_\alpha} J_{\zeta_\alpha,\beta})\left(\frac{x-z}{t}\right)$. We then have that

$$|\psi_t(z,x)| \leq \frac{C}{t^n}, \qquad |\nabla_x \psi_t(z,x)| \leq \frac{C}{t^{n+1}}$$

and both terms are zero if $|x - z| > t/2$. Finally, observe that $\int \psi_t(z,x)\,dx = 0$ and so $\Theta_t 1(x) = 0$; thus, we have the estimate

$$\int_0^\infty\int_{\mathbb{R}^n} |W_{\alpha,r}(z,t)|^2\,\frac{dz\,dt}{t} \leq C\|v_\alpha(\,\cdot\,,r)\|^2_{L^2(\mathbb{R}^n)}.$$

Thus,

$$\left|\int_{\mathbb{R}^{n+1}_+} t\,\partial_{j_\alpha} T_{\zeta_\alpha,\beta}\partial^\beta H(x,t)\int_t^\infty v_\alpha(x,r)\,dr\,dx\,dt\right| \\ = \left|\int_0^\infty\int_0^s\int_{\mathbb{R}^n} W_{\alpha,r}(z,t)\,\partial^\beta H(z,t)\,dz\,dt\,dr\right|.$$

By the bound (3.5) and the above bound on $W_{\alpha,r}$,

$$\left|\int_0^\infty\int_0^s\int_{\mathbb{R}^n} W_{\alpha,r}(z,t)\,\partial^\beta H(z,t)\,dz\,dt\,dr\right| \leq \int_0^\infty \|v_\alpha(\,\cdot\,,r)\|_{L^2(\mathbb{R}^n)}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}\,dr$$

and by the bounds (10.11) and (10.14),

$$\left|\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\,w_\alpha^-(x,t)\,dx\,dt\right| \leq \frac{C}{\kappa^{n/2}\sqrt{|Q|}}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)},$$
$$\left|\int_{\mathbb{R}^{n+1}_+} t\,\partial^\alpha\partial_t G(x,t)\,\widetilde{w}_\alpha(x,t)\,dx\,dt\right| \leq \frac{C\kappa}{\sqrt{|Q|}}\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}$$

as desired. □

We now have that $\dot{\boldsymbol{b}}_Q^S$ is a bounded linear operator on $\dot{W}A^2_{m-1}(\mathbb{R}^n)$, a subspace of $L^2(\mathbb{R}^n)$. We extend $\dot{\boldsymbol{b}}_Q^S$ to an operator on $L^2(\mathbb{R}^n)$ using a similar projection as in Section 6.1; the difference in this case is that we use only two projection operators rather than countably many.

Let W_n and W_f be the closure in $L^2(\mathbb{R}^n)$ of, respectively,

$$\widetilde{W}_n = \{\mathbf{1}_{(1/2)Q}\,\dot{\mathbf{Tr}}_{m-1}\,\varphi + (1-\mathbf{1}_{(1/2)Q})\dot{\boldsymbol{f}} : \varphi\in C_0^\infty,\ \dot{\boldsymbol{f}}\in L^2(\mathbb{R}^n)\},$$
$$\widetilde{W}_f = \{\mathbf{1}_{(1/4)Q}\dot{\boldsymbol{f}} + (1-\mathbf{1}_{(1/4)Q})\,\dot{\mathbf{Tr}}_{m-1}\,\varphi : \varphi\in C_0^\infty,\ \dot{\boldsymbol{f}}\in L^2(\mathbb{R}^n)\}.$$

Let O_n and O_f denote orthogonal projection from $L^2(\mathbb{R}^n)$ onto the subspaces W_n and W_f; observe that $O_n\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}$ outside of $(1/2)Q$ and that $O_f\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}$ inside $(1/4)Q$. Furthermore, $O_n(\dot{\mathbf{Tr}}_{m-1}\,\varphi) = O_f(\dot{\mathbf{Tr}}_{m-1}\,\varphi) = \dot{\mathbf{Tr}}_{m-1}\,\varphi$ for any nice (e.g., smooth and compactly supported) function φ.

Let η be smooth and satisfy

$$\eta\equiv 1 \text{ in } (1/4)Q\times(-\ell(Q)/8,\ell(Q)/8), \qquad \eta\equiv 0 \text{ outside } (1/2)Q\times(-\ell(Q)/4,\ell(Q)/4)$$

with $|\nabla^j\eta|\leq C_j\ell(Q)^{-j}$ for any $j\geq 0$.

