Maximal Function Characterizations of Hardy Spaces Associated to Homogeneous Higher Order Elliptic Operators

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Abstract Let L be a homogeneous divergence form higher order elliptic operator with complex bounded measurable coefficients and $(q_-(L),\, q_+(L))$ be the maximal interval of exponents $q \in [1,\, \infty]$ such that the gradient semigroup $\{\sqrt{t}\nabla^m e^{-tL}\}_{t>0}$ is bounded on $L^q(\mathbb{R}^n)$. In this article, the authors establish the non-tangential maximal function characterizations of the associated Hardy spaces $H^p_L(\mathbb{R}^n)$ for all $p \in (0,\, q_+(L))$, which, when $p = 1$, answers a question asked by Deng et al. in [J. Funct. Anal. 263 (2012), 604-674]. Moreover, the authors characterize $H^p_L(\mathbb{R}^n)$ via various versions of square functions and Lusin-area functions associated to the operator L.

1 Introduction

Let L be a homogeneous higher order elliptic operator of the form

$$L := \sum_{|\alpha|=m=|\beta|} (-1)^m \partial^\alpha \left(a_{\alpha,\beta}\partial^\beta\right), \tag{1.1}$$

where $\{a_{\alpha,\beta}\}_{|\alpha|=m=|\beta|}$ are bounded measurable functions mapping $\mathbb{R}^n$ into $\mathbb{C}$ (see Subsection 2.1 below for the exact definition of L in (1.1)). The aim of this article is to establish the maximal function characterizations of the associated Hardy space $H^p_L(\mathbb{R}^n)$ adapted to L, which when $p = 1$ answers a question asked by Deng et al. in [15]. It is now well known that such a Hardy space adapted to L, which has appeared in [15, 9], is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$, for smaller p, when studying the regularity of the solution to the corresponding elliptic equation (see, for example, [11, 12, 10, 13, 17, 28, 8, 20, 19, 18, 25]).

Notice that, if $L \equiv -\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator, the Hardy space $H^p_{-\Delta}(\mathbb{R}^n)$ is just the classical Hardy space $H^p(\mathbb{R}^n)$ which has been systematically studied

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2010 *Mathematics Subject Classification.* Primary: 42B30; Secondary: 42B20, 42B35, 42B37, 46E30, 35J48, 47B06, 47B38.

Key words and phrases. higher order elliptic operator, off-diagonal estimate, Hardy space, maximal function, square function, molecule, Riesz transform.

Dachun Yang is supported by the National Natural Science Foundation of China (Grant Nos. 11171027 and 11361020) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant Nos. 2012LYB26 and 2012CXQT09).

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by Fefferman and Stein in their seminal paper [21]. In the same paper, Fefferman and Stein also established various real-variable characterizations of $H^p(\mathbb{R}^n)$, including their non-tangential maximal function characterization and Littlewood-Paley function characterizations. Recall that, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *non-tangential maximal function* $\mathcal{N}_\Delta(f)(x)$ is defined by

$$\mathcal{N}_\Delta(f)(x) := \sup_{(y,\,t)\in\Gamma(x)} \left|e^{-t\sqrt{\Delta}}(f)(y)\right|. \tag{1.2}$$

Recall also that, if $n = 1$, the nontangential maximal function characterization of $H^p(\mathbb{R}^n)$ has already been proved by Burkholder, Gundy and Silverstein [16] more early, which constitutes one of the motivations for Fefferman and Stein to study the real-variable theory of $H^p(\mathbb{R}^n)$.

Let $L \equiv -\mathrm{div}(A\nabla)$ be the second order elliptic operator, where $\nabla := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ and $A := A(x)$ is an $n \times n$ matrix of complex bounded measurable coefficients defined on $\mathbb{R}^n$ and satisfies the *ellipticity condition*

$$\lambda|\xi|^2 \le \Re e\left(A\xi\cdot\overline{\xi}\right) \quad \text{and} \quad |A\xi\cdot\overline{\zeta}| \le \Lambda|\xi||\zeta|,$$

for all $\xi,\,\zeta \in \mathbb{C}^n$ and for some positive constants $0 < \lambda \le \Lambda < \infty$ independent of ξ and ζ. Hofmann and Mayboroda [23] (for $p = 1$), and Jiang and Yang [26] (for $p \in (0, 1]$) established the non-tangential maximal function characterization of the associated Hardy space $H^p_L(\mathbb{R}^n)$. Here, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *non-tangential maximal function* is defined by

$$\mathcal{N}_L(f)(x) := \sup_{(y,\,t)\in\Gamma(x)} \left\{\frac{1}{t^n}\int_{B(y,\,t)} \left|e^{-t^2L}(f)(z)\right|^2\,dz\right\}^{\frac{1}{2}}. \tag{1.3}$$

Observe that the non-tangential maximal function (1.3) is a little bit different from (1.2). The main reason for adding an extra averaging in the space variable in (1.3) is that we need to compensate for the lack of pointwise estimates of the heat semigroup (see [23] for more details).

Now, let L be a homogenous $2m$-th order elliptic operator of the form (1.1), where $\{a_{\alpha,\,\beta}\}_{|\alpha|=m=|\beta|}$ are bounded measurable functions mapping $\mathbb{R}^n$ into $\mathbb{C}$ satisfying the Ellipticity condition $(\mathcal{E}_0)$ or the Strong ellipticity condition $(\mathcal{E}_1)$ (see Subsection 2.1 for their definitions). Some properties of Hardy spaces $H^p_L(\mathbb{R}^n)$ associated with a homogeneous higher order elliptic operator L as in (1.1), for $p \in (0, 1]$, have already been established in [9, 15]. To be precise, let L be the homogeneous higher order operator defined as in (1.1) that satisfies the Ellipticity condition $(\mathcal{E}_0)$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *L-adapted square function* $S_L(f)$ is defined by

$$S_L(f)(x) := \left\{\iint_{\Gamma(x)} \left|t^{2m}Le^{-t^{2m}L}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}, \tag{1.4}$$

where, for all $x \in \mathbb{R}^n$,

$$\Gamma(x) := \left\{(y,\,t) \in \mathbb{R}^{n+1}_+ :\ |y-x| < t\right\} \tag{1.5}$$

is the *cone with vertex* x. Here and hereafter, $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$.

The following definition of Hardy spaces is motivated by [23, 26, 24]; see also [15, 9] for the case when $p \in (0, 1]$.

Definition 1.1. Let $p \in (0, 2]$, L be as in (1.1) and satisfy the Ellipticity condition $(\mathcal{E}_0)$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in the *space* $\mathbb{H}^p_L(\mathbb{R}^n)$ if $S_L(f) \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{H^p_L(\mathbb{R}^n)} := \|S_L(f)\|_{L^p(\mathbb{R}^n)}$. The *Hardy space* $H^p_L(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{H}^p_L(\mathbb{R}^n)$ with respect to the *quasi-norm* $\|\cdot\|_{H^p_L(\mathbb{R}^n)}$.

For $p \in (2, \infty)$, the *Hardy space* $H^p_L(\mathbb{R}^n)$ is defined as the dual space of the Hardy space $H^{p'}_{L^*}(\mathbb{R}^n)$, where L^* denotes the *adjoint operator* of L in $L^2(\mathbb{R}^n)$ and $p' := \frac{p}{p-1} \in (1, 2)$ denotes the *conjugate exponent* of p.

For the Hardy space $H^p_L(\mathbb{R}^n)$ with $p \in (0, 1]$, the authors in [9] established various characterizations of $H^p_L(\mathbb{R}^n)$ in terms of molecules, the generalized square function or the Riesz transform. Moreover, Deng et al. in [15] also established some other interesting characterizations of these Hardy spaces in the case of $p = 1$. However, neither of the above articles gives the maximal function characterizations of $H^p_L(\mathbb{R}^n)$ and it has been raised by Deng et al. [15] as an open question whether $H^1_L(\mathbb{R}^n)$ has the maximal function characterizations or not.

Motivated by the above articles, the main purpose of this article is to establish the maximal function characterizations of the Hardy space $H^p_L(\mathbb{R}^n)$ associated with L as in (1.1). Based on [23], we first introduce the following versions of maximal functions associated with L. For $\lambda \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *radial maximal function*, $\mathcal{R}^\lambda_{h,L}(f)$, associated with the heat semigroup generated by L, is defined by

$$\mathcal{R}^\lambda_{h,L}(f)(x) := \sup_{t\in(0,\infty)} \left\{ \frac{1}{(\lambda t)^n} \int_{B(x,\lambda t)} \sum_{k=0}^{m-1} \left| (t\nabla)^k e^{-t^{2m}L}(f)(y) \right|^2 dy \right\}^{\frac{1}{2}}. \tag{1.6}$$

Similarly, the *non-tangential maximal function*, $\mathcal{N}^\lambda_{h,L}(f)$, associated with the heat semigroup generated by L, is defined by

$$\mathcal{N}^\lambda_{h,L}(f)(x) := \sup_{(y,t)\in\Gamma^\lambda(x)} \left\{ \frac{1}{(\lambda t)^n} \int_{B(y,\lambda t)} \sum_{k=0}^{m-1} \left| (t\nabla)^k e^{-t^{2m}L}(f)(z) \right|^2 dz \right\}^{\frac{1}{2}}, \tag{1.7}$$

where $\Gamma^\lambda(x)$ for all $x \in \mathbb{R}^n$ is defined by setting

$$\Gamma^\lambda(x) := \left\{(y, t) \in \mathbb{R}^{n+1}_+ :\ |y - x| < \lambda t\right\}. \tag{1.8}$$

In what follows, when $\lambda = 1$, we remove the superscript λ from $\mathcal{R}^\lambda_{h,L}(f)$ and $\mathcal{N}^\lambda_{h,L}(f)$ for simplicity.

Definition 1.2. Let L be as in (1.1) and satisfy the Ellipticity condition $(\mathcal{E}_0)$. For all $p \in (0, \infty)$, the *Hardy space* $H^p_{\mathcal{N}_{h,L}}(\mathbb{R}^n)$ is defined as the completion of

$$\left\{f \in L^2(\mathbb{R}^n) :\ \mathcal{N}_{h,L}(f) \in L^p(\mathbb{R}^n)\right\}$$

with respect to the *quasi-norm*

$$\|f\|_{H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)} := \|\mathcal{N}_{h,\,L}(f)\|_{L^p(\mathbb{R}^n)}\,.$$

The *Hardy space* $H^p_{\mathcal{R}_{h,\,L}}(\mathbb{R}^n)$ is defined in the way same as $H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$ with $\mathcal{N}_{h,\,L}(f)$ in (1.7) replaced by $\mathcal{R}_{h,\,L}(f)$ in (1.6).

Remark 1.3. By the argument that used in the proof of [23, (6.50)] with a small modification, we know that, for all $p \in (0,\,\infty)$ and $f \in L^2(\mathbb{R}^n)$,

$$\|\mathcal{N}_{h,\,L}(f)\|_{L^p(\mathbb{R}^n)} \sim \|\mathcal{R}_{h,\,L}(f)\|_{L^p(\mathbb{R}^n)}\,,$$

which implies that, for all $p \in (0,\,\infty)$, $H^p_{\mathcal{R}_{h,\,L}}(\mathbb{R}^n) = H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$ with equivalent quasi-norms.

Let $(q_-(L),\,q_+(L))$ be the *maximal interval* of exponents $q \in [1,\,\infty]$ such that the family $\{\sqrt{t}\nabla^m e^{-tL}\}_{t>0}$ of operators is bounded on $L^q(\mathbb{R}^n)$. The following theorem establishes the maximal function characterizations of $H^p_L(\mathbb{R}^n)$.

Theorem 1.4. *Let L be as in* (1.1) *and satisfy the Strong ellipticity condition* $(\mathcal{E}_1)$ *(see Subsection 2.1 below for its definition). Then, for all $p \in (0,\,q_+(L))$, $H^p_L(\mathbb{R}^n) = H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$ with equivalent quasi-norms, where $H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$ is defined as in Definition 1.2.*

Moreover, let $H^p_{\mathcal{R}_{h,\,L}}(\mathbb{R}^n)$ be as in Definition 1.2. Then, for all $p \in (0,\,q_+(L))$,

$$H^p_L(\mathbb{R}^n) = H^p_{\mathcal{R}_{h,\,L}}(\mathbb{R}^n)$$

with equivalent quasi-norms.

The proof of Theorem 1.4 will be given in Section 3 of this article.

Before describing our method to prove Theorem 1.4, let us first recall some key points of the methods used to establish the maximal function characterizations in [21, 23].

For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S_\Delta(f)(x) := \left\{\iint_{\Gamma(x)} \left|t\nabla e^{-t\sqrt{\Delta}}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac{1}{2}} \tag{1.9}$$

be the *Lusin-area function* of f *associated to* Δ, where $\Gamma(x)$ for all $x \in \mathbb{R}^n$ is as in (1.5). For convenience, throughout the article, we distinguish in terminology the *square function with gradient* from the one without gradient via calling the former the *Lusin-area function*.

Recall that Fefferman and Stein [21] established the maximal function characterizations of $H^p(\mathbb{R}^n)$ by developing the equivalence of the $L^p(\mathbb{R}^n)$ quasi-norms between $\mathcal{N}_\Delta(f)$ in (1.2) and $S_\Delta(f)$ in (1.9). The heart of their proof is to control the integral $\int_E [S_\Delta(f)(x)]^2\,dx$ for some set E. By Fubini's theorem, this is reduced to the corresponding estimates on a saw-tooth region $\mathcal{R} := \cup_{x\in E}\Gamma(x)$ based on E, namely, we need to control

$$\iint_{\mathcal{R}} t\left|\nabla e^{-t\sqrt{\Delta}}(f)(y)\right|^2 dy\,dt. \tag{1.10}$$

The main tool that they used to estimate (1.10) is Green's theorem. To this end, they first replaced the region $\mathcal{R}$ by an approximating family $\{\mathcal{R}_\epsilon\}_{\epsilon>0}$ of regions whose boundaries have certain uniform smoothness, and then applied Green's theorem to reduce the estimates on $\mathcal{R}_\epsilon$ to its boundary. Finally, they used some properties of harmonic functions to estimate the corresponding integral on the boundary.