Define $\pi_n : W_n\mapsto \dot{W}A^2_{m-1}(\mathbb{R}^n)$ and $\pi_f : W_f\mapsto \dot{W}A^2_{m-1}(\mathbb{R}^n)$ as follows. Suppose that $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}\,\varphi$ in $(1/2)Q$ or $\mathbb{R}^n\setminus(1/4)Q$ for some smooth function φ. We may renormalize φ so that $\int_{(1/2)Q\setminus(1/4)Q}\operatorname{Tr}\partial^\zeta\varphi = 0$ for all $|\zeta|\leq m-1$. Let $\pi_n\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}(\eta\varphi)$ and let $\pi_f\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}((1-\eta)\varphi)$. Observe that π_n and π_f are well-defined, that $\pi_n\dot{\boldsymbol{f}} = 0$ outside $(1/2)Q$ and that $\pi_f\dot{\boldsymbol{f}} = 0$ in $(1/4)Q$, and that by the Poincaré inequality $\pi_n : W_n\mapsto \dot{W}A^2_{m-1}(\mathbb{R}^n)$ and $\pi_f : W_f\mapsto \dot{W}A^2_{m-1}(\mathbb{R}^n)$ are bounded operators. Finally, notice that $\pi_n(\dot{\mathbf{Tr}}_{m-1}\,\varphi)+\pi_f(\dot{\mathbf{Tr}}_{m-1}\,\varphi) = \dot{\mathbf{Tr}}_{m-1}\,\varphi$ for any smooth, compactly supported φ.

We let $\dot{\boldsymbol{b}}_Q^S$ satisfy

$$\langle\dot{\boldsymbol{f}},\dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n} = \langle\pi_n O_n\dot{\boldsymbol{f}} + \pi_f O_f\dot{\boldsymbol{f}},\dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n}$$

where the right-hand side is given by formula (10.4). Notice that if $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}\,\varphi$, then $O_n\dot{\boldsymbol{f}} = O_f\dot{\boldsymbol{f}} = \dot{\boldsymbol{f}}$. Thus, this definition is consistent with formula (10.4).

By the bound (10.6) and boundedness of π_n and π_f, we see that

$$|\langle\dot{\boldsymbol{f}},\dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n}| \leq \frac{C}{\kappa^{n/2}}\|\dot{\boldsymbol{f}}\|_{L^2(\mathbb{R}^n)}\sqrt{|Q|}$$

and so the bound (5.11) is established with $C_0 = C\kappa^{-n}$. We are left with the bounds (5.12) and (5.13).

Observe that by the bound (10.7),

$$|\langle\pi_f O_f\dot{\boldsymbol{f}},\dot{\boldsymbol{b}}_Q^S\rangle_{\mathbb{R}^n}| \leq C\|\dot{\boldsymbol{f}}\|_{L^2(\mathbb{R}^n)}\kappa\sqrt{|Q|}.$$

Furthermore, if $\dot{\boldsymbol{f}} = 0$ in $(1/2)Q$ then $\pi_n O_n\dot{\boldsymbol{f}} = 0$; thus, we have that

$$\|\dot{\boldsymbol{b}}_Q^S\|_{L^2(\mathbb{R}^n\setminus(1/2)Q)} \leq C\kappa\sqrt{|Q|}. \tag{10.15}$$

Fix some γ with $|\gamma| = m-1$, and let $b_Q^\gamma = (b_Q^S)_\gamma$ for some $|\gamma| = m-1$. Then $b_Q^\perp = b_Q^{\gamma_\perp}$. We seek to show that

$$\operatorname{Re}\frac{1}{\int_Q\phi_Q}\int_Q b_Q^\gamma(x)\phi_Q(x)\,dx \geq \sigma \quad\text{or}\quad \left|\frac{1}{\int_Q\phi_Q}\int_Q b_Q^\gamma(x)\phi_Q(x)\,dx\right| \leq \eta\sigma$$

for some constant σ independent of Q and some η depending on C_1. Notice that $\int_Q \phi_Q = c|Q|$ for some constant c depending on ϕ_Q, with $1/2 \leq c \leq (1+\omega)^n$.

Let

$$\Phi_Q^\gamma(x,t) = \frac{1}{\gamma!}(x-y_Q,t)^\gamma \phi_Q(x)\,\rho(t)$$

where $\rho(t)=1$ for $|t|<\ell(Q)$ and $\rho(t)=0$ for $|t|>2\ell(Q)$.

Notice that if $x\in(1/2)Q$ and $|t|<\ell(Q)$, then $\nabla^{m-1}\Phi_Q^\gamma(x,t) = \dot{\boldsymbol{e}}_\gamma = \phi_Q(x)\dot{\boldsymbol{e}}_\gamma$. In particular, $b_Q^\gamma\phi_Q = \dot{\mathbf{Tr}}_{m-1}\,\Phi_Q^\gamma\cdot\dot{\boldsymbol{b}}_Q$ in $(1/2)Q$. Furthermore, $b_Q^\gamma\,\phi_Q=0$ and $\dot{\mathbf{Tr}}_{m-1}\,\Phi_Q^\gamma\cdot\dot{\boldsymbol{b}}_Q=0$ outside of $(1+\omega)Q$. Thus, by the bound (10.15),

$$\left|\frac{1}{|Q|}\int_Q b_Q^\gamma(x)\phi_Q(x)\,dx - \frac{1}{|Q|}\int_{\mathbb{R}^n}\dot{\mathbf{Tr}}_{m-1}\,\Phi_Q^\gamma\cdot\dot{\boldsymbol{b}}_Q\right| \leq \frac{C}{|Q|}\int_{(1+\omega)Q\setminus(1/2)Q}|\dot{\boldsymbol{b}}_Q| \leq C\kappa.$$

Thus, to establish the bounds (5.12) and (5.13), it suffices to bound the quantity

$$\frac{1}{|Q|}\int_{\mathbb{R}^n}\dot{\mathbf{Tr}}_{m-1}\,\Phi_Q^\gamma\cdot\dot{\boldsymbol{b}}_Q^S$$

from above or from below.