Hofmann and Mayboroda [23] used the strategy similar to that of Fefferman and Stein [21]. However, there do exist some differences between these two methods. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S_L(f)(x) := \left\{ \iint_{\Gamma(x)} \left| t^2 L e^{-t^2 L}(f)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}} \tag{1.11}$$

be the *square function* of f *associated to* L, which was used in [23] to introduce the Hardy space $H^p_L(\mathbb{R}^n)$ associated with $L \equiv -\mathrm{div}(A\nabla)$. Notice that this square function, which is more convenient when defining $H^p_L(\mathbb{R}^n)$, is different from the Lusin-area function (1.9). Then, to obtain the non-tangential maximal characterization of $H^p_L(\mathbb{R}^n)$, Hofmann and Mayboroda used a Caccioppoli's inequality to control $S_L(f)$ by another, Lusin-type, area function defined in a way similar to (1.9) with $e^{-t\sqrt{\Delta}}$ replaced by $e^{-t\sqrt{L}}$. Furthermore, they used the truncated cone to approximate the cone in (1.11) before applying Fubini's theorem. This reduces to estimating the following integral

$$\iint_{\mathcal{R}^{\alpha\epsilon,\,\alpha\epsilon,\,\frac{1}{\alpha}}(E^*)} t \left| \nabla e^{-t^2 L}(f)(y) \right|^2 dy\,dt, \tag{1.12}$$

where $\mathcal{R}^{\alpha\epsilon,\,\alpha\epsilon,\,\frac{1}{\alpha}}(E^*)$ denotes a truncated saw-tooth region. Finally, in the estimate of (1.12), since $e^{-t^2 L}(f)$ is no longer a harmonic function and hence Green's theorem cannot be used directly, Hofmann and Mayboroda [23] made full use of the ellipticity condition of the operator $-\mathrm{div}(A\nabla)$ and the divergence theorem to reduce the corresponding estimates to the boundary of $\mathcal{R}^{\alpha\epsilon,\,\alpha\epsilon,\,\frac{1}{\alpha}}(E^*)$.

To prove Theorem 1.4, we first point out that the proof of the inclusion

$$H^p_L(\mathbb{R}^n) \subset H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$$

is relatively easy. Indeed, for $p \in (0, 1]$, by the molecular characterization of $H^p_L(\mathbb{R}^n)$ (see Theorem 2.11 below), we only need to consider the action of the non-tangential maximal function $\mathcal{N}_{h,\,L}$ on each molecule of $H^p_L(\mathbb{R}^n)$. For $p \in [2, q_+(L))$, using the $L^2(\mathbb{R}^n)$ off-diagonal estimates, we show that the radial maximal function $\mathcal{R}_{h,\,L}$ is bounded on $L^p(\mathbb{R}^n)$, which, together with relations between $H^p_L(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ (see Lemma 2.12 below), the complex interpolation of $H^p_L(\mathbb{R}^n)$ (see Proposition 2.8 below) and Remark 1.3, implies that, for all $p \in (1, q_+(L))$, $\mathcal{N}_{h,\,L}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. This furnishes the proof of the inclusion $H^p_L(\mathbb{R}^n) \subset H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$.

For the proof of the converse inclusion of Theorem 1.4, when $p \in (0, 2]$, we adapt the strategy of [21, 23]. The higher order setting produces new problems and requires new tools. To be precise, let S_L and $S_{h,L}$ are, respectively, the square function and the Lusin-area function as in (1.4) and (1.16). We obtain the converse inclusion by showing that, for all $p \in (0, 2]$ and $f \in L^2(\mathbb{R}^n)$,

$$\|S_L(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|S_{h,L}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \left\|\mathcal{N}^{\gamma}_{h,L}(f)\right\|_{L^p(\mathbb{R}^n)}, \tag{1.13}$$

where $\gamma \in (0, \infty)$ and the implicit positive constants are independent of f. However, in the proof of the first inequality of (1.13), we need a new higher order parabolic Caccioppoli's inequality (see (3.1) below). Unlike in the case of [23], this parabolic Caccioppoli's inequality necessarily has an "ϵ-term", which makes the argument more delicate (see also its elliptic analogy in [4]). Second, in the proof of the second inequality of (1.13), in order to avoid the estimates on the boundary when applying the divergence theorem, we use some special cut-off functions. In this argument, another new parabolic Caccioppoli's inequality is needed (see (3.2) below), whose proof rests on an interpolation inequality from [1] (see also (3.9) below) to bound various intermediate order gradient terms. We also point out that it is the lower order terms, appearing in the parabolic Caccioppoli's inequality (3.2), which force us to add lower order terms in the definitions of maximal functions (1.6) and (1.7). This artifact, however, can be avoided using somewhat more delicate definitions of maximal functions (see Proposition 1.5 below). The case $p \in (2, q_+(L))$ of the first inequality in (1.13) is obtained via duality; see Proposition 3.5 and Corollary 3.6 below.

Now we characterize $H^p_L(\mathbb{R}^n)$ by using the non-tangential maximal function with only the $(m-1)$-order gradients of the heat semigroup generated by L.

Let $\psi \in C^\infty_c(B(0, 2))$ satisfy $0 \le \psi(x) \le 1$, $\psi \equiv 1$ on $B(0, 1)$ and, for all $k \in \{0, \ldots, m\}$,

$$\left\|\nabla^k \psi\right\|_{L^\infty(\mathbb{R}^n)} \lesssim 1.$$

For all $(x, t) \in \mathbb{R}^{n+1}_+$ and $y \in \mathbb{R}^n$, let

$$\psi_{x,t}(y) := \frac{1}{t^n}\psi\left(\frac{y-x}{t}\right). \tag{1.14}$$

Then $\psi_{x,t} \in C^\infty_c(B(x, 2t))$ and $0 \le \psi_{x,t}(y) \le 1$, $\psi_{x,t} \equiv 1$ on $B(x, t)$ and, for all $k \in \{0, \ldots, m\}$,

$$\left\|\nabla^k \psi_{x,t}\right\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-k}.$$

Having fixed any ψ as above, for any $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $\lambda \in (0, \infty)$, we introduce the following version of the *non-tangential maximal function associated with the heat semigroup*

$$\begin{aligned}&\mathcal{N}^{\lambda}_{h,\psi,L}(f)(x)\\ &\quad:= \sup_{(y,t)\in\Gamma^\lambda(x)} \left\{\frac{1}{(\lambda t)^n}\int_{B(y,\lambda t)} \left|(t\nabla)^{m-1}\left(\psi_{x,t} e^{-t^{2m}L}(f)\right)(z)\right|^2 dz\right\}^{\frac12}.\end{aligned}$$

When $\lambda = 1$, we remove the supscript λ from $\mathcal{N}^{\lambda}_{h,\psi,L}(f)$ for simplicity.

Proposition 1.5. *Let L be as in* (1.1) *and satisfy the Strong ellipticity condition* $(\mathcal{E}_1)$*, and let ψ be a cut-off function defined as in* (1.14)*. For any $p \in (0,\, q_+(L))$, denote by $H^p_{\mathcal{N}_{h,\,\psi,\,L}}(\mathbb{R}^n)$ the Hardy space defined as $H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$ with $\mathcal{N}_{h,\,L}$ replaced by $\mathcal{N}_{h,\,\psi,\,L}$. Then $H^p_L(\mathbb{R}^n) = H^p_{\mathcal{N}_{h,\,\psi,\,L}}(\mathbb{R}^n)$ with equivalent quasi-norms. In particular, different choices of ψ in the definition of $H^p_{\mathcal{N}_{h,\,\psi,\,L}}(\mathbb{R}^n)$ above yield equivalent quasi-norms.*

The proof of Proposition 1.5 will be given in Section 3 of this article.

By the method used in the proof of Theorem 1.4, we are able to characterize $H^p_L(\mathbb{R}^n)$ via some more general square functions and Lusin-area functions.

To be precise, for all $\lambda \in (0,\, \infty)$, $k \in \mathbb{Z}_+$ and $f \in L^2(\mathbb{R}^n)$, the *L-adapted square function* $S^\lambda_{L,\,k}(f)$ is defined by setting, for all $x \in \mathbb{R}^n$,

$$S^\lambda_{L,\,k}(f)(x) := \left\{\iint_{\Gamma^\lambda(x)} \left|\left(t^{2m}L\right)^k e^{-t^{2m}L}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2} \tag{1.15}$$

and the *Lusin-area function* $S_{h,\,L,\,k}(f)$ by setting, for all $x \in \mathbb{R}^n$,

$$S^\lambda_{h,\,L,\,k}(f)(x) := \left\{\iint_{\Gamma^\lambda(x)} \left|(t\nabla)^m \left(t^{2m}L\right)^k e^{-t^{2m}L}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{1/2}, \tag{1.16}$$

where Γ^λ is as in (1.8). For simplicity, if $k = 1$, we remove the subscript k from $S^\lambda_{L,\,k}(f)$ and, if $k = 0$, we remove the subscript k from $S^\lambda_{h,\,L,\,k}(f)$. Also, if $\lambda = 1$, we remove the superscript λ from both $S^\lambda_{L,\,k}(f)$ and $S^\lambda_{h,\,L,\,k}(f)$.

Definition 1.6. Let L be as in (1.1) and satisfy the Ellipticity condition $(\mathcal{E}_0)$. For all $k \in \mathbb{N}$ and $p \in (0,\, \infty)$, the *Hardy space* $H^p_{S_{L,\,k}}(\mathbb{R}^n)$ is defined as the completion of

$$\left\{f \in L^2(\mathbb{R}^n) :\ S_{L,\,k}(f) \in L^p(\mathbb{R}^n)\right\}$$

with respect to the *quasi-norm*

$$\|f\|_{H^p_{S_{L,\,k}}(\mathbb{R}^n)} := \|S_{L,\,k}(f)\|_{L^p(\mathbb{R}^n)}\,.$$

Moreover, for all $k \in \mathbb{Z}_+$ and $p \in (0,\, \infty)$, the *Hardy space* $H^p_{S_{h,\,L,\,k}}(\mathbb{R}^n)$ is defined in the way same as $H^p_{S_{L,\,k}}(\mathbb{R}^n)$ with $S_{L,\,k}(f)$ in (1.15) replaced by $S_{h,\,L,\,k}(f)$ in (1.16).

The following theorem establishes the characterization of $H^p_L(\mathbb{R}^n)$ via, respectively, some square functions and some Lusin-area functions.

Theorem 1.7. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*. Then*

(i) *for all $k \in \mathbb{N}$ and $p \in (0,\, p_+(L))$, $H^p_L(\mathbb{R}^n) = H^p_{S_{L,\,k}}(\mathbb{R}^n)$ with equivalent quasi-norms;*

(ii) *for all $k \in \mathbb{Z}_+$ and $p \in (0,\, q_+(L))$, $H^p_L(\mathbb{R}^n) = H^p_{S_{h,\,L,\,k}}(\mathbb{R}^n)$ with equivalent quasi-norms.*

The proof of Theorem 1.7 will be given in Section 3 of this article.

Let us end this section by making some conventions on the notation. Throughout the paper, we always let $\mathbb{N} := \{1, 2, \cdots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Denote the *differential operator* $\frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ simply by ∂^α, where $\alpha := (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. We use C to denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line, and $C_{(\alpha,\ldots)}$ to denote a *positive constant* depending on the parameters $\alpha, \ldots$. *Constants with subscripts*, such as C_1, do not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $\lambda \in (0, \infty)$, let $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ and $\lambda B := B(x, \lambda r)$. Also, for any set $E \subset \mathbb{R}^n$, χ_E denotes its *characteristic function* and, for all $z \in \mathbb{C}$, $\Re e\, z$ denotes its *real part*.

2 The Hardy space $H_L^p(\mathbb{R}^n)$

In this section, we study the Hardy space $H_L^p(\mathbb{R}^n)$ associated with the homogeneous higher order elliptic operator L in (1.1). To this end, we first collect some known basic facts on L in Subsection 2.1; then, in Subsection 2.2, we present the real-variable theory of the Hardy space $H_L^p(\mathbb{R}^n)$ associated with L for $p \in (0, \infty)$. Recall that, for $p \in (0, 1]$, $H_L^p(\mathbb{R}^n)$ has been studied in [9, 15]. Our results here also include the case $p \in (1, \infty)$.

2.1 Homogeneous higher order elliptic operators

Let $m \in \mathbb{N}$ and $\dot{W}^{m,2}(\mathbb{R}^n)$ be the *m-order homogeneous Sobolev space* equipped with the usual *norm*

$$\|f\|_{\dot{W}^{m,2}(\mathbb{R}^n)} := \left[\sum_{|\alpha|=m} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^2\right]^{1/2}.$$

For all multi-indices α and β satisfying $|\alpha| = m = |\beta|$, let $a_{\alpha,\beta}$ be a complex valued L^∞ function on $\mathbb{R}^n$. For all f and $g \in \dot{W}^{m,2}(\mathbb{R}^n)$, define the *sesquilinear form* $\mathfrak{a}_0$, mapping $\dot{W}^{m,2}(\mathbb{R}^n) \times \dot{W}^{m,2}(\mathbb{R}^n)$ into $\mathbb{C}$, by

$$\mathfrak{a}_0(f, g) := \sum_{|\alpha|=m=|\beta|} \int_{\mathbb{R}^n} a_{\alpha,\beta}(x) \partial^\beta f(x) \overline{\partial^\alpha g(x)}\, dx. \tag{2.1}$$

The following ellipticity condition on $\{a_{\alpha,\beta}\}_{|\alpha|=m=|\beta|}$ is necessary.

Ellipticity condition $(\mathcal{E}_0)$**.** There exist constants $0 < \lambda_0 \le \Lambda_0 < \infty$ such that, for all f and $g \in \dot{W}^{m,2}(\mathbb{R}^n)$,

$$\left|\sum_{|\alpha|=m=|\beta|} \int_{\mathbb{R}^n} a_{\alpha,\beta}(x) \partial^\beta f(x) \overline{\partial^\alpha g(x)}\, dx\right| \le \Lambda_0 \|\nabla^m f\|_{L^2(\mathbb{R}^n)} \|\nabla^m g\|_{L^2(\mathbb{R}^n)}$$

and

$$\Re e\left\{\sum_{|\alpha|=m=|\beta|}\int_{\mathbb{R}^n} a_{\alpha,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha f(x)}\,dx\right\} \geq \lambda_0 \left\|\nabla^m f\right\|^2_{L^2(\mathbb{R}^n)},$$

where

$$\left\|\nabla^m f\right\|_{L^2(\mathbb{R}^n)} := \left[\sum_{|\alpha|=m}\int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2\,dx\right]^{1/2}.$$

We also need the following strong ellipticity condition on $\{a_{\alpha,\beta}\}_{|\alpha|=m=|\beta|}$.

Strong ellipticity condition $(\mathcal{E}_1)$**.** There exists a positive constant λ_1 such that, for all $\xi := \{\xi_\alpha\}_{|\alpha|=m}$ with $\xi_\alpha \in \mathbb{C}$ and almost every $x \in \mathbb{R}^n$,

$$\Re e\left\{\sum_{|\alpha|=m=|\beta|} a_{\alpha,\beta}(x)\xi_\beta\overline{\xi_\alpha}\right\} \geq \lambda_1|\xi|^2 = \lambda_1\left\{\sum_{|\alpha|=m}|\xi_\alpha|^2\right\}.$$

Moreover, for all multi-indices α and β with $|\alpha| = m = |\beta|$, $a_{\alpha,\beta} \in L^\infty(\mathbb{R}^n)$.

Remark 2.1. It is easy to see that the Strong ellipticity condition $(\mathcal{E}_1)$ implies the Ellipticity condition $(\mathcal{E}_0)$. However, the equivalence between $(\mathcal{E}_1)$ and $(\mathcal{E}_0)$ is a specific feature of second order operators (see, for example, [5, p. 15]). For more relationships on these two kinds of ellipticity conditions, we refer the reader to [3, p. 365].

Let us recall some basic facts on sesquilinear forms from [30, p. 3, Section 1.2.1].

Definition 2.2 ([30])**.** Assume that $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ is a sesquilinear form in the Hilbert space $\mathcal{H}$.

(i) $\mathfrak{a}$ is said to be *densely defined* if the domain of $\mathfrak{a}$, $D(\mathfrak{a})$, is dense in $\mathcal{H}$;

(ii) $\mathfrak{a}$ is said to be *accretive* if, for all $u \in D(\mathfrak{a})$,

$$\Re e\ (\mathfrak{a}(u, u)) \geq 0;$$

(iii) $\mathfrak{a}$ is said to be *continuous* if there exists a nonnegative constant M such that, for all $u, v \in D(\mathfrak{a})$,

$$|\mathfrak{a}(u, v)| \leq M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}},$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\Re e\,(\mathfrak{a}(u, u)) + \|u\|^2_{\mathcal{H}}}$;

(iv) $\mathfrak{a}$ is said to be *closed* if $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is a complete space.

For a densely defined, accretive, continuous and closed sesquilinear form in the Hilbert space $\mathcal{H}$, we have the following conclusion from [30, Proposition 1.22]. Recall that $\|\cdot\|_{\mathcal{H}}$ and $(\cdot, \cdot)_{\mathcal{H}}$ denote, respectively, the *inner product* and the *norm* of $\mathcal{H}$.

Proposition 2.3 ([30]). *Assume that* $\mathfrak{a}$ *is a densely defined, accretive, continuous and closed sesquilinear form in the Hilbert space* $\mathcal{H}$. *Then there exists a densely defined operator* T, *defined by setting*

$$D(T) := \{u \in \mathcal{H} : \ \exists\, v \in \mathcal{H} \text{ such that, for all } \phi \in D(\mathfrak{a}),\ \mathfrak{a}(u, \phi) = (v, \phi)_{\mathcal{H}}\}$$

and $Tu := v$ *for all* $u \in D(T)$, *such that, for all* $\lambda \in (0, \infty)$, $\lambda I + T$ *is invertible (from* $D(T)$ *into* $\mathcal{H}$*) and* $(\lambda I + T)^{-1}$ *is bounded on* $\mathcal{H}$. *Moreover, for all* $\lambda \in (0, \infty)$ *and* $f \in \mathcal{H}$,

$$\left\|\lambda\,(\lambda I + T)^{-1}(f)\right\|_{\mathcal{H}} \le \|f\|_{\mathcal{H}}.$$

For $\mathfrak{a}_0$ defined as in (2.1), from the fact that $\dot{W}^{m,\,2}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and the Ellipticity condition $(\mathcal{E}_0)$, we deduce that $\mathfrak{a}_0$ is a densely defined, accretive and continuous sesquilinear form.

Moreover, let $W^{m,\,2}(\mathbb{R}^n)$ be the *m-order inhomogeneous Sobolev space* equipped with the usual *norm*

$$\|f\|_{W^{m,\,2}(\mathbb{R}^n)} := \left[\sum_{0\le|\alpha|\le m} \|\partial^\alpha f\|^2_{L^2(\mathbb{R}^n)}\right]^{1/2}. \tag{2.2}$$

For all $f \in D(\mathfrak{a}_0)$, by the Ellipticity condition $(\mathcal{E}_0)$ and Plancherel's theorem, it is easy to see

$$\|f\|_{\mathfrak{a}_0} := \sqrt{\Re e\ (\mathfrak{a}_0(f, f)) + \|f\|^2_{L^2(\mathbb{R}^n)}} \sim \|f\|_{W^{m,\,2}(\mathbb{R}^n)}.$$

This, combined with the fact that $W^{m,\,2}(\mathbb{R}^n)$ is a Banach space, further implies that $(\dot{W}^{m,\,2}(\mathbb{R}^n), \|\cdot\|_{\mathfrak{a}_0})$ is complete. Thus, $\mathfrak{a}_0$ is closed. Using Proposition 2.3, we know that there exists a densely defined operator L in $L^2(\mathbb{R}^n)$ associated with $\mathfrak{a}_0$, which is formally written as in (1.1).

Let $\omega \in [0, \pi/2)$. Recall that an operator T in the Hilbert space $\mathcal{H}$ is said to be *m-ω-accretive* (or *maximal ω-accretive*) if

(i) the range of the operator $T + I$, $R(T + I)$, is dense in $\mathcal{H}$;

(ii) for all $u \in D(T)$, $|\arg(T(u), u)_{\mathcal{H}}| \le \omega$,

where $\arg(T(u), u)_{\mathcal{H}}$ denotes the *argument* of $(T(u), u)_{\mathcal{H}}$; see [22, p. 173].

It is known that, by [22, Proposition 7.1.1], every closed m-ω-accretive operator is of *type* ω in $L^2(\mathbb{R}^n)$, namely, the spectrum of T, $\sigma(T)$, is contained in the *sector*

$$S_\omega := \{z \in \mathbb{C} : \ |\arg z| \le \omega\}$$

and, for each $\theta \in (\omega, \pi)$, there exists a nonnegative constant C such that, for all $z \in \mathbb{C}\setminus S_\theta$, $\|(T - zI)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \le C|z|^{-1}$, where $\|S\|_{\mathcal{L}(\mathcal{H})}$ denotes the *operator norm* of the linear operator S on the normed linear space $\mathcal{H}$.

Moreover, by [22], we know that, if T is of type ω, then $-T$ generates a semigroup $\{e^{-tT}\}_{t>0}$, which can be extended to a bounded holomorphic semigroup $\{e^{-zT}\}_{z\in S^0_{\pi/2-\omega}}$ in the *open sector*

$$S^0_{\pi/2-\omega} := \{z \in \mathbb{C} \setminus \{0\} : \ |\arg z| < \pi/2 - \omega\}.$$

Recall that, by the Ellipticity condition $(\mathcal{E}_0)$, we know that L is an m-arctan $\frac{\Lambda}{\lambda}$-accretive operator in $L^2(\mathbb{R}^n)$. Thus, $-L$ generates a bounded holomorphic semigroup in the open sector $S^0_{\pi/2-\arctan\frac{\Lambda}{\lambda}}$.

The following $L^2(\mathbb{R}^n)$ off-diagonal estimates of $\{e^{-zL}\}_{z\in S^0_{\pi/2-\arctan\frac{\Lambda}{\lambda}}}$ are well known (see, for example, [2, p. 66], [15, Theorem 3.2] or [9, Lemma 3.1]).

Proposition 2.4. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*, and let* $\omega := \arctan\frac{\Lambda_0}{\lambda_0}$*, where Λ_0 and λ_0 are as in the Ellipticity condition* $(\mathcal{E}_0)$*. Then, for all $\ell \in (0, 1)$, $k \in \mathbb{Z}_+$, the family of operators, $\{(zL)^k e^{-zL}\}_{z\in S^0_{\ell(\frac{\pi}{2}-\omega)}}$, satisfies the m-Davies-Gaffney estimates in z. That is, there exist positive constants C and $\widetilde{C}$ such that, for all $f \in L^2(\mathbb{R}^n)$ supported in E and $z \in S^0_{\ell(\frac{\pi}{2}-\omega)}$,*

$$\|(zL)^k e^{-zL}(f)\|_{L^2(F)} \le C \exp\left\{-\widetilde{C}\frac{[\operatorname{dist}(E, F)]^{2m/(2m-1)}}{|z|^{1/(2m-1)}}\right\}\|f\|_{L^2(E)}.$$

We now consider the $L^p(\mathbb{R}^n)$ theory of $\{e^{-tL}\}_{t>0}$. Let $(p_-(L), p_+(L))$ be the *maximal interval* of exponents $p \in [1, \infty]$ such that $\{e^{-tL}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Let $(q_-(L), q_+(L))$ be the *maximal interval* of exponents $q \in [1, \infty]$ such that $\{\sqrt{t}\nabla^m e^{-tL}\}_{t>0}$ is bounded on $L^q(\mathbb{R}^n)$. By [2, pp. 66-67] and [15, Theorem 3.2], we have the following conclusion.

Proposition 2.5 ([2, 15])**.** *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*. Then*

(i)

$$\begin{cases} (p_-(L), p_+(L)) = (1, \infty), & \text{when } n \le 2m, \\ \left[\dfrac{n}{n+2m}, \dfrac{n}{n-2m}\right] \subset (p_-(L), p_+(L)), & \text{when } n > 2m. \end{cases}$$

(ii) *$q_-(L) = p_-(L)$, $q_+(L) > 2$ and $p_+(L) \ge (q_+(L))^{*m}$, where, for any $q \in (1, \infty)$,*

$$q^* := \begin{cases} \dfrac{np}{n-p}, & \text{when } p < n, \\ \infty, & \text{when } p \ge n \end{cases}$$

denotes the Sobolev exponent of q and q^{*m} means the m-th iteration of the operation $q \mapsto q^*$.

(iii) *For all $k \in \mathbb{Z}_+$ and $p_-(L) < p \le q < p_+(L)$, the family $\{(tL)^k e^{-tL}\}_{t>0}$ of operators satisfies the following m-L^p-L^q off-diagonal estimates: there exist positive constants C and $\widetilde{C}$ such that, for any closed sets E, F in $\mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ supported in E,*

$$\left\|(tL)^k e^{-tL}(f)\right\|_{L^q(F)} \le C t^{\frac{n}{2m}(\frac{1}{q}-\frac{1}{p})} \exp\left\{-\widetilde{C}\frac{[d(E, F)]^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\} \|f\|_{L^p(\mathbb{R}^n)}.$$

(iv) *For all $p_-(L) < p \le q < q_+(L)$, the family $\{(t^{1/(2m)}\nabla)^k e^{-tL}\}_{t>0}$ of operators satisfies the following m-L^p-L^q off-diagonal estimates: there exist positive constants C and $\widetilde{C}$ such that, for any closed sets E, F in $\mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ supported in E,*

$$\left\|\left(t^{1/(2m)}\nabla\right)^k e^{-tL}(f)\right\|_{L^q(F)} \le C t^{\frac{n}{2m}(\frac{1}{q}-\frac{1}{p})} \exp\left\{-\widetilde{C}\frac{[d(E, F)]^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\} \|f\|_{L^p(\mathbb{R}^n)}.$$

Finally, we recall some results on the square root of L. Let L be defined as in (1.1). It is known that L is one-to-one and m-ω-accretive. By [5, p. 8], we know that L has a bounded holomorphic functional calculus in $L^2(\mathbb{R}^n)$. Thus, its square root $L^{1/2}$ is well defined on $L^2(\mathbb{R}^n)$.

Auscher et al. proved the following result on Kato's square root problem of $L^{1/2}$ (see [3, Theorem 1.1]).

Proposition 2.6 ([3]). *Let L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$). *The square root of L has a domain equal to the Sobolev space $W^{m,2}(\mathbb{R}^n)$ defined as in* (2.2). *Moreover, there exists a positive constant C such that, for all $f \in W^{m,2}(\mathbb{R}^n)$,*

$$\frac{1}{C}\left\|\sqrt{L}(f)\right\|_{L^2(\mathbb{R}^n)} \le \|\nabla^m f\|_{L^2(\mathbb{R}^n)} \le C\left\|\sqrt{L}(f)\right\|_{L^2(\mathbb{R}^n)}.$$

Proposition 2.6 implies immediately that the Riesz transform $\nabla^m L^{-1/2}$ associated with L is bounded on $L^2(\mathbb{R}^n)$. Moreover, Auscher proved the following boundedness of $\nabla^m L^{-1/2}$ on $L^p(\mathbb{R}^n)$ (see [2, p. 68]).

Proposition 2.7 ([2]). *Let L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$). *Then, for all $p \in (q_-(L)), q_+(L))$, $\nabla^m L^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$.*

We also refer the reader to [6, Theorem 1.2] for a related result on the boundedness of $\nabla^m L^{-1/2}$.

2.2 The Hardy space $H^p_L(\mathbb{R}^n)$

Let L be the homogeneous higher order operator defined as in (1.1) that satisfies the Ellipticity condition ($\mathcal{E}_0$). Let $H^p_L(\mathbb{R}^n)$ be the Hardy space associated with L defined as in Definition 1.1. In this subsection, we give some real-variable properties of $H^p_L(\mathbb{R}^n)$ for $p \in (0, \infty)$. Our first result is the following complex interpolation of $H^p_L(\mathbb{R}^n)$. Recall ([14])

that, for all $p \in (0, \infty)$, a function F on $\mathbb{R}_+^{n+1}$ is said to be in the *tent space* $T^p(\mathbb{R}_+^{n+1})$, if $\|F\|_{T^p(\mathbb{R}_+^{n+1})} =: \|\mathcal{A}(F)\|_{L^p(\mathbb{R}^n)} < \infty$, where

$$\mathcal{A}(F)(x) := \left\{ \iint_{\Gamma(x)} |F(y, t)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac{1}{2}},$$

with $\Gamma(x)$ for all $x \in \mathbb{R}^n$ as in (1.5), denotes the $\mathcal{A}$-*functional* of F (see [14] for more properties of tent spaces).

Proposition 2.8. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$). *Then, for each $\theta \in (0, 1)$ and $0 < p_1 < p_2 < \infty$,*

$$\left[H_L^{p_1}(\mathbb{R}^n),\, H_L^{p_2}(\mathbb{R}^n)\right]_\theta = H_L^p(\mathbb{R}^n),$$

where p satisfies $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $[\cdot, \cdot]_\theta$ denotes the complex interpolation (see, for example, [27, Section 7]).

Proof. The proof of Proposition 2.8 is a consequence of the complex interpolation of tent spaces $T^p(\mathbb{R}_+^{n+1})$ and the fact that $H_L^p(\mathbb{R}^n)$ is a retract of $T^p(\mathbb{R}_+^{n+1})$ (see [24, Lemma 4.20] for more details in the case when $m = 1$), the details being omitted. □

For $H_L^p(\mathbb{R}^n)$ with $p \in (0, 1]$, one of its most useful properties is its molecular characterization. To state it, we first recall the following notion of $(p, 2, M, q)_L$-molecules.

Definition 2.9. Let L be as in (1.1) and satisfy the Ellipticity condition ($\mathcal{E}_0$), $p \in (0, 1]$, $\epsilon \in (0, \infty)$ and $M \in \mathbb{N}$. A function $\alpha \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M, \epsilon)_L$-*molecule*, if there exists a ball $B \subset \mathbb{R}^n$ such that, for each $\ell \in \{0, \ldots, M\}$, α belongs to the range of L^ℓ in $L^2(\mathbb{R}^n)$ and, for all $i \in \mathbb{Z}_+$ and $\ell \in \{0, \ldots, M\}$,

$$\left\| \left(r_B^{2m} L\right)^{-\ell} (\alpha) \right\|_{L^2(S_i(B))} \le 2^{-i\epsilon} |2^i B|^{\frac{1}{2} - \frac{1}{p}}.$$

Assume that $\{\alpha_j\}_j$ is a sequence of $(p, 2, M, \epsilon)_L$-molecules and $\{\lambda_j\}_j \in l^p$. For any $f \in L^2(\mathbb{R}^n)$, if $f = \sum_j \lambda_j \alpha_j$ in $L^2(\mathbb{R}^n)$, then $\sum_j \lambda_j \alpha_j$ is called a *molecular* $(p, 2, M, \epsilon)_L$-*representation* of f.