Applying the definition of $\dot{\boldsymbol{b}}_Q^S$, we see that

$$\begin{aligned}\frac{1}{|Q|}\int_{\mathbb{R}^n}\dot{\mathbf{Tr}}_{m-1}\,\Phi_Q^\gamma\cdot\dot{\boldsymbol{b}}_Q &= \int_{\mathbb{R}^{n+1}_+}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}(x)\nabla^m F_-(x,t)\,dx\,dt\\ &\quad+\int_{\mathbb{R}^{n+1}_-}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}(x)\nabla^m F_+(x,t)\,dx\,dt.\end{aligned}$$

Now, observe that $\nabla^m\Phi_Q^\gamma=0$ in $(1/2)Q\times(-\ell(Q),\ell(Q))$. Applying the definition of $F_\pm$ and the bounds (9.9) or (10.5), we see that

$$\left|\int_{\mathbb{R}^{n+1}_-}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}(x)\big(\nabla^m F_+(x,t)-\nabla^m F_-(x,t)\big)\,dx\,dt\right|\leq C\kappa$$

and so we may consider

$$\int_{\mathbb{R}^{n+1}_+}\nabla^m\Phi_Q^\gamma\cdot\boldsymbol{A}\nabla^m F_- + \int_{\mathbb{R}^{n+1}_-}\nabla^m\Phi_Q^\gamma\cdot\boldsymbol{A}\nabla^m F_- = \int_{\mathbb{R}^{n+1}}\nabla^m\Phi_Q^\gamma\cdot\boldsymbol{A}\nabla^m F_-.$$

Now, recall that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}\nabla^m F_-(x,t)\,dx\,dt\\ &\qquad= \int_{\mathbb{R}^{n+1}}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}(x)\nabla^m_{x,t}\partial_s^{m-1}E^L(x,t,y_Q,-\kappa\ell(Q))\,dx\,dt.\end{aligned}$$

Applying the symmetry property (2.19), we see that

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}\nabla^m F_-(x,t)\,dx\,dt\\ &\qquad= \int_{\mathbb{R}^{n+1}}\overline{\partial_s^{m-1}\nabla^m_{x,t}E^{L^*}(y_Q,-\kappa\ell(Q),x,t)\cdot\boldsymbol{A}^*(x)\nabla^m\Phi_Q^\gamma(x,t)}\,dx\,dt.\end{aligned}$$

By formula (2.22),

$$\int_{\mathbb{R}^{n+1}}\nabla^m\Phi_Q^\gamma(x,t)\cdot\boldsymbol{A}\nabla^m F_-(x,t)\,dx\,dt = \overline{\partial_s^{m-1}\Pi^{L^*}(\boldsymbol{A}^*\nabla^m\Phi_Q^\gamma)(y_Q,-\kappa\ell(Q))}.$$

Recall (formula (2.17)) that $\Pi^{L^*}(\boldsymbol{A}^*\nabla^m\Phi_Q^\gamma) = \Phi_Q^\gamma$. Thus

$$\int_{\mathbb{R}^{n+1}} \nabla^m\Phi_Q^\gamma(x,t)\cdot \boldsymbol{A}\nabla^m F_-(x,t)\,dx\,dt = \partial_s^{m-1}\Phi_Q^\gamma(y_Q,-\kappa\ell(Q)).$$

The right-hand side is equal to one if $\gamma=\gamma_\perp$ and is zero otherwise, and so the bounds (5.12) and (5.13) are established.

11 Reduction to operators of higher order

We have now shown that Θ_t^D and Θ_t^S satisfy the bounds (5.1) and (5.2). We have established that whenever $2m>n$, the condition (5.8) is valid, and there exist functions $\dot{\boldsymbol{b}}_Q$ such that the conditions (5.10), (5.11), (5.12) and (5.13) are valid.

Thus, if $2m>n$, then by Theorem 5.4, Θ_t^D and Θ_t^S satisfy the bound (5.14); this implies that the bounds (1.8) and (2.30) are valid.

We now must establish these bounds for operators of order $2m\leq n$. We use a fairly standard technique in the theory of higher-order differential equations; see [44, Section 2.2] and [45, Section 5.4].

Fix some operator L of order $2m\leq n$, and choose some number M such that $2m+4M>n$. Now, there are constants a_ζ such that

$$\Delta^M = \sum_{|\zeta|=M} a_\zeta\,\partial^{2\zeta}.$$

In fact, $a_\zeta = m!/\zeta!$, and so we have that $a_\zeta\geq 1$ for all $|\zeta|=M$.