Definition 2.10. Let L be as in (1.1) and satisfy the Ellipticity condition ($\mathcal{E}_0$), and let $p \in (0, 1]$, $\epsilon \in (0, \infty)$ and $M \in \mathbb{N}$. The *molecular Hardy space* $H^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)$ is defined as the completion of the space

$$\mathbb{H}^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n) :\ f \ \text{ has a molecular } (p, 2, M, \epsilon)_L\text{-representation}\}$$

with respect to the *quasi-norm*

$$\|f\|_{H^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)} := \inf \left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} :\ f = \sum_j \lambda_j \alpha_j \text{ is a molecular} \right.$$

$$(p,\,2,\,M,\,\epsilon)_L\text{-representation}\Bigg\},$$

where the infimum is taken over all the molecular $(p,\,2,\,M,\,\epsilon)_L$-representations of f as above.

Theorem 2.11 ([9, 15])**.** *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*, $p\in(0,\,1]$, $\epsilon\in(0,\,\infty)$ and $M\in\mathbb{N}$ such that $M>\frac{n}{2m}(\frac1p-\frac12)$. Then $H^p_L(\mathbb{R}^n)=H^p_{L,\,\mathrm{mol},\,M,\,\epsilon}(\mathbb{R}^n)$ with equivalent quasi-norms.*

For more characterizations of $H^p_L(\mathbb{R}^n)$ with $p\in(0,\,1]$, we refer the reader to [9, 15].

We now study the relationship between $H^p_L(\mathbb{R}^n)$ and the Lebesgue space $L^p(\mathbb{R}^n)$.

Lemma 2.12. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*, and let $p_-(L)$ and $p_+(L)$ be as in Proposition 2.5. Then, for all $p\in(p_-(L),\,p_+(L))$,*

$$H^p_L(\mathbb{R}^n)=L^p(\mathbb{R}^n)$$

with equivalent norms.

Proof. We prove Lemma 2.12 by borrowing some ideas from the proof of [24, Proposition 9.1(v)]. First, from [8, Propositions 2.10 and 2.13], it follows that, for all $p\in(p_-(L),\,p_+(L))$, S_L is bounded on $L^p(\mathbb{R}^n)$. This, together with Definition 1.1, shows that, for all $p\in(p_-(L),\,2]$ and $f\in L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$,

$$\|f\|_{H^p_L(\mathbb{R}^n)}:=\|S_L(f)\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)},$$

which immediately implies that, for all $p\in(p_-(L),\,2]$,

$$\left(L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)\right)\subset\left(L^2(\mathbb{R}^n)\cap H^p_L(\mathbb{R}^n)\right).$$

On the other hand, recall the following Calderón reproducing formula for L (since L has a bounded holomorphic functional calculus in $L^2(\mathbb{R}^n)$): for all $g\in L^2(\mathbb{R}^n)$,

$$g=\widetilde{C}\int_0^\infty (t^{2m}L)^{M+2}e^{-2t^{2m}L}(g)\,\frac{dt}{t}=:\widetilde{C}\pi_{L,\,M}\circ Q_{L,\,1,\,t}(g) \tag{2.3}$$

holds true in $L^2(\mathbb{R}^n)$, where $\widetilde{C}$ is a positive constant such that $\widetilde{C}\int_0^\infty t^{2m(M+2)}e^{-2t^{2m}}\,\frac{dt}{t}=1$, $M\in\mathbb{N}$ is sufficiently large,

$$\pi_{L,\,M}:=\int_0^\infty (t^{2m}L)^{M+1}e^{-t^{2m}L}\,\frac{dt}{t}$$

and, for all $k\in\mathbb{N}$,

$$Q_{L,\,k,\,t}:=(t^{2m}L)^k e^{-t^{2m}L}.$$

Thus, for $p\in(p_-(L),\,2]$, if $f\in L^2(\mathbb{R}^n)\cap H^p_L(\mathbb{R}^n)$, then, for all $g\in L^2(\mathbb{R}^n)\cap L^{p'}(\mathbb{R}^n)$, by (2.3), duality between $T^p(\mathbb{R}^{n+1}_+)$ and $T^{p'}(\mathbb{R}^{n+1}_+)$ with $1/p+1/p'=1$, and Hölder's inequality, we see that

$$\left|\int_{\mathbb{R}^n}f(x)\overline{g(x)}\,dx\right|=\widetilde{C}\left|\int_{\mathbb{R}^n}\pi_{L,\,M}\circ Q_{L,\,1,\,t}(f)(x)\overline{g(x)}\,dx\right|$$

$$
\begin{aligned}
&= \widetilde{C}\left|\iint_{\mathbb{R}^{n+1}_+} Q_{L,1,t}(f)(x)\overline{Q_{L^*,M+1,t}(g)(x)}\,\frac{dx\,dt}{t}\right| \\
&\lesssim \|Q_{L,1,t}(f)\|_{T^p(\mathbb{R}^{n+1}_+)}\,\|Q_{L^*,M+1,t}(g)\|_{T^{p'}(\mathbb{R}^{n+1}_+)} \\
&\sim \|f\|_{H^p_L(\mathbb{R}^n)}\left\|\left\{\iint_{\Gamma(\cdot)}\left|\left(t^{2m}L^*\right)^{M+1}e^{-t^{2m}L^*}(g)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right\}^{1/2}\right\|_{L^{p'}(\mathbb{R}^n)},
\end{aligned}
$$

where $T^p(\mathbb{R}^{n+1}_+)$ denotes the tent space and L^* the adjoint operator of L in $L^2(\mathbb{R}^n)$. Since $p'\in[2,\,p_+(L^*))$, similar to the boundedness of S_L on $L^{p'}(\mathbb{R}^n)$ for all $p'\in(p_-(L),\,p_+(L))$, we have

$$\|Q_{L^*,M+1,t}(g)\|_{T^{p'}(\mathbb{R}^{n+1}_+)}\lesssim\|g\|_{L^{p'}(\mathbb{R}^n)},$$

which, combined with the arbitrariness of g, implies that $f\in L^p(\mathbb{R}^n)$ and

$$\|f\|_{L^p(\mathbb{R}^n)}\lesssim\|f\|_{H^p_L(\mathbb{R}^n)},$$

and hence $(L^2(\mathbb{R}^n)\cap H^p_L(\mathbb{R}^n))\subset(L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n))$. By density, this finishes the proof of Lemma 2.12 for $p\in(p_-(L),\,2]$. The case $p\in[2,\,p_+(L))$ follows from Definition 1.1 and a dual argument, the details being omitted. This finishes the proof of Lemma 2.12. □

Combining Lemma 2.12, Propositions 2.7 and 2.8, together with the fact that $\nabla^m L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ for all $p\in(\frac{n}{n+m},\,1]$ (see [9, Theorem 6.2]), we conclude the following proposition, the details being omitted.

Proposition 2.13. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$. *Then, for all $p\in(\frac{n}{n+m},\,q_+(L))$, $\nabla^m L^{-1/2}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.*

Now, we establish the generalized square function characterization of $H^p_L(\mathbb{R}^n)$, which is available in [24] for $m=1$ and $p\in(0,\,\infty)$ and in [9] for $m\in\mathbb{N}$ and $p\in(0,\,1]$. Let $p\in(0,\,\infty)$, $\omega\in[0,\,\pi/2)$ be the type of L, $\alpha\in(0,\,\infty)$, $\beta\in(\frac{n}{2m}(\max\{\frac{1}{p},\,1\}-\frac12),\,\infty)$ and $\psi\in\Psi_{\alpha,\beta}(S^0_\mu)$ with $\mu\in(\omega,\,\pi/2)$, where

$$S^0_\mu:=\{z\in\mathbb{C}\setminus\{0\}:\ |\arg z|<\mu\}$$

and

$$
\begin{aligned}
\Psi_{\alpha,\beta}(S^0_\mu):=\Big\{&f\text{ is analytic on }S^0_\mu:\ \text{there exists a positive constant }C\text{ such that}\\
&|f(\xi)|\le C\inf\{|\xi|^\alpha,\,|\xi|^{-\beta}\}\text{ for all }\xi\in S^0_\mu\Big\}.
\end{aligned}
$$

For all $f\in L^2(\mathbb{R}^n)$ and $(x,\,t)\in\mathbb{R}^{n+1}_+$, define the operator $Q_{\psi,L}(f)$ by

$$Q_{\psi,L}(f)(x,\,t):=\psi\left(t^{2m}L\right)(f)(x).$$

Definition 2.14. Let $p \in (0, \infty)$, L be as in (1.1) and satisfy the Ellipticity condition $(\mathcal{E}_0)$, $\alpha \in (0, \infty)$, $\beta \in (\frac{n}{2m}(\max\{\frac{1}{p}, 1\} - \frac{1}{2}), \infty)$, $\mu \in (\omega, \pi/2)$ and

$$\psi \in \begin{cases} \Psi_{\alpha,\beta}(S_\mu^0) & \text{when } p \in (0, 2]; \\ \Psi_{\beta,\alpha}(S_\mu^0) & \text{when } p \in (2, \infty). \end{cases}$$

The *generalized square function Hardy space* $H^p_{\psi,L}(\mathbb{R}^n)$ is defined as the completion of the space

$$\mathbb{H}^p_{\psi,L}(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : \ Q_{\psi,L}(f) \in T^p(\mathbb{R}^{n+1}_+) \right\}$$

with respect to the quasi-norm $\|f\|_{H^p_{\psi,L}(\mathbb{R}^n)} := \|Q_{\psi,L}(f)\|_{T^p(\mathbb{R}^{n+1}_+)}$.

The following result establishes the generalized square function characterization of $H^p_L(\mathbb{R}^n)$ for $p \in (0, \infty)$.

Proposition 2.15. *Let $p \in (0, \infty)$, L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*, $\alpha \in (0, \infty)$, $\beta \in (\frac{n}{2m}(\max\{\frac{1}{p}, 1\} - \frac{1}{2}), \infty)$, $\mu \in (\omega, \pi/2)$ and*

$$\psi \in \begin{cases} \Psi_{\alpha,\beta}(S_\mu^0) & \textit{when } p \in (0, 2]; \\ \Psi_{\beta,\alpha}(S_\mu^0) & \textit{when } p \in (2, \infty). \end{cases}$$

Then the Hardy space $H^p_L(\mathbb{R}^n) = H^p_{\psi,L}(\mathbb{R}^n)$ with equivalent quasi-norms.

Proof. If $p \in (0, 1]$, Proposition 2.15 is just [9, Theorem 5.2], where $\beta \in (\frac{n}{2m}(\frac{1}{p} - \frac{1}{2}), \infty)$ is needed to guarantee $H^p_L(\mathbb{R}^n) \subset H^p_{\psi,L}(\mathbb{R}^n)$, via an application of the Calderón reproducing formula.

If $p \in (1, \infty)$ and $m = 1$, Proposition 2.15 is just [24, Corollary 4.17], where $\beta \in (\frac{n}{4}, \infty)$ is used to guarantee $H^p_L(\mathbb{R}^n) \subset H^p_{\psi,L}(\mathbb{R}^n)$, via an application of the Calderón reproducing formula. If $p \in (1, \infty)$ and $m \in \mathbb{N} \cap [2, \infty)$, an argument similar to that used in the proof of [24, Corollary 4.17], together with an application of the Calderón reproducing formula, also gives us the desired conclusion of Proposition 2.15, where we need $\beta \in (\frac{n}{4m}, \infty)$ to guarantee $H^p_L(\mathbb{R}^n) \subset H^p_{\psi,L}(\mathbb{R}^n)$, the details being omitted, which completes the proof of Proposition 2.15. $\square$

3 Proofs of Theorem 1.4, Proposition 1.5 and Theorem 1.7

In this section, we give the proofs of Theorem 1.4, Proposition 1.5 and Theorem 1.7. To this end, we first establish the following parabolic Caccioppoli's inequalities, resonating with [23, Lemma 2.8] and, in a different way, with [4, Proposition 40].

Proposition 3.1. *Let L be as in* (1.1) *and satisfy the Strong ellipticity condition* $(\mathcal{E}_1)$*, and let $f \in L^2(\mathbb{R}^n)$, $t \in (0, \infty)$ and $u(x, t) := e^{-t^{2m}L}(f)(x)$ for all $x \in \mathbb{R}^n$. For all*

$\epsilon \in (0, \infty)$, there exist positive constants $C_{(\epsilon)}$, depending on ϵ, and C such that, for all $x_0 \in \mathbb{R}^n$, $r \in (0, \infty)$ and $t_0 \in (3r, \infty)$,

$$\begin{aligned}(3.1)\quad &\int_{t_0-r}^{t_0+r}\int_{B(x_0,\, r)} |\nabla^m u(x,\, t)|^2\, dx\, dt\\ &\le \epsilon\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\, 2r)} |\nabla^m u(x,\, t)|^2\, dx\, dt + \frac{C_{(\epsilon)}}{r^{2m}}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\, 2r)} |u(x,\, t)|^2\, dx\, dt\end{aligned}$$

and

$$\begin{aligned}(3.2)\quad &\int_{t_0-r}^{t_0+r}\int_{B(x_0,\, r)} |\nabla^m u(x,\, t)|^2\, dx\, dt\\ &\le \frac{C}{r^2}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\, 2r)} \left|\nabla^{m-1} u(x,\, t)\right|^2\, dx\, dt + \frac{C}{r^{2m}}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\, 2r)} |u(x,\, t)|^2\, dx\, dt,\end{aligned}$$

where $C_{(\epsilon)}$ and C are independent of f.

Proof. We first prove (3.2). To this end, we introduce two smooth cut-off functions. Let $\eta \in C_c^\infty(B(x_0,\, 2r))$ satisfy $0 \le \eta(x) \le 1$, $\eta \equiv 1$ on $B(x_0,\, r)$ and, for all $k \in \{0,\, \ldots,\, m\}$,

$$\left\|\nabla^k \eta\right\|_{L^\infty(\mathbb{R}^n)} \lesssim r^{-k}.$$

Let $\gamma \in C_c^\infty(t_0 - 2r,\, t_0 + 2r)$ satisfy $0 \le \gamma(t) \le 1$, $\gamma \equiv 1$ on $(t_0 - r,\, t_0 + r)$ and

$$\|\partial_t \gamma\|_{L^\infty(\mathbb{R})} \lesssim \frac{1}{r}.$$

We first write

$$\begin{aligned}(3.3)\quad &\int_{t_0-r}^{t_0+r}\int_{B(x_0,\, r)} |\nabla^m u(x,\, t)|^2\, dx\, dt\\ &\le \int_{t_0-2r}^{t_0+2r}\left[\int_{\mathbb{R}^n} |\nabla^m u(x,\, t)|^2\, [\eta(x)]^{2m}\, dx\right]\gamma(t)\, dt =: \mathcal{A}.\end{aligned}$$

To bound $\mathcal{A}$, by the Strong ellipticity condition $(\mathcal{E}_1)$, we know that

$$\begin{aligned}(3.4)\quad \mathcal{A} &\le \frac{1}{\lambda_1}\Re e\left(\int_{t_0-2r}^{t_0+2r}\left\{\int_{\mathbb{R}^n} A(x)\nabla^m u(x,\, t)\overline{\nabla^m u(x,\, t)}\, [\eta(x)]^{2m}\, dx\right\}\gamma(t)\, dt\right)\\ &= \frac{1}{\lambda_1}\Re e\left\{\int_{t_0-2r}^{t_0+2r}\left[\int_{\mathbb{R}^n} A(x)\nabla^m u(x,\, t)\right.\right.\\ &\quad \left.\left.\times\left\{\overline{\nabla^m u(x,\, t)\, [\eta(x)]^{2m} - \nabla^m(u\eta^{2m})(x,\, t)}\right\}\, dx\right]\gamma(t)\, dt\right\}\\ &\quad + \frac{1}{\lambda_1}\Re e\left\{\int_{t_0-2r}^{t_0+2r}\left[\int_{\mathbb{R}^n} A(x)\nabla^m u(x,\, t)\overline{\nabla^m(u\eta^{2m})(x,\, t)}\, dx\right]\gamma(t)\, dt\right\}\\ &=: \mathrm{I} + \mathrm{J},\end{aligned}$$

where

$$A(x) := \{a_{\alpha,\,\beta}(x)\}_{|\alpha|=m=|\beta|} \quad \text{for all } x \in \mathbb{R}^n \tag{3.5}$$

is a (properly arranged) coefficient matrix of L so that, for all $f,\, g \in \dot{W}^{m,2}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$A(x)\nabla^m f(x)\overline{\nabla^m g(x)} := \sum_{|\alpha|=m=|\beta|} a_{\alpha,\,\beta}(x)\partial^\beta f(x)\overline{\partial^\alpha g(x)}.$$

We first bound J. For all $(x,\, \widetilde{t}) \in \mathbb{R}^{n+1}_+$, let $F(x,\, \widetilde{t}) := e^{-\widetilde{t}L}(f)(x)\overline{e^{-\widetilde{t}L}(f)(x)}[\eta(x)]^{2m}$. Using $\partial_{\widetilde{t}} e^{-\widetilde{t}L} = -Le^{-\widetilde{t}L}$ and

$$\overline{Le^{-\widetilde{t}L}(f)(x)\overline{e^{-\widetilde{t}L}(f)(x)}[\eta(x)]^{2m}} = e^{-\widetilde{t}L}(f)(x)\overline{Le^{-\widetilde{t}L}(f)(x)}[\eta(x)]^{2m},$$

we know that

$$\begin{aligned}\partial_{\widetilde{t}} F(x,\, \widetilde{t}) &= -Le^{-\widetilde{t}L}(f)(x)\overline{e^{-\widetilde{t}L}(f)(x)}[\eta(x)]^{2m} - e^{-\widetilde{t}L}(f)(x)\overline{Le^{-\widetilde{t}L}(f)(x)}[\eta(x)]^{2m}\\ &= -2\Re e\left\{Le^{-\widetilde{t}L}(f)(x)\overline{e^{-\widetilde{t}L}(f)(x)}[\eta(x)]^{2m}\right\},\end{aligned}$$

which, together with integration by parts, and the definition of the cut-off function γ, shows

$$\begin{aligned}\mathrm{J} &= \frac{1}{\lambda_1}\Re e\left(\int_{t_0-2r}^{t_0+2r}\left\{\int_{\mathbb{R}^n} Le^{-t^{2m}L}(f)(x)\overline{e^{-t^{2m}L}(f)(x)\,[\eta(x)]^{2m}}\,dx\right\}\gamma(t)\,dt\right)\\
&= \frac{1}{2m\lambda_1}\Re e\left\{\int_{(t_0-2r)^{2m}}^{(t_0+2r)^{2m}}\left[\int_{\mathbb{R}^n} Le^{-\widetilde{t}L}(f)(x)\overline{e^{-\widetilde{t}L}(f)(x)\,[\eta(x)]^{2m}}\,dx\right]\right.\\
&\qquad \left.\times\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\widetilde{t}^{\frac{1}{2m}-1}\,d\widetilde{t}\right\}\\
&= -\frac{1}{4m\lambda_1}\int_{(t_0-2r)^{2m}}^{(t_0+2r)^{2m}}\partial_{\widetilde{t}}\left(\int_{B(x_0,\,2r)} F(x,\,\widetilde{t})\,dx\right)\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\widetilde{t}^{\frac{1}{2m}-1}\,d\widetilde{t}\\
&= \frac{1}{4m\lambda_1}\int_{(t_0-2r)^{2m}}^{(t_0+2r)^{2m}}\left[\int_{B(x_0,\,2r)} F(x,\,\widetilde{t})\,dx\right]\partial_{\widetilde{t}}\left(\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\widetilde{t}^{\frac{1}{2m}-1}\right)\,d\widetilde{t}\\
&\quad -\frac{1}{4m\lambda_1}\int_{(t_0-2r)^{2m}}^{(t_0+2r)^{2m}}\partial_{\widetilde{t}}\left(\left[\int_{B(x_0,\,2r)} F(x,\,\widetilde{t})\,dx\right]\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\widetilde{t}^{\frac{1}{2m}-1}\right)\,d\widetilde{t}\\
&= \frac{1}{4m\lambda_1}\int_{(t_0-2r)^{2m}}^{(t_0+2r)^{2m}}\left[\int_{B(x_0,\,2r)} F(x,\,\widetilde{t})\,dx\right]\partial_{\widetilde{t}}\left(\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\widetilde{t}^{\frac{1}{2m}-1}\right)\,d\widetilde{t}.\end{aligned} \tag{3.6}$$

This, together with the change of variables, the size condition of γ and $t_0 \in (3r,\, \infty)$, implies that

$$|\mathrm{J}| \lesssim \int_{(t_0-2r)^{2m}}^{(t_0+2r)^{2m}}\left\{\int_{B(x_0,\,2r)} e^{-\widetilde{t}L}(f)(x)\overline{e^{-\widetilde{t}L}(f)(x)}\,[\eta(x)]^{2m}\,dx\right\} \tag{3.7}$$

$$\times \left[\left|\partial_{\widetilde{t}}\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\right|\widetilde{t}^{\frac{1}{2m}-1}+\gamma\left(\widetilde{t}^{\frac{1}{2m}}\right)\widetilde{t}^{\frac{1}{2m}-2}\right]\,d\widetilde{t}$$
$$\lesssim \frac{1}{r^{2m}}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)}|u(x,\,t)|^2\,dx\,dt.$$

We now turn to the estimate of I. Notice that, for all multi-indices α,

$$\partial^\alpha(u\eta^{2m})(x,\,t)-\partial^\alpha u(x,\,t)\,[\eta(x)]^{2m}=\sum_{\theta\le\beta<\alpha}C_{(\alpha,\,\beta)}\partial^\beta u(x,\,t)\partial^{\alpha-\beta}(\eta^{2m})(x),$$

where $\theta := (0,\,\ldots,\,0)\in\mathbb{N}^n$ and, for each α and β, $C_{(\alpha,\,\beta)}$ is a positive constant depending on α and β, and $\beta<\alpha$ means that each component of β is not larger than the corresponding component of α and $|\beta|<|\alpha|$.

Moreover, from $0<|\alpha-\beta|\le m$ and Leibnitz's rule, we deduce that, for each α and β, there exists a smooth function $\eta_{\alpha,\,\beta}$ such that $\partial^{\alpha-\beta}(\eta^{2m})=\eta^m\eta_{\alpha,\,\beta}$ and

$$\|\eta_{\alpha,\,\beta}\|_{L^\infty(\mathbb{R}^n)}\lesssim\frac{1}{r^{|\alpha-\beta|}}. \tag{3.8}$$

Indeed, if $\alpha-\beta\equiv(2,\,0,\,\ldots,\,0)$, then we have

$$\begin{aligned}\partial^{\alpha-\beta}(\eta^{2m}) &= \partial^2_{x_1}(\eta^{2m})=\partial_{x_1}(2m\eta^{2m-1}\partial_{x_1}\eta)\\ &= 2m(2m-1)\eta^{2m-2}\left(\partial_{x_1}\eta\right)^2+2m\eta^{2m-1}\partial^2_{x_1}\eta\\ &= \eta^m\left(2m(2m-1)\eta^{m-2}\left(\partial_{x_1}\eta\right)^2+2m\eta^{m-1}\partial^2_{x_1}\eta\right)=:\eta^m\eta_{\alpha,\,\beta},\end{aligned}$$

where the fact that $\eta_{\alpha,\,\beta}$ satisfies (3.8) is an easy consequence of properties of η. The general cases follow from a similar calculation, the details being omitted.

Combined (3.8) with Cauchy's inequality with ϵ and the size condition of η, we see that, for every $\epsilon\in(0,\,\infty)$, there exists a positive constant $C_{(\epsilon)}$ such that

$$\begin{aligned}|\mathrm{I}| &\lesssim \epsilon\int_{t_0-2r}^{t_0+2r}\left\{\int_{\mathbb{R}^n}|\nabla^m u(x,\,t)|^2\,[\eta(x)]^{2m}\,dx\right\}\gamma(t)\,dt\\ &\quad+C_{(\epsilon)}\sum_{|\alpha|=m}\sum_{\theta\le\beta<\alpha}\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}\left|\partial^\beta u(x,\,t)\eta_{\alpha,\,\beta}(x)\right|^2\,dx\right]\gamma(t)\,dt\\ &\lesssim \epsilon\mathcal{A}+\sum_{k=0}^{m-1}\frac{C_{(\epsilon)}}{r^{2(m-k)}}\sum_{|\alpha|=m}\sum_{|\beta|=k,\,\beta<\alpha}\int_{t_0-2r}^{t_0+2r}\left[\int_{B(x_0,\,2r)}\left|\partial^\beta u(x,\,t)\right|^2\,dx\right]\gamma(t)\,dt.\end{aligned}$$

Before going further, we first observe that, from [1, Theorem 5.2(3)] with a small modification, we easily deduce that there exists a positive constant $C_{(n,\,m)}$, depending only on n and m, such that, for all balls B, $f\in W^{m,\,p}(B)$ and $k\in\{0,\,\ldots,\,m\}$,

$$\left\|\nabla^k f\right\|_{L^2(B)}\le C_{(n,\,m)}\left\|\nabla^m f\right\|_{L^2(B)}^{k/m}\left\|f\right\|_{L^2(B)}^{1-k/m}, \tag{3.9}$$

which, together with Young's inequality, implies that

$$
\begin{aligned}
|\mathrm{I}| &\lesssim \epsilon\mathcal{A} + C_{(\epsilon)} \sum_{k=0}^{m-1} \int_{t_0-2r}^{t_0+2r} \left[\frac{1}{r^2}\int_{B(x_0,\,2r)} \left|\nabla^{m-1}u(x,\,t)\right|^2\,dx\right]^{\frac{k}{m-1}} \\
&\quad \times \left[\frac{1}{r^{2m}}\int_{B(x_0,\,2r)} |u(x,\,t)|^2\,dx\right]^{\frac{m-k-1}{m-1}} \gamma(t)\,dt \\
&\lesssim \epsilon\mathcal{A} + \frac{C_{(\epsilon)}}{r^2}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)} \left|\nabla^{m-1}u(x,\,t)\right|^2\,dx\,dt \\
&\quad + \frac{C_{(\epsilon)}}{r^{2m}}\int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)} |u(x,\,t)|^2\,dx\,dt,
\end{aligned}
$$

which, combined with (3.3) through (3.7), shows that Caccioppoli's inequality (3.2) holds true.

Observe also that (3.1) is an immediate consequence of (3.2), the interpolation inequality (3.9) and Young's inequality with ϵ. This finishes the proof of Proposition 3.1. □

Remark 3.2. We point out that, if L satisfies the Ellipticity condition $(\mathcal{E}_0)$, then

$$u(x,\,t) := e^{-t^{2m}L}(f)(x),\ \ (x,\,t) \in \mathbb{R}^n \times (0,\,\infty)$$

still satisfies the parabolic Caccioppoli's inequality (3.1). Indeed, letting η and γ be the cut-off functions defined as in the proof of Proposition 3.1, by the Ellipticity condition $(\mathcal{E}_0)$, we conclude that

$$
\begin{aligned}
&\int_{t_0-r}^{t_0+r}\int_{B(x_0,\,r)} |\nabla^m u(x,\,t)|^2\,dx\,dt \\
&\leq \int_{t_0-2r}^{t_0+2r}\int_{B(x_0,\,2r)} |\nabla^m(u\eta^m)(x,\,t)|^2\,dx\,\gamma(t)\,dt \\
&\lesssim \Re e\left\{\int_{t_0-2r}^{t_0+2r}\left[\int_{\mathbb{R}^n} A(x)\nabla^m\left(u\eta^m\right)(x,\,t)\overline{\nabla^m\left(u\eta^m\right)(x,\,t)}\,dx\right]\gamma(t)\,dt\right\} \\
&\lesssim \left|\Re e\left(\int_{t_0-2r}^{t_0+2r}\left\{\int_{\mathbb{R}^n}\left[A(x)\nabla^m\left(u\eta^m\right)(x,\,t)\overline{\nabla^m\left(u\eta^m\right)(x,\,t)}\right.\right.\right.\right. \\
&\quad \left.\left.\left.\left.-A(x)\nabla^m u(x,\,t)\overline{\nabla^m(u\eta^{2m})(x,\,t)}\right]\,dx\right\}\gamma(t)\,dt\right)\right| \\
&\quad + \left|\Re e\left\{\int_{t_0-2r}^{t_0+2r}\left[\int_{\mathbb{R}^n} A(x)\nabla^m u(x,\,t)\overline{\nabla^m(u\eta^{2m})(x,\,t)}\,dx\right]\gamma(t)\,dt\right\}\right| \\
&=: \mathrm{I} + \mathrm{J},
\end{aligned}
$$

where A is as in (3.5). The estimates of I and J are similar to those of I and J in the proof of Proposition 3.1, respectively, and lead to (3.1). We omit the details here (see also [4, Proposition 40] for a similar Caccioppoli's inequality in the elliptic case). Observe however that we cannot obtain the parabolic Caccioppoli's inequality (3.2) directly via this argument.