Define the differential operator $\widetilde L = \Delta^M L\Delta^M$; that is, $\langle\varphi,\widetilde L\psi\rangle = \langle\Delta^M\varphi, L\Delta^M\psi\rangle$ for all nice test functions φ and ψ. We remark that $\widetilde L$ is associated to coefficients $\widetilde{\boldsymbol{A}}$ that satisfy

$$\widetilde A_{\delta\varepsilon}(x) = \sum_{\substack{\alpha+2\zeta=\delta\\ \beta+2\xi=\varepsilon}} a_\zeta\,a_\xi\,A_{\alpha\beta}(x) = \sum_{\substack{|\zeta|=M,\,2\zeta<\delta\\ |\xi|=M,\,2\xi<\varepsilon}} a_\zeta\,a_\xi\,A_{(\delta-2\zeta)(\varepsilon-2\xi)}(x) \tag{11.1}$$

for all $|\delta|=|\varepsilon|=m+2M$.

Observe that $\widetilde{\boldsymbol{A}}$ is t-independent and satisfies the bounds (2.4) and (2.5). It was shown in the proof of [45, Theorem 62] that

$$E^L(x,t,y,s) = \sum_{|\zeta|=|\xi|=M} a_\zeta\,a_\xi\,\partial_{x,t}^{2\zeta}\partial_{y,s}^{2\xi}E^{\widetilde L}(x,t,y,s).$$

Now, by formula (2.32), if $|\alpha|=m$ then

$$\begin{aligned}\partial^\alpha\mathcal{S}^L\dot{\boldsymbol{g}}(x,t) &= \sum_{|\gamma|=m-1}\int_{\mathbb{R}^n}\partial_{x,t}^\alpha\partial_{y,s}^\gamma E^L(x,t,y,0)\,g_\gamma(y)\,dy\\ &= \sum_{|\gamma|=m-1}\sum_{|\zeta|=M}\sum_{|\xi|=M} a_\zeta\int_{\mathbb{R}^n}\partial_{x,t}^{\alpha+2\zeta}\partial_{y,s}^{\gamma+2\xi}E^{\widetilde L}(x,t,y,0)\,a_\xi\,g_\gamma(y)\,dy.\end{aligned}$$

Let $\widetilde g_\varepsilon(x) = \sum_{\gamma+2\xi=\varepsilon} a_\xi\,g_\gamma(x)$. Notice that $|\dot{\widetilde{\boldsymbol{g}}}(x)|\leq C|\dot{\boldsymbol{g}}(x)|$. Then

$$\partial^\alpha\mathcal{S}^L\dot{\boldsymbol{g}}(x,t) = \sum_{|\zeta|=M} a_\zeta\partial^{\alpha+2\zeta}\mathcal{S}^{\widetilde L}\dot{\widetilde{\boldsymbol{g}}}(x,t). \tag{11.2}$$

Thus, because the bound (1.8) is valid for operators $\widetilde L$ of order $2m+4M$ for M large enough, we have that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_0^\infty|\nabla^m\partial_t\mathcal{S}^L\dot{\boldsymbol{g}}(x,t)|^2\,t\,dt\,dx &\leq C\int_{\mathbb{R}^n}\int_0^\infty|\nabla^{m+2M}\partial_t\mathcal{S}^{\widetilde L}\dot{\widetilde{\boldsymbol{g}}}(x,t)|^2\,t\,dt\,dx\\ &\leq C\|\dot{\widetilde{\boldsymbol{g}}}\|_{L^2(\mathbb{R}^n)}^2 \leq C\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}^2\end{aligned}$$

and so the bound (1.8) is valid even for operators of order $2m \leq n$.

The argument for $\mathcal{D}^{\boldsymbol{A}}$ is somewhat more involved. In this case we will use Theorem 5.2; observe that Θ_t^D satisfies the bounds (5.1) and (5.2), and so we need only establish the bound (5.3), that is, to bound $\Theta_t^D \dot{\boldsymbol{e}}_\beta$ for multiindices β with $|\beta| = m$.

Recall from formula (9.1) that

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = -\sum_{|\alpha|=m} t^k \int_{\mathbb{R}^{n+1}_-} \partial_t^{m+k} \partial_{y,s}^\alpha E^L(x,t,y,s)\, A_{\alpha\beta}(y)\, ds\, dy$$

and so

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = -\sum_{|\zeta|=M} a_\zeta \sum_{|\delta|=m+2M} t^k \int_{\mathbb{R}^{n+1}_-} \partial_t^{m+k} \partial_{x,t}^{2\zeta} \partial_{y,s}^\delta E^{\widetilde{L}}(x,t,y,s)\, B_{\delta\beta}(y)\, ds\, dy$$

where

$$B_{\delta\beta}(y) = \sum_{\alpha+2\xi=\delta} a_\xi\, A_{\alpha\beta}(y) = \sum_{|\xi|=M,\, 2\xi<\delta} a_\xi\, A_{(\delta-2\xi)\beta}(y).$$

We would like to write the right-hand side in terms of $\widetilde{\boldsymbol{A}}$ rather than $\boldsymbol{A}$ and $\boldsymbol{B}$. Recall our formula (11.1) for the coefficients $\widetilde{\boldsymbol{A}}$ of $\widetilde{L}$. We then have that

$$\widetilde{A}_{\delta\varepsilon}(y) = \sum_{|\xi|=m,\, 2\xi<\varepsilon} a_\xi\, B_{\delta(\varepsilon-2\xi)}(y).$$