Now, let $(p_-(L),\, p_+(L))$ be the maximal interval of exponents $p \in [1,\, \infty]$ such that $\{e^{-tL}\}_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$, and let $(q_-(L),\, q_+(L))$ be the maximal interval of exponents $q \in [1,\, \infty]$ such that the family $\{\sqrt{t}\nabla^m e^{-tL}\}_{t>0}$ of operators is bounded on $L^q(\mathbb{R}^n)$. For $p \in (0,\, q_+(L))$, we want to control the $H^p_L(\mathbb{R}^n)$ quasi-norm, by the $L^p(\mathbb{R}^n)$ quasi-norm of $S_{h,\, L}$ in (1.16), via the parabolic Caccioppoli's inequality (3.1), which, together with Remark 3.2, implies that we only need the Ellipticity condition $(\mathcal{E}_0)$ in this case. To this end, we point out that, in the remainder of this section, including the proofs of Propositions 3.3 and 3.7, and Theorem 1.4, we borrow some ideas from the corresponding parts of [23], in which the authors considered the case when $m = 1$ and $p = 1$.

We first need the following notation. For all $\lambda \in (0,\, \infty)$, $k \in \mathbb{Z}_+$ and $f \in L^2(\mathbb{R}^n)$, let $S^\lambda_{L,\, k}(f)$ and $S^\lambda_{h,\, L,\, k}(f)$ be the same, respectively, as in (1.15) and (1.16). For any $0 < \epsilon \ll R < \infty$ and $x \in \mathbb{R}^n$, let $\Gamma^{\epsilon,\, R,\, \lambda}(x)$ be the *truncated cone* defined by setting,

$$\Gamma^{\epsilon,\, R,\, \lambda}(x) := \{(y,\, t) \in \mathbb{R}^n \times (\epsilon,\, R) :\ |x - y| < \lambda t\}\,. \tag{3.10}$$

We replace $S^\lambda_{L,\, k}(f)(x)$ and $S^\lambda_{L,\, h,\, k}(f)(x)$, respectively, by $S^{\epsilon,\, R,\, \lambda}_{L,\, k}(f)(x)$ and $S^{\epsilon,\, R,\, \lambda}_{L,\, h,\, k}(f)(x)$ when the cone Γ^λ, in (1.15) and (1.16), is replaced by $\Gamma^{\epsilon,\, R,\, \lambda}(x)$.

From [14, Proposition 4], it follows that, for all $k \in \mathbb{Z}_+$, $\lambda \in (0,\, \infty)$, $p \in (0,\, \infty)$ and $f \in L^2(\mathbb{R}^n)$,

$$\left\|S^\lambda_{h,\, L,\, k}(f)\right\|_{L^p(\mathbb{R}^n)} \sim \left\|S_{h,\, L,\, k}(f)\right\|_{L^p(\mathbb{R}^n)} \tag{3.11}$$

and

$$\left\|S^\lambda_{L,\, k}(f)\right\|_{L^p(\mathbb{R}^n)} \sim \left\|S_{L,\, k}(f)\right\|_{L^p(\mathbb{R}^n)}\,, \tag{3.12}$$

where the implicit positive constants are independent of f.

For $p \in (0,\, \infty)$, we can control the $L^p(\mathbb{R}^n)$ quasi-norm of S_L in (1.15) by the that of $S_{h,\, L}$ in (1.16) as follows.

Proposition 3.3. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$*, and let $p \in (0,\, \infty)$. Then there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\left\|S_L(f)\right\|_{L^p(\mathbb{R}^n)} \le C \left\|S_{h,\, L}(f)\right\|_{L^p(\mathbb{R}^n)}\,.$$

Proof. We begin the proof of Proposition 3.3 by first introducing some smooth cut-off functions supported in truncated cones. For all $0 < \epsilon \ll R < \infty$, $\lambda \in (0,\, \infty)$ and $x \in \mathbb{R}^n$, let $\Gamma^{\epsilon,\, R,\, \lambda}(x)$ be the truncated cone defined as in (3.10).

Let $\eta \in C^\infty_c(\Gamma^{\epsilon/2,\, 2R,\, 3/2}(x))$ satisfy $\eta \equiv 1$ on $\Gamma^{\epsilon,\, R,\, 1}(x)$, $0 \le \eta \le 1$ and, for all $k \in \mathbb{N}$ with $k \le m$ and $(y,\, t) \in \Gamma^{\epsilon/2,\, 2R,\, 3/2}(x)$,

$$\left|\nabla^k \eta(y,\, t)\right| \lesssim \frac{1}{t^k}.$$

From the definition of L and Minkowski's inequality, we deduce that

$$\left\{\iint_{\Gamma^{\epsilon,\, R,\, 1}(x)} \left|t^{2m} L e^{-t^{2m} L}(f)(y)\right|^2 \frac{dy\, dt}{t^{n+1}}\right\}^{\frac{1}{2}} \tag{3.13}$$

$$\leq \left| \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} t^{2m} L e^{-t^{2m}L}(f)(y) \overline{t^{2m} L e^{-t^{2m}L}(f)(y)} \eta(y,\,t)\,\frac{dy\,dt}{t^{n+1}} \right|^{\frac12}$$
$$= \left| \sum_{|\alpha|=m=|\beta|} \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} a_{\alpha,\,\beta}(y) t^m \partial^\beta \left(e^{-t^{2m}L}(f)\right)(y) \right.$$
$$\left. \times \overline{t^m \partial^\alpha \left(t^{2m} L e^{-t^{2m}L}(f)\eta\right)(y,\,t)}\,\frac{dy\,dt}{t^{n+1}} \right|^{\frac12}$$
$$\lesssim \sum_{k=0}^m \left| \sum_{|\alpha|=m=|\beta|} \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} a_{\alpha,\,\beta}(y) t^m \partial^\beta \left(e^{-t^{2m}L}(f)\right)(y) \right.$$
$$\left. \times \left[t^m \sum_{|\widetilde{\alpha}|=k,\,\widetilde{\alpha}\leq\alpha} C_{(\alpha,\,\widetilde{\alpha})} \overline{\partial^{\widetilde{\alpha}} \left(t^{2m} L e^{-t^{2m}L}(f)\right)(y)\, \partial^{\alpha-\widetilde{\alpha}} \eta(y,\,t)} \right] \frac{dy\,dt}{t^{n+1}} \right|^{\frac12}$$
$$=: \sum_{k=0}^m \mathrm{I}_k.$$

We first bound I_0. By Hölder's inequality, the size condition of η and the Ellipticity condition $(\mathcal{E}_0)$, we see that

$$\mathrm{I}_0 \lesssim \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left| t^m \nabla^m \left(e^{-t^{2m}L}(f)\right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac14} \tag{3.14}$$
$$\times \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left| t^{2m} L e^{-t^{2m}L}(f)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac14}$$
$$\lesssim \left[S_{h,\,L}^{\epsilon/2,\,2R,\,3/2}(f)(x) \right]^{\frac12} \left[S_L^{\epsilon/2,\,2R,\,3/2}(f)(x) \right]^{\frac12},$$

where $S_{h,\,L}^{\epsilon/2,\,2R,\,3/2}(f)$ and $S_L^{\epsilon/2,\,2R,\,3/2}(f)$ are defined, respectively, similar to $S_{h,\,L}(f)$ in (1.16) and $S_L(f)$ in (1.15), with $\Gamma(x)$ replaced by $\Gamma^{\epsilon/2,\,2R,\,3/2}(x)$.

The estimate of I_m is similar to that of I_0. Similar to (3.14), we have

$$\mathrm{I}_m \lesssim \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left| t^m \nabla^m \left(e^{-t^{2m}L}(f)\right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac14} \tag{3.15}$$
$$\times \left\{ \iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left| t^m \nabla^m \left(t^{2m} L e^{-t^{2m}L}(f)\right)(y) \right|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{\frac14}$$
$$\sim \left[S_{h,\,L}^{\epsilon/2,\,2R,\,3/2}(f)(x) \right]^{\frac12} \times \left[S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x) \right]^{\frac12}.$$

To bound $S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)$, let $Q(z,\,2r)$ be the cube with center z and sidelength $2r$ in $\mathbb{R}_+^{n+1}$. Write $z := (z^*,\,t)$ with $z^* \in \mathbb{R}^n$ and $t \in (0,\,\infty)$. Assume that $\{Q(z_j,\,2r_j)\}_{j\in\mathbb{N}}$ is a covering of $\Gamma^{\epsilon/2,\,2R,\,3/2}(x)$ satisfying

$$\Gamma^{\epsilon/2,\,2R,\,3/2}(x)\subset\bigcup_{j\in\mathbb{N}}Q(z_j,\,2r_j)\subset\bigcup_{j\in\mathbb{N}}Q(z_j,\,4\sqrt{n}r_j)\subset\Gamma^{\epsilon/4,\,3R,\,2}(x),$$

$$d\left(z_j,\,\left(\Gamma^{\epsilon/4,\,3R,\,2}(x)\right)^{\complement}\right)\sim r_j\sim d\left(z_j,\,\{t=0\}\right),\qquad j\in\mathbb{N}$$

and the collection $\{B(z_j^*,\,\sqrt{n}r_j)\times(t_j-\sqrt{n}r_j,\,t_j+\sqrt{n}r_j)\}_{j\in\mathbb{N}}$ has a bounded overlap, where, for all $j\in\mathbb{N}$, $z_j:=(z_j^*,\,t_j)$. This kind of covering is based on Whitney's decomposition; see [23, (5.26)] for a covering of similar nature consisting of balls.

It is easy to see that

$$\begin{aligned}\Gamma^{\epsilon/2,\,2R,\,3/2}(x)&\subset\bigcup_{j\in\mathbb{N}}Q\left(z_j,\,2r_j\right)\subset\bigcup_{j\in\mathbb{N}}B\left(z_j^*,\,\sqrt{n}r_j\right)\times\left(t_j-\sqrt{n}r_j,\,t_j+\sqrt{n}r_j\right)\\&\subset\bigcup_{j\in\mathbb{N}}B\left(z_j^*,\,2\sqrt{n}r_j\right)\times\left(t_j-2\sqrt{n}r_j,\,t_j+2\sqrt{n}r_j\right)\subset\bigcup_{j\in\mathbb{N}}Q\left(z_j,\,4\sqrt{n}r_j\right)\\&\subset\Gamma^{\epsilon/4,\,3R,\,2}(x).\end{aligned}$$

From these and the parabolic Caccioppoli's inequality (3.1), we deduce that, for all $\widetilde{\epsilon}\in(0,\,\infty)$, there exists a positive constant $C_{(\widetilde{\epsilon})}$ such that

$$\begin{aligned}&\left[S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^2\\&\quad\lesssim\sum_{j\in\mathbb{N}}\int_{t_j-\sqrt{n}r_j}^{t_j+\sqrt{n}r_j}\int_{B(z_j^*,\,\sqrt{n}r_j)}\left|t^m\nabla^m\left(t^{2m}Le^{-t^{2m}L}(f)\right)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\\&\quad\le\sum_{j\in\mathbb{N}}\left[(\widetilde{\epsilon})^4\int_{t_j-2\sqrt{n}r_j}^{t_j+2\sqrt{n}r_j}\int_{B(z_j^*,\,2\sqrt{n}r_j)}\left|t^m\nabla^m\left(t^{2m}Le^{-t^{2m}L}(f)\right)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right.\\&\qquad\left.+\left(C_{(\widetilde{\epsilon})}\right)^4\frac{1}{r_j^{2m}}\int_{t_j-2\sqrt{n}r_j}^{t_j+2\sqrt{n}r_j}\int_{B(z_j^*,\,2\sqrt{n}r_j)}\left|t^{3m}Le^{-t^{2m}L}(f)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right]\\&\quad\le(\widetilde{\epsilon})^4\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)}\left|t^m\nabla^m\left(t^{2m}Le^{-t^{2m}L}(f)\right)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\\&\qquad+\left(C_{(\widetilde{\epsilon})}\right)^4\iint_{\Gamma^{\epsilon/4,\,3R,\,2}(x)}\left|t^{2m}Le^{-t^{2m}L}(f)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\\&\quad=(\widetilde{\epsilon})^4\left[S_{h,\,L,\,1}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2+\left(C_{(\widetilde{\epsilon})}\right)^4\left[S_L^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2,\end{aligned}\tag{3.16}$$

which together with letting $\epsilon\to0$ and $R\to\infty$, (3.11) and (3.12), implies that, for all $q\in(0,\,\infty)$ and $f\in L^2(\mathbb{R}^n)$,

$$\left\|S_{h,\,L,\,1}(f)\right\|_{L^q(\mathbb{R}^n)}\lesssim\left\|S_L(f)\right\|_{L^q(\mathbb{R}^n)},\tag{3.17}$$

Moreover, by (3.15) and (3.16), we conclude that

$$\text{(3.18)} \qquad \mathrm{I}_m \le \left[S_{h,L}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12} \times \left\{\widetilde{\epsilon}\left[S_{h,L,1}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^{\frac12} + C_{(\widetilde{\epsilon})}\left[S_{L}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^{\frac12}\right\}.$$

We now turn to the estimates of I_k for all $k \in \{1, \ldots, m-1\}$. Again, by Hölder's inequality, the Ellipticity condition $(\mathcal{E}_0)$ and the size condition of η, we see that

$$\begin{aligned}\mathrm{I}_k &\lesssim \left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left|t^m\nabla^m\left(e^{-t^{2m}L}(f)\right)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\ &\quad\times\left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left|t^k\nabla^k\left(t^{2m}Le^{-t^{2m}L}(f)\right)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\ &=: \left[S_{h,L}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12} \times (\mathrm{II}_k)^{\frac14}.\end{aligned}$$

To bound II_k, using again the interpolation inequality (3.9) and Hölder's inequality, we conclude that

$$\begin{aligned}\mathrm{II}_k &\sim \int_{\epsilon/2}^{2R} \left\|t^k\nabla^k\left(t^{2m}Le^{-t^{2m}L}(f)\right)\right\|^2_{L^2(B(x,\,(3/2)t))}\frac{dt}{t^{n+1}}\\ &\lesssim \int_{\epsilon/2}^{2R} \left\|t^m\nabla^m\left(t^{2m}Le^{-t^{2m}L}(f)\right)\right\|^{2k/m}_{L^2(B(x,\,(3/2)t))}\\ &\quad\times\left\|t^{2m}Le^{-t^{2m}L}(f)\right\|^{2(m-k)/m}_{L^2(B(x,\,(3/2)t))}\frac{dt}{t^{n+1}}\\ &\lesssim \left\{\int_{\epsilon/2}^{2R} \left\|t^m\nabla^m\left(t^{2m}Le^{-t^{2m}L}(f)\right)\right\|^2_{L^2(B(x,\,(3/2)t))}\frac{dt}{t^{n+1}}\right\}^{k/m}\\ &\quad\times\left\{\int_{\epsilon/2}^{2R} \left\|t^{2m}Le^{-t^{2m}L}(f)\right\|^2_{L^2(B(x,\,(3/2)t))}\frac{dt}{t^{n+1}}\right\}^{(m-k)/m}\\ &\sim \left[S_{h,L,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{2k/m} \times \left[S_{L}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{2(m-k)/m}.\end{aligned}$$

From (3.16), we deduce that

$$\mathrm{II}_k \lesssim \left\{\widetilde{\epsilon}\left[S_{h,L,1}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2 + C_{(\widetilde{\epsilon})}\left[S_{L}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2\right\}^{k/m}\left[S_{L}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{2(m-k)/m}.$$

Thus, it holds true that

$$\text{(3.19)} \quad \mathrm{I}_k \lesssim \left[S_{h,L}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12}\left\{\widetilde{\epsilon}\left[S_{h,L,1}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2 + C_{(\widetilde{\epsilon})}\left[S_{L}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2\right\}^{k/(4m)}$$

$$\times \left[S_L^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{(m-k)/(2m)}.$$

Combining the estimates of (3.13) through (3.19) and letting $\epsilon \to 0$ and $R \to \infty$, we conclude that

$$\begin{aligned}(3.20)\qquad S_L(f)(x) &\lesssim \left[S_{h,\,L}^{3/2}(f)(x)\right]^{\frac12}\left[S_L^{3/2}(f)(x)\right]^{\frac12}\\ &\quad+\left[S_{h,\,L}^{3/2}(f)(x)\right]^{\frac12}\left\{\widetilde{\epsilon}\left[S_{h,\,L,\,1}^{2}(f)(x)\right]^{\frac12}+C_{(\widetilde{\epsilon})}\left[S_L^{2}(f)(x)\right]^{\frac12}\right\}\\ &\quad+\sum_{k=1}^{m-1}\left[S_{h,\,L}^{3/2}(f)(x)\right]^{\frac12}\left\{\widetilde{\epsilon}\left[S_{h,\,L,\,1}^{2}(f)(x)\right]^{2}+C_{(\widetilde{\epsilon})}\left[S_L^{2}(f)(x)\right]^{2}\right\}^{\frac{k}{4m}}\\ &\quad\times\left[S_L^{3/2}(f)(x)\right]^{\frac{m-k}{2m}}.\end{aligned}$$

This, combined with (3.11) and (3.12), immediately implies that, for all $p \in (0,\,\infty)$ and $\epsilon \in (0,\,\infty)$, there exists a positive constant $C_{(\epsilon,p)}$ such that, for all $f \in L^2(\mathbb{R}^n)$,

$$\|S_L(f)\|_{L^p(\mathbb{R}^n)} \le \epsilon\,\|S_{h,\,L,\,1}(f)\|_{L^p(\mathbb{R}^n)} + C_{(\epsilon,p)}\,\|S_{h,\,L}(f)\|_{L^p(\mathbb{R}^n)}\,,$$

which, together with (3.17) and via choosing ϵ sufficient small, shows that, for all $f \in L^2(\mathbb{R}^n)$,

$$\|S_L(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|S_{h,\,L}(f)\|_{L^p(\mathbb{R}^n)}.$$

This finishes the proof of Proposition 3.3. □

We first have the boundedness of $S_{L,\,k}$ and $S_{h,\,L,\,k}$ in $L^q(\mathbb{R}^n)$ as follows, which when $k = 1$ was pointed out in [2, p. 68] without any details.

Lemma 3.4. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$)*. Let $S_{L,\,k}$ and $S_{h,\,L,\,k}$ be the same, respectively, as in* (1.15) *and* (1.16)*. Then*

(i) *for all $k \in \mathbb{N}$ and $q \in (p_-(L),\,p_+(L))$, there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,*

$$\|S_{L,\,k}(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)};$$

(ii) *for all $k \in \mathbb{Z}_+$ and $q \in (q_-(L),\,q_+(L))$, there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,*

$$\|S_{h,\,L,\,k}(f)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)}.$$

Proof. Observe that, by Proposition 2.5(iii), we know that L satisfies all the assumptions of [7, Theorem 2.13] and, as a consequence, we obtain (i) of Lemma 3.4.

The proof of (ii) of this lemma is similar to that of (i). We only need to replace the m-L^p-L^q off-diagonal estimates from Proposition 2.5(iii), in the proof of [7, Theorem 2.13], by the corresponding m-L^p-L^q off-diagonal estimates of the gradient semigroups from Proposition 2.5(iv), the details being omitted. This finishes the proof of Lemma 3.4. □

The next proposition presents an equivalence between the $H^p_L(\mathbb{R}^n)$ norm, defined via the square function S_L, and the $L^p(\mathbb{R}^n)$ norm when $p \in (p_+(L), p_+(L))$. Recall that this conclusion was pointed out in [2, p. 68] without giving any details.

Proposition 3.5. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$)*, and let $p \in (p_-(L), p_+(L))$, with $p_-(L)$ and $p_+(L)$ as in Proposition 2.5, and S_L be as in* (1.15) *with $k = 1$ and $\lambda = 1$. Then there exists a positive constant C such that, for all $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$,*

$$\frac{1}{C}\|f\|_{L^p(\mathbb{R}^n)} \le \|S_L(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.21}$$

Proof. It is easy to see that the second inequality of (3.21) is a direct consequence of Lemma 3.4(i) in the case when $k = 1$.

We now prove the first inequality of (3.21). Let $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. For all $p \in (p_-(L), p_+(L))$ and $g \in L^2(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$ satisfying $\|g\|_{L^{p'}(\mathbb{R}^n)} \le 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$, by the Calderón reproducing formula (2.3) with $M = 0$, duality, Fubini's theorem, Hölder's inequality and Lemma 3.4(i), we know that

$$\begin{aligned}
\left|\langle f, g\rangle_{L^2(\mathbb{R}^n)}\right| &\sim \left|\left\langle \int_0^\infty \left(t^{2m}L\right)^2 e^{-2t^{2m}L}(f)\,\frac{dt}{t}, g\right\rangle_{L^2(\mathbb{R}^n)}\right| \\
&\sim \left|\int_0^\infty \left\langle t^{2m}Le^{-t^{2m}L}(f), t^{2m}L^*e^{-t^{2m}L^*}(g)\right\rangle_{L^2(\mathbb{R}^n)}\frac{dt}{t}\right| \\
&\sim \|S_L(f)\|_{L^p(\mathbb{R}^n)}\|S_{L^*}(g)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|S_L(f)\|_{L^p(\mathbb{R}^n)}
\end{aligned} \tag{3.22}$$

and hence

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|S_L(f)\|_{L^p(\mathbb{R}^n)}.$$

This finishes the proof of the first inequality of (3.21) and hence Proposition 3.5. □

With the help of Propositions 3.3 and 3.5, we obtain the following corollary.

Corollary 3.6. *Let L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$)*. Then, for all $p \in (0, p_+(L))$, there exists a positive constant $C_{(p)}$, depending on p, such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\|f\|_{H^p_L(\mathbb{R}^n)} \le C_{(p)}\|S_{h,L}(f)\|_{L^p(\mathbb{R}^n)}. \tag{3.23}$$

Proof. If $p \in (0, 2]$, Corollary 3.6 is an immediately consequence of Proposition 3.3 and Definition 1.1.

If $p \in (2, p_+(L))$, Corollary 3.6 is follows from Lemma 2.12 and Propositions 3.3 and 3.5. This finishes the proof of Corollary 3.6. □

The next proposition shows that the $L^p(\mathbb{R}^n)$ quasi-norm of $S_{h,L}$, as in (1.16) with $\lambda = 1$, can be controlled by that of the non-tangential maximal function $\mathcal{N}^\gamma_{h,L}$ as in (1.7).

Proposition 3.7. *Let L be as in* (1.1) *and satisfy the Strong ellipticity condition* $(\mathcal{E}_1)$*, and let $p\in(0,\infty)$. Then there exist positive constants γ and C such that, for all $f\in L^2(\mathbb{R}^n)$,*

$$\|S_{h,L}(f)\|_{L^p(\mathbb{R}^n)}\le C\left\|\mathcal{N}_{h,L}^{\gamma}(f)\right\|_{L^p(\mathbb{R}^n)}.$$

Proof. To prove Proposition 3.7, we first introduce some notation. Let $\sigma\in(0,\infty)$. Assume that $\gamma\in(0,\infty)$, whose exact value will be determined later. Let

$$E:=\left\{x\in\mathbb{R}^n:\ \mathcal{N}_{h,L}^{\gamma}(f)(x)\le\sigma\right\}.$$

Its *subset E^* of global* $1/2$ *density* is defined by

$$E^*:=\left\{x\in\mathbb{R}^n:\ \text{for all balls } B(x,r) \text{ in } \mathbb{R}^n,\quad \frac{|E\cap B(x,r)|}{|B(x,r)|}\ge\frac12\right\}.$$

For all $0<\epsilon\ll R<\infty$, let $\mathcal{R}^{\epsilon,R,\gamma}(E^*):=\cup_{x\in E^*}\Gamma^{\epsilon,R,\gamma}(x)$ be the *sawtooth region based on* E^* and $\mathcal{B}^{\epsilon,R,\gamma}(E^*)$ the *boundary* of $\mathcal{R}^{\epsilon,R,\gamma}(E^*)$. Moreover, for all $y\in\mathbb{R}^n$ and $t\in(0,\infty)$, let $u(y,t):=e^{-t^{2m}L}(f)(y)$. By Fubini's theorem, we find that

$$\int_{E^*}\left[S_{h,L}^{\epsilon,R,1/2}(f)(x)\right]^2dx\sim\iint_{\mathcal{R}^{\epsilon,R,1/2}(E^*)}t^{2m}\left|\nabla^m u(y,t)\right|^2\frac{dy\,dt}{t},$$

where $S_{h,L}^{\epsilon,R,1/2}(f)(x)$ is defined as in (3.15).

Now, let $\eta\in C_c^\infty(\mathcal{R}^{\epsilon/2,2R,3/2}(E^*))$ be a smooth cut-off function satisfying $0\le\eta\le1$, $\eta\equiv1$ on $\mathcal{R}^{\epsilon,R,1/2}(E^*)$ and, for all $k\in\mathbb{N}$ with $k\le m$ and $(x,t)\in\mathcal{R}^{\epsilon/2,2R,3/2}(E^*)$,

$$\left|\nabla_x^k\eta(x,t)\right|\lesssim\frac{1}{t^k}$$

and $|\partial_t\eta(x,t)|\lesssim\frac1t$. These assumptions, together with the Strong ellipticity condition $(\mathcal{E}_1)$, imply that

$$\begin{aligned}&\iint_{\mathcal{R}^{\epsilon,R,1/2}(E^*)}t^{2m}\left|\nabla^m u(y,t)\right|^2\frac{dy\,dt}{t}\\&\lesssim\Re e\left\{\iint_{\mathcal{R}^{\epsilon,R,1/2}(E^*)}\left[t^{2m}\sum_{|\alpha|=m=|\beta|}a_{\alpha,\beta}(y)\partial^\beta u(y,t)\overline{\partial^\alpha u(y,t)}\right]\frac{dy\,dt}{t}\right\}\\&\lesssim\Re e\left\{\iint_{\mathbb{R}^{n+1}_+}\left[t^{2m}\sum_{|\alpha|=m=|\beta|}a_{\alpha,\beta}(y)\partial^\beta u(y,t)\overline{\partial^\alpha u(y,t)}\eta(y,t)\right]\frac{dy\,dt}{t}\right\}.\end{aligned}$$

From this, we further deduce that

$$(3.24)\qquad \iint_{\mathcal{R}^{\epsilon,R,1/2}(E^*)}t^{2m}\left|\nabla^m u(y,t)\right|^2\frac{dy\,dt}{t}$$

$$
\begin{aligned}
&\lesssim \left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+}\left[t^{2m}\sum_{|\alpha|=m=|\beta|}(-1)^m\partial^\alpha\left(\eta a_{\alpha,\beta}\partial^\beta u\right)(y,\,t)\overline{u(y,\,t)}\right]\frac{dy\,dt}{t}\right\}\right|\\
&\lesssim \sum_{k=0}^m\left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+}\left[t^{2m}\sum_{|\alpha|=m=|\beta|}\sum_{|\widetilde{\alpha}|=k,\,\widetilde{\alpha}\le\alpha}C_{(\alpha,\,\widetilde{\alpha})}(-1)^m\partial^{\widetilde{\alpha}}\eta(y,\,t)\right.\right.\right.\\
&\quad\left.\left.\left.\times\partial^{\alpha-\widetilde{\alpha}}\left(a_{\alpha,\beta}\partial^\beta u\right)(y,\,t)\overline{u(y,\,t)}\right]\frac{dy\,dt}{t}\right\}\right|\\
&=:\sum_{k=0}^m \mathrm{J}_k,
\end{aligned}
$$

where, for all multi-indices α and $\widetilde{\alpha}$, $C_{(\alpha,\,\widetilde{\alpha})}$ are positive constants depending on α and $\widetilde{\alpha}$.

We first bound J_0. Since, for all $(y,\,t)\in\mathbb{R}^{n+1}_+$, $\frac{\partial}{\partial t}u(y,\,t)=-2mt^{2m-1}L(u)(y,\,t)$, we know that

$$
\begin{aligned}
\frac{\partial}{\partial t}|u(y,\,t)|^2&=-2mt^{2m-1}L(u)(y,\,t)\overline{u(y,\,t)}-2mt^{2m-1}u(y,\,t)\overline{L(u)(y,\,t)}\\
&=-4mt^{2m-1}\Re e\left\{L(u)(y,\,t)\overline{u(y,\,t)}\right\},
\end{aligned}
$$

which, together with integration by parts and properties of the cut-off function η, shows that

$$
\begin{aligned}
\mathrm{J}_0&\sim\left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+}t^{2m-1}\eta(y,\,t)L(u)(y,\,t)\overline{u(y,\,t)}\,dy\,dt\right\}\right|\\
&\sim\left|\iint_{\mathbb{R}^{n+1}_+}\eta(y,\,t)\frac{\partial}{\partial t}|u(y,\,t)|^2\,dy\,dt\right|\\
&\lesssim\iint_{\mathcal{R}^{\epsilon/2,\,2R,\,3/2}(E^*)\setminus\mathcal{R}^{\epsilon,\,R,\,1/2}(E^*)}|u(y,\,t)|^2\,\frac{dy\,dt}{t}.
\end{aligned}
$$

To estimate the last term in the above formulae, we let

$$
\widetilde{\mathcal{B}}^\epsilon(E^*):=\left\{(x,\,t)\in\mathbb{R}^n\times(\epsilon/2,\,\epsilon):\ d(x,\,E^*)<\frac{3}{2}t\right\},\tag{3.25}
$$

$$
\widetilde{\mathcal{B}}^R(E^*):=\left\{(x,\,t)\in\mathbb{R}^n\times(R,\,2R):\ d(x,\,E^*)<\frac{3}{2}t\right\},\tag{3.26}
$$

and

$$
\widetilde{\mathcal{B}}_0(E^*):=\left\{(x,\,t)\in\mathbb{R}^n\times(\epsilon/2,\,2R):\ \frac{1}{2}t\le d(x,\,E^*)<\frac{3}{2}t\right\}.\tag{3.27}
$$