Let $\Psi(\dot{\boldsymbol{F}})_\varepsilon = \sum_{|\xi|=M,\, 2\xi<\varepsilon} a_\xi\, F_{\varepsilon-2\xi}$. Then $\widetilde{A}_{\delta\varepsilon} = \Psi(B_\delta)$, where $(B_\delta)_\xi = B_{\delta\xi}$. We claim that Ψ has a left inverse; this means that there exists a matrix of constants $b_{\beta\varepsilon}$ such that

$$B_{\delta\beta} = \sum_{|\varepsilon|=m+2M} b_{\beta\varepsilon}\, \widetilde{A}_{\delta\varepsilon}.$$

To establish this, we need only show that Ψ is one-to-one. Suppose that $\Psi(\dot{\boldsymbol{F}}) = 0$. Because Ψ is a linear operator, it suffices to show that $\dot{\boldsymbol{F}} = 0$. Begin with indices α such that $\alpha_j \leq 1$ for all but one value j_0 of j. Let $\xi = M\vec{e}_{j_0}$ and let $\varepsilon = \alpha + 2\xi$. Then ξ is the only multiindex with $|\xi| = M$ and with $2\xi < \varepsilon$. Thus $0 = \Psi(\dot{\boldsymbol{F}})_\varepsilon = a_\xi\, F_\alpha = F_\alpha$. Next, consider indices α with α_{j_0} arbitrary, $2 \leq \alpha_{j_1} \leq 3$, and $\alpha_j \leq 1$ for all other values of j. Let ξ be as before and let $\varepsilon = \alpha + 2\xi$. Then $0 = \Psi(\dot{\boldsymbol{F}})_\varepsilon = a_\xi\, F_\alpha + a_\zeta\, F_{\varepsilon-2\zeta}$, where $\zeta = \vec{e}_{j_1} + (M-1)\vec{e}_{j_0}$. Since $F_{\varepsilon-2\zeta} = 0$, we also have that $F_\alpha = 0$. Continuing in this fashion, we see that $F_\alpha = 0$ for all multiindices α.

Thus

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = -\sum_{|\zeta|=M} a_\zeta \sum_{\substack{|\delta|=m+2M \\ |\varepsilon|=m+2M}} b_{\beta\varepsilon}\, t^k \int_{\mathbb{R}^{n+1}_-} \partial_t^{m+k} \partial_{x,t}^{2\zeta} \partial_{y,s}^\delta E^{\widetilde{L}}(x,t,y,s)\, \widetilde{A}_{\delta\varepsilon}(y)\, ds\, dy.$$

By formula (2.28) for $\mathcal{D}$ and (2.29) for $\widetilde{\mathcal{D}}$, this equals

$$\Theta_t^D \dot{\boldsymbol{e}}_\beta(x) = \sum_{|\zeta|=M} a_\zeta \sum_{|\varepsilon|=m+2M} b_{\beta\varepsilon}\, t^k \partial_t^{m+k}\, \partial_{x,t}^{2\zeta} \widetilde{D}^{\widetilde{\boldsymbol{A}}} \dot{\boldsymbol{e}}_\varepsilon(x,t).$$

Recall from formula (4.3) that $\Theta_t^D \dot{\boldsymbol{f}}(x) = t^k \partial_t^{m+k} \widetilde{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}}(x,t)$. Define

$$\widetilde{\Theta}_t^D \dot{\boldsymbol{f}}(x) = t^{k'} \partial_t^{m+2M+k'} \widetilde{D}^{\widetilde{\boldsymbol{A}}} \dot{\boldsymbol{f}}(x,t)$$

for some k' to be chosen momentarily.

Because $m+2M>n$, if k' is large enough then we have that $\widetilde{\Theta}_t^D$ satisfies the estimates (5.14) and (5.1), and so by Lemma 9.1,

$$\sup_Q \frac{1}{|Q|}\int_Q\int_0^{\ell(Q)} |t^{k'}\partial_t^{m+2M+k'}\widetilde{D}^{\widetilde{\boldsymbol{A}}}\dot{\boldsymbol{e}}_\varepsilon(x,t)|^2\frac{dt\,dx}{t}\leq C.$$

Fix some cube Q and observe that

$$\int_0^{\ell(Q)}\int_Q |\Theta_t^D\dot{\boldsymbol{e}}_\beta(x)|^2\frac{dx\,dt}{t}\leq C\sum_{|\varepsilon|=m+2M}\int_0^{\ell(Q)}\int_Q |t^k\partial_t^{m+k}\nabla^{2M}\widetilde{D}^{\widetilde{\boldsymbol{A}}}\dot{\boldsymbol{e}}_\varepsilon(x,t)|^2\frac{dx\,dt}{t}.$$

Applying the Caccioppoli inequality in Whitney boxes, we see that

$$\int_0^{\ell(Q)}\int_Q |\Theta_t^D\dot{\boldsymbol{e}}_\beta(x)|^2\frac{dx\,dt}{t}\leq C\sum_{|\varepsilon|=m+2M}\int_0^{2\ell(Q)}\int_{2Q} |t^{k-2M}\partial_t^{m+k}\widetilde{D}^{\widetilde{\boldsymbol{A}}}\dot{\boldsymbol{e}}_\varepsilon(x,t)|^2\frac{dx\,dt}{t}.$$

If we let $k=2M+k'$, we see that

$$\int_0^{\ell(Q)}\int_Q |\Theta_t^D\dot{\boldsymbol{e}}_\beta(x)|^2\frac{dx\,dt}{t}\leq C|Q|$$

and so Θ_t^D satisfies the bound (5.3). Thus, by Theorem 5.2 we have that Θ_t^D satisfies the bound (5.4). Thus, by Lemma 4.2 we have that $\widetilde{D}^{\boldsymbol{A}}$ satisfies the bound (2.30), as desired.

Acknowledgements We would like to thank the American Institute of Mathematics for hosting the SQuaRE workshop on "Singular integral operators and solvability of boundary problems for elliptic equations with rough coefficients," at which many of the results and techniques of this paper were discussed.

References

[1] G. Verchota, The Dirichlet problem for the polyharmonic equation in Lipschitz domains, Indiana Univ. Math. J. **39**(3), 671–702 (1990). 2, 4, 15
[2] J. Pipher and G. C. Verchota, Maximum principles for the polyharmonic equation on Lipschitz domains, Potential Anal. **4**(6), 615–636 (1995). 2, 15
[3] G. C. Verchota, Potentials for the Dirichlet problem in Lipschitz domains, in: Potential theory—ICPT 94 (Kouty, 1994), (de Gruyter, Berlin, 1996), pp. 167–187. 2, 6, 15
[4] Z. Shen, The L^p Dirichlet problem for elliptic systems on Lipschitz domains, Math. Res. Lett. **13**(1), 143–159 (2006). 2, 15
[5] Z. Shen, Necessary and sufficient conditions for the solvability of the L^p Dirichlet problem on Lipschitz domains, Math. Ann. **336**(3), 697–725 (2006). 2, 15
[6] J. Kilty and Z. Shen, The L^p regularity problem on Lipschitz domains, Trans. Amer. Math. Soc. **363**(3), 1241–1264 (2011). 2, 15
[7] I. Mitrea and M. Mitrea, Multi-layer potentials and boundary problems for higher-order elliptic systems in Lipschitz domains, Lecture Notes in Mathematics, Vol. 2063 (Springer, Heidelberg, 2013). 2, 5, 11, 15
[8] A. Barton and S. Mayboroda, Higher-order elliptic equations in non-smooth domains: a partial survey, in: Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory. Celebrating Cora Sadosky's life, , AWM-Springer, Vol. 1 (Springer-Verlag, 2016), to appear. 2
[9] M. S. Agranovich, On the theory of Dirichlet and Neumann problems for linear strongly elliptic systems with Lipschitz domains, Funktsional. Anal. i Prilozhen. **41**(4), 1–21, 96 (2007), English translation: Funct. Anal. Appl. **41** (2007), no. 4, 247–263. 2, 5, 11, 15
[10] V. Maz'ya, M. Mitrea, and T. Shaposhnikova, The Dirichlet problem in Lipschitz domains for higher order elliptic systems with rough coefficients, J. Anal. Math. **110**, 167–239 (2010). 2, 15
[11] A. Barton, Perturbation of well-posedness and layer potentials for higher-order elliptic systems with rough coefficients, `arXiv:1604.00062v1 [math.AP]`, March 2016. 2, 5, 11, 15, 41
[12] E. B. Fabes, M. JodeitJr.., and N. M. Rivière, Potential techniques for boundary value problems on C^1-domains, Acta Math. **141**(3-4), 165–186 (1978). 2, 3

[13] G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains, J. Funct. Anal. **59**(3), 572–611 (1984). 2, 3
[14] B. E. J. Dahlberg and C. E. Kenig, Hardy spaces and the Neumann problem in L^p for Laplace's equation in Lipschitz domains, Ann. of Math. (2) **125**(3), 437–465 (1987). 2
[15] E. Fabes, O. Mendez, and M. Mitrea, Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains, J. Funct. Anal. **159**(2), 323–368 (1998). 2
[16] D. Z. Zanger, The inhomogeneous Neumann problem in Lipschitz domains, Comm. Partial Differential Equations **25**(9-10), 1771–1808 (2000). 2
[17] M. A. Alfonseca, P. Auscher, A. Axelsson, S. Hofmann, and S. Kim, Analyticity of layer potentials and L^2 solvability of boundary value problems for divergence form elliptic equations with complex L^∞ coefficients, Adv. Math. **226**(5), 4533–4606 (2011). 2, 3, 4, 16, 19, 34, 40
[18] A. Barton, Elliptic partial differential equations with almost-real coefficients, Mem. Amer. Math. Soc. **223**(1051), vi+108 (2013). 2, 3
[19] A. Rosén, Layer potentials beyond singular integral operators, Publ. Mat. **57**(2), 429–454 (2013). 2, 4
[20] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, The regularity problem for second order elliptic operators with complex-valued bounded measurable coefficients, Math. Ann. **361**(3-4), 863–907 (2015). 2, 3
[21] S. Hofmann, S. Mayboroda, and M. Mourgoglou, Layer potentials and boundary value problems for elliptic equations with complex L^∞ coefficients satisfying the small Carleson measure norm condition, Adv. Math. **270**, 480–564 (2015). 2, 4
[22] S. Hofmann, M. Mitrea, and A. J. Morris, The method of layer potentials in L^p and endpoint spaces for elliptic operators with L^∞ coefficients, Proc. Lond. Math. Soc. (3) **111**(3), 681–716 (2015). 2, 4
[23] A. Barton and S. Mayboroda, Layer potentials and boundary-value problems for second order elliptic operators with data in Besov spaces, Mem. Amer. Math. Soc. **243**(1149), iv+109 (2016). 2, 3
[24] C. E. Kenig and D. J. Rule, The regularity and Neumann problem for non-symmetric elliptic operators, Trans. Amer. Math. Soc. **361**(1), 125–160 (2009). 2, 3
[25] D. J. Rule, Non-symmetric elliptic operators on bounded Lipschitz domains in the plane, Electron. J. Differential Equations pp. No. 144, 1–8 (2007). 2, 3
[26] D. Mitrea, A generalization of Dahlberg's theorem concerning the regularity of harmonic Green potentials, Trans. Amer. Math. Soc. **360**(7), 3771–3793 (2008). 2
[27] M. S. Agranovich, Potential-type operators and conjugation problems for second-order strongly elliptic systems in domains with a Lipschitz boundary, Funktsional. Anal. i Prilozhen. **43**(3), 3–25 (2009). 2
[28] D. Mitrea and I. Mitrea, On the regularity of Green functions in Lipschitz domains, Comm. Partial Differential Equations **36**(2), 304–327 (2011). 2
[29] P. Auscher and M. Mourgoglou, Boundary layers, Rellich estimates and extrapolation of solvability for elliptic systems, Proc. Lond. Math. Soc. (3) **109**(2), 446–482 (2014). 2, 3
[30] J. Pipher and G. Verchota, The Dirichlet problem in L^p for the biharmonic equation on Lipschitz domains, Amer. J. Math. **114**(5), 923–972 (1992). 2, 4
[31] A. Barton and S. Mayboroda, The Dirichlet problem for higher order equations in composition form, J. Funct. Anal. **265**(1), 49–107 (2013). 2, 3, 4
[32] L. A. Caffarelli, E. B. Fabes, and C. E. Kenig, Completely singular elliptic-harmonic measures, Indiana Univ. Math. J. **30**(6), 917–924 (1981). 3
[33] D. S. Jerison and C. E. Kenig, The Dirichlet problem in nonsmooth domains, Ann. of Math. (2) **113**(2), 367–382 (1981). 3
[34] E. B. Fabes, D. S. Jerison, and C. E. Kenig, Necessary and sufficient conditions for absolute continuity of elliptic-harmonic measure, Ann. of Math. (2) **119**(1), 121–141 (1984). 3
[35] C. E. Kenig and J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients, Invent. Math. **113**(3), 447–509 (1993). 3
[36] P. Auscher, A. Axelsson, and S. Hofmann, Functional calculus of Dirac operators and complex perturbations of Neumann and Dirichlet problems, J. Funct. Anal. **255**(2), 374–448 (2008). 3, 4
[37] P. Auscher, A. Axelsson, and A. McIntosh, Solvability of elliptic systems with square integrable boundary data, Ark. Mat. **48**(2), 253–287 (2010). 3, 4
[38] P. Auscher and A. Axelsson, Weighted maximal regularity estimates and solvability of non-smooth elliptic systems I, Invent. Math. **184**(1), 47–115 (2011). 3
[39] P. Auscher and A. Rosén, Weighted maximal regularity estimates and solvability of nonsmooth elliptic systems, II, Anal. PDE **5**(5), 983–1061 (2012). 3
[40] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators, J. Amer. Math. Soc. **28**(2), 483–529 (2015). 3, 4
[41] C. Kenig, H. Koch, J. Pipher, and T. Toro, A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations, Adv. Math. **153**(2), 231–298 (2000). 3

[42] R. R. Coifman, A. McIntosh, and Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes, Ann. of Math. (2) **116**(2), 361–387 (1982). 3
[43] A. Grau de la Herran and S. Hofmann, A local Tb theorem with vector-valued testing functions, preprint. 4, 6, 20, 21, 22, 25, 26, 28, 30, 40, 42
[44] P. Auscher, S. Hofmann, A. McIntosh, and P. Tchamitchian, The Kato square root problem for higher order elliptic operators and systems on $\mathbb{R}^n$, J. Evol. Equ. **1**(4), 361–385 (2001), Dedicated to the memory of Tosio Kato. 4, 30, 31, 50
[45] A. Barton, Gradient estimates and the fundamental solution for higher-order elliptic systems with rough coefficients, Manuscripta Math. (2016), doi:10.1007/s00229-016-0839-x. 4, 11, 12, 13, 16, 50
[46] B. E. J. Dahlberg, C. E. Kenig, and G. C. Verchota, The Dirichlet problem for the biharmonic equation in a Lipschitz domain, Ann. Inst. Fourier (Grenoble) **36**(3), 109–135 (1986). 4
[47] J. Cohen and J. Gosselin, Adjoint boundary value problems for the biharmonic equation on C^1 domains in the plane, Ark. Mat. **23**(2), 217–240 (1985). 5, 10
[48] G. C. Verchota, The biharmonic Neumann problem in Lipschitz domains, Acta Math. **194**(2), 217–279 (2005). 5, 10, 11
[49] Z. Shen, A relationship between the Dirichlet and regularity problems for elliptic equations, Math. Res. Lett. **14**(2), 205–213 (2007). 5
[50] I. Mitrea and M. Mitrea, Boundary value problems and integral operators for the bi-Laplacian in non-smooth domains, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **24**(3), 329–383 (2013). 5
[51] G. C. Verchota, Boundary coerciveness and the Neumann problem for 4th order linear partial differential operators, in: Around the research of Vladimir Maz'ya. II, , Int. Math. Ser. (N. Y.), Vol. 12 (Springer, New York, 2010), pp. 365–378. 5, 11
[52] J. Cohen and J. Gosselin, The Dirichlet problem for the biharmonic equation in a C^1 domain in the plane, Indiana Univ. Math. J. **32**(5), 635–685 (1983). 5
[53] S. Agmon, Multiple layer potentials and the Dirichlet problem for higher order elliptic equations in the plane. I, Comm. Pure Appl. Math **10**, 179–239 (1957). 5
[54] J. Pipher and G. C. Verchota, Dilation invariant estimates and the boundary Gårding inequality for higher order elliptic operators, Ann. of Math. (2) **142**(1), 1–38 (1995). 6, 15
[55] M. Christ and J. L. Journé, Polynomial growth estimates for multilinear singular integral operators, Acta Math. **159**(1-2), 51–80 (1987). 6, 20
[56] L. C. Evans, Partial differential equations, Graduate Studies in Mathematics, Vol. 19 (American Mathematical Society, Providence, RI, 1998). 8, 16, 32
[57] G. Sweers, A survey on boundary conditions for the biharmonic, Complex Var. Elliptic Equ. **54**(2), 79–93 (2009). 9
[58] A. Nadai, Theory of Flow and Fracture of Solids (McGraw-Hill, 1963). 9
[59] Z. Shen, The L^p boundary value problems on Lipschitz domains, Adv. Math. **216**(1), 212–254 (2007). 11
[60] V. Adolfsson and J. Pipher, The inhomogeneous Dirichlet problem for Δ^2 in Lipschitz domains, J. Funct. Anal. **159**(1), 137–190 (1998). 15
[61] I. Mitrea, M. Mitrea, and M. Wright, Optimal estimates for the inhomogeneous problem for the bi-Laplacian in three-dimensional Lipschitz domains, J. Math. Sci. (N. Y.) **172**(1), 24–134 (2011), Problems in mathematical analysis. No. 51. 15
[62] S. Campanato, Sistemi ellittici in forma divergenza. Regolarità all'interno, Quaderni. [Publications] (Scuola Normale Superiore Pisa, Pisa, 1980). 16
[63] P. Auscher and M. Qafsaoui, Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form, J. Funct. Anal. **177**(2), 310–364 (2000). 16
[64] G. David and J. L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators, Ann. of Math. (2) **120**(2), 371–397 (1984). 20
[65] S. Hofmann, Local $T(b)$ theorems and applications in PDE, in: Harmonic analysis and partial differential equations, , Contemp. Math., Vol. 505 (Amer. Math. Soc., Providence, RI, 2010), pp. 29–52. 22
[66] A. McIntosh and Y. Meyer, Algèbres d'opérateurs définis par des intégrales singulières, C. R. Acad. Sci. Paris Sér. I Math. **301**(8), 395–397 (1985). 22
[67] G. David, J. L. Journé, and S. Semmes, Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation, Rev. Mat. Iberoamericana **1**(4), 1–56 (1985). 22
[68] S. Semmes, Square function estimates and the $T(b)$ theorem, Proc. Amer. Math. Soc. **110**(3), 721–726 (1990). 22
[69] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. **60/61**(2), 601–628 (1990). 22
[70] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland Mathematical Library, Vol. 18 (North-Holland Publishing Co., Amsterdam, 1978). 27
[71] H. Triebel, Theory of function spaces, Monographs in Mathematics, Vol. 78 (Birkhäuser Verlag, Basel, 1983). 27
[72] E. B. Davies, Uniformly elliptic operators with measurable coefficients, J. Funct. Anal. **132**(1), 141–169 (1995). 31

[73] P. Auscher and P. Tchamitchian, Square root problem for divergence operators and related topics, Astérisque(249), viii+172 (1998). 31

[74] E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, Vol. 43 (Princeton University Press, Princeton, NJ, 1993), With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. 38

[75] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math. **129**(3-4), 137–193 (1972). 38