It is easy to see that $(\mathcal{R}^{\epsilon/2,\,2R,\,3/2}(E^*)\setminus\mathcal{R}^{\epsilon,\,R,\,1/2}(E^*))\subset(\widetilde{\mathcal{B}}^\epsilon(E^*)\cup\widetilde{\mathcal{B}}^R(E^*)\cup\widetilde{\mathcal{B}}_0(E^*))$. For any $(y,\,t)\in\widetilde{\mathcal{B}}^\epsilon(E^*)$, we find that there exists $x\in E^*$ such that $|x-y|<\frac{3}{2}t$. Moreover, from the definition of E^*, it follows that, for all $t\in(0,\,\infty)$,

$$
|E\cap B(x,\,t)|\ge\frac{1}{2}|B(x,\,t)|=\frac{1}{2}\omega_n t^n,
$$

where $\omega_n := |B(x, 1)| = |B(0, 1)|$. Thus, $|E \cap B(y, 3t)| \geq \frac{1}{2}\omega_n t^n$, which, combined with Fubini's theorem, implies that

$$
\begin{aligned}
(3.28)\qquad &\iint_{\widetilde{B}^\epsilon(E^*)} |u(y, t)|^2 \,\frac{dy\,dt}{t}\\
&\lesssim \iint_{\widetilde{B}^\epsilon(E^*)} \left[\int_{E\cap B(y,3t)} |u(y, t)|^2 \,dz\right] \frac{dy\,dt}{t^{n+1}}\\
&\sim \int_{\epsilon/2}^{\epsilon}\int_E \left[\frac{1}{t^n}\int_{B(z,3t)} \left|e^{-t^{2m}L}(f)(y)\right|^2 dy\right]\frac{dz\,dt}{t}\\
&\lesssim \int_{\epsilon/2}^{\epsilon}\int_E \left|\sup_{(x,t)\in\Gamma_3(z)} \left\{\frac{1}{\omega_n(3t)^n}\int_{B(x,3t)} \left|e^{-t^{2m}L}(f)(y)\right|^2 dy\right\}^{\frac{1}{2}}\right|^2 \frac{dz\,dt}{t}\\
&\sim \int_{\epsilon/2}^{\epsilon}\int_E \left|\mathcal{N}_{h,L}^3(f)(z)\right|^2 \frac{dz\,dt}{t} \sim \int_E \left|\mathcal{N}_{h,L}^3(f)(z)\right|^2 dz.
\end{aligned}
$$

Similarly, we have

$$
\iint_{\widetilde{B}^R(E^*)} |u(y, t)|^2 \,\frac{dy\,dt}{t} \lesssim \int_E \left|\mathcal{N}_{h,L}^3(f)(z)\right|^2 dz. \tag{3.29}
$$

To estimate the integrand on the region $\widetilde{\mathcal{B}}_0(E^*)$, let $\{B(x_k, r_k)\}_k$ be Whitney's covering of B^*, where $B^* := (E^*)^{\complement}$. Then we see that

(i) $\cup_k B(x_k, r_k) = B^*$;

(ii) there exist positive constants C_1 and C_2 such that, for all k,

$$
C_1 d(x_k, E^*) \leq r_k \leq C_2 d(x_k, E^*);
$$

(iii) there exists a positive constant C_3 such that, for all $x \in B^*$, $\sum_k \chi_{B(x_k,r_k)}(x) \leq C_3$.

From these, we deduce that

$$
\begin{aligned}
\iint_{\widetilde{\mathcal{B}}_0(E^*)} |u(y, t)|^2 \,\frac{dy\,dt}{t} &\lesssim \sum_k \int_{r_k(\frac{1}{C_2}-1)}^{2r_k(\frac{1}{C_1}+1)} \int_{B(x_k,r_k)} |u(y, t)|^2 \,\frac{dy\,dt}{t}\\
&\lesssim \sum_k \int_{r_k(\frac{1}{C_2}-1)}^{2r_k(\frac{1}{C_1}+1)} r_k^n \left[\frac{1}{t^n}\int_{B(x_k,r_k)} |u(y, t)|^2 \,dy\right]\frac{dt}{t}.
\end{aligned}
$$

By the fact $E^* \subset E$, we know that $d(x_k, E) \leq d(x_k, E^*) \leq \frac{C_2}{(1-C_2)C_1}t$. Thus, by taking $\gamma \in (\frac{C_2}{(1-C_2)C_1}, \infty)$, we conclude that

$$
\iint_{\widetilde{\mathcal{B}}_0(E^*)} |u(y, t)|^2 \,\frac{dy\,dt}{t} \lesssim \sum_k r_k^n \left[\sup_{z\in E}\mathcal{N}_{h,L}^{\gamma}(f)(z)\right]^2 \lesssim |B^*| \left[\sup_{z\in E}\mathcal{N}_{h,L}^{\gamma}(f)(z)\right]^2. \tag{3.30}
$$

Combining the estimates of (3.28), (3.29) and (3.30), we see that

$$\mathrm{J}_0 \lesssim \int_E \left|\mathcal{N}^3_{h,\,L}(f)(z)\right|^2\,dz + |B^*|\left[\sup_{z\in E}\mathcal{N}^\gamma_{h,\,L}(f)(z)\right]^2. \tag{3.31}$$

Now, we turn to the estimates of J_k for all $k\in\{1,\ldots,m\}$. Using integration by parts and Hölder's inequality, we first write

$$\begin{aligned}\mathrm{J}_k &\sim \left|\Re e\left\{\iint_{\mathbb{R}^{n+1}_+} t^{2m}\sum_{|\alpha|=m=|\beta|}\ \sum_{|\widetilde{\alpha}|=k,\,\widetilde{\alpha}\le\alpha}\left[a_{\alpha,\,\beta}(y)\partial^\beta u(y,\,t)\right.\right.\right.\\
&\qquad\left.\left.\left.\times\overline{\partial^{\alpha-\widetilde{\alpha}}\left(\partial^{\widetilde{\alpha}}(\eta)u\right)(y,\,t)}\right]\frac{dy\,dt}{t}\right\}\right|\\
&\lesssim \sum_{|\alpha|=m=|\beta|}\ \sum_{|\widetilde{\alpha}|=k,\,\widetilde{\alpha}\le\alpha}\left\{\iint_{\mathcal{R}^{\epsilon/2,\,2R,\,3/2}(E^*)\setminus\mathcal{R}^{\epsilon,\,R,\,1/2}(E^*)}\left|t^m\partial^\beta u(y,\,t)\right|^2\frac{dy\,dt}{t}\right\}^{\frac12}\\
&\quad\times\left\{\int_{\mathcal{R}^{\epsilon/2,\,2R,\,3/2}(E^*)\setminus\mathcal{R}^{\epsilon,\,R,\,1/2}(E^*)}\left|t^{m-k}\partial^{\alpha-\widetilde{\alpha}}\left(t^k\partial^{\widetilde{\alpha}}(\eta)u\right)(y,\,t)\right|^2\frac{dy\,dt}{t}\right\}^{\frac12}\\
&=: \sum_{|\alpha|=m=|\beta|}\ \sum_{|\widetilde{\alpha}|=k,\,\widetilde{\alpha}\le\alpha}\mathrm{J}_{\alpha,\,\widetilde{\alpha},\,\beta,\,1}\times\mathrm{J}_{\alpha,\,\widetilde{\alpha},\,\beta,\,2}.\end{aligned} \tag{3.32}$$

We first control $\mathrm{J}_{\alpha,\,\widetilde{\alpha},\,\beta,\,1}$. Let $\widetilde{\mathcal{B}}^\epsilon(E^*)$, $\widetilde{\mathcal{B}}^R(E^*)$ and $\widetilde{\mathcal{B}}_0(E^*)$ be, respectively, as in (3.25), (3.26) and (3.27). Similar to (3.28), we first obtain

$$\iint_{\widetilde{\mathcal{B}}^\epsilon(E^*)}|t^m\nabla^m u(y,\,t)|^2\,\frac{dy\,dt}{t} \lesssim \int_E\left[\int_{\epsilon/2}^{\epsilon}\frac{1}{t^n}\int_{B(z,\,3t)}|t^m\nabla^m u(y,\,t)|^2\,\frac{dy\,dt}{t}\right]dz. \tag{3.33}$$

This, together with the parabolic Caccioppoli's inequality (3.2), implies that

$$\begin{aligned}&\iint_{\widetilde{\mathcal{B}}^\epsilon(E^*)}|t^m\nabla^m u(y,\,t)|^2\,\frac{dy\,dt}{t}\\
&\quad\lesssim \int_E\int_{\epsilon/4}^{2\epsilon}\frac{1}{t^{n-2m}}\int_{B(z,\,6t)}\left[\frac{1}{t^2}\left|\nabla^{m-1}u(y,\,t)\right|^2+\frac{1}{t^{2m}}\left|u(y,\,t)\right|^2\right]\frac{dy\,dt\,dz}{t}\\
&\quad\lesssim \int_E\left[\mathcal{N}^6_{h,\,L}(f)(y)\right]^2\,dy.\end{aligned} \tag{3.34}$$

Similarly, resting on estimates (3.29), (3.30), (3.34) and the parabolic Caccioppoli's inequality (3.2), we conclude that there exists a positive constant $\gamma\in(0,\,\infty)$ large enough such that

$$\begin{aligned}&\iint_{\widetilde{\mathcal{B}}^R(E^*)\cup\widetilde{\mathcal{B}}_0(E^*)}|t^m\nabla^m u(y,\,t)|^2\,\frac{dy\,dt}{t}\\
&\quad\lesssim \int_E\left[\mathcal{N}^\gamma_{h,\,L}(f)(y)\right]^2\,dy + |B^*|\left[\sup_{x\in E}\mathcal{N}^\gamma_{h,\,L}(f)(x)\right]^2.\end{aligned}$$

This, combined with (3.34), implies that

$$\mathrm{J}_{\alpha,\widetilde{\alpha},\beta,1} \lesssim \left\{ \int_E \left[\mathcal{N}_{h,L}^{\gamma}(f)(y)\right]^2 dy + |B^*| \left[\sup_{x\in E} \mathcal{N}_{h,L}^{\gamma}(f)(x)\right]^2 \right\}^{\frac{1}{2}}. \tag{3.35}$$

The estimate of $\mathrm{J}_{\alpha,\widetilde{\alpha},\beta,2}$ can be obtained by using the definition of η, the interpolation inequality (3.9) and the estimates of J_0 and $\mathrm{J}_{\alpha,\widetilde{\alpha},\beta,1}$. This, together with (3.24) and the estimates of (3.31) through (3.35), implies that

$$\begin{aligned}
&\int_{\mathcal{R}^{2\epsilon,R,1/2}(E^*)} t^{2m} |\nabla^m u(y,t)|^2 \frac{dy\,dt}{t} \\
&\quad \lesssim \int_E \left[\mathcal{N}_{h,L}^{\gamma}(f)(y)\right]^2 dy + |B^*| \left[\sup_{x\in E} \mathcal{N}_{h,L}^{\gamma}(f)(x)\right]^2,
\end{aligned}$$

where $\gamma \in (0,\infty)$ is a sufficiently large constant. This and the argument similar to that used in [23, (6.31) through (6.37)], finish the proof of Proposition 3.7. □

We are now in a position to prove our main result of this article.

Proof of Theorem 1.4. The inclusion $H^p_{\mathcal{N}_{h,L}}(\mathbb{R}^n) \subset H^p_L(\mathbb{R}^n)$ is a direct consequence of Propositions 3.3 and 3.7, and Corollary 3.6.

To prove the inclusion $H^p_L(\mathbb{R}^n) \subset H^p_{\mathcal{N}_{h,L}}(\mathbb{R}^n)$, we consider two cases. If $p \in (0,1]$, by Theorem 2.11 and Remark 1.3, we see that it suffices to show that, for all $(p,2,M,\epsilon)_L$-molecules α,

$$\|\mathcal{R}_{h,L}(\alpha)\|_{L^p(\mathbb{R}^n)} \lesssim 1,$$

where $\mathcal{R}_{h,L}$ is the radial heat maximal function defined as in (1.6). The latter estimate can be obtained by using the same method as that used in the proof of [23, Theorem 6.3], the details being omitted here. This finishes the proof of Theorem 1.4 for $p \in (0,1]$.

If $p \in (1, q_+(L))$, for any locally integrable function f, let $\mathcal{M}(f)$ be the *Hardy-Littlewood maximal function* defined by setting, for all $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |f(x)|\,dx,$$

where the supremum is taken over all the balls in $\mathbb{R}^n$ containing x. Let $\mathcal{R}_{h,L}$ be the radial maximal function defined as in (1.6) and, for any ball B and $j \in \mathbb{N}$, let $S_j(B) := 2^jB\backslash(2^{j-1}B)$ and $S_0(B) := B$. Then, for any $q \in (2, q_+(L))$, using Minkowski's inequality, Proposition 2.5, and the boundedness of $\mathcal{M}$ on $L^{q/2}(\mathbb{R}^n)$, we know that there exists a positive constant η such that

$$\begin{aligned}
&\|\mathcal{R}_{h,L}(f)\|_{L^q(\mathbb{R}^n)} \\
&= \left\| \sup_{t\in(0,\infty)} \left\{ \frac{1}{t^n} \int_{B(\cdot,t)} \sum_{k=0}^{m-1} \left| (t\nabla)^k e^{-t^{2m}L} \left(\sum_{j\in\mathbb{Z}_+} \chi_{S_j(B(\cdot,t))} f \right)(y) \right|^2 dy \right\}^{\frac{1}{2}} \right\|_{L^q(\mathbb{R}^n)}
\end{aligned}$$

$$\lesssim \sum_{j\in\mathbb{Z}_+} \left\| \sup_{t\in(0,\infty)} \left[\frac{1}{t^{\frac{n}{2}}} \exp\left\{ -\frac{[d(B(\cdot,\,t),\,S_j(B(\cdot,\,t)))]^{2m/(2m-1)}}{t^{2m/(2m-1)}} \right\} \|f\|_{L^2(S_j(B(\cdot,\,t)))} \right] \right\|_{L^q(\mathbb{R}^n)}$$

$$\lesssim \sum_{j\in\mathbb{Z}_+} 2^{-j\eta} \left\| \sup_{t\in(0,\infty)} \left[\frac{1}{(2^j t)^n} \int_{2^j B(\cdot,\,t)} |f(x)|^2\,dx \right]^{\frac12} \right\|_{L^q(\mathbb{R}^n)}$$

$$\lesssim \left\| \left[\mathcal{M}\left(|f|^2\right)\right]^{q/2} \right\|_{L^q(\mathbb{R}^n)}^{\frac12} \lesssim \|f\|_{L^q(\mathbb{R}^n)},$$

which implies that $\mathcal{R}_{h,\,L}$ is bounded on $L^q(\mathbb{R}^n)$. This, together with Remark 1.3, further shows that the non-tangential maximal function $\mathcal{N}_{h,\,L}$ is also bounded on $L^q(\mathbb{R}^n)$. By the case $p\in(0,\,1]$, Lemma 2.12 and the complex interpolation of $H^p_L(\mathbb{R}^n)$ (see Proposition 2.8), we see that, for all $p\in(0,\,q_+(L))$, $\mathcal{N}_{h,\,L}$ is bounded from $H^p_L(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. This implies the inclusion $H^p_L(\mathbb{R}^n)\subset H^p_{\mathcal{N}_{h,\,L}}(\mathbb{R}^n)$ for all $p\in(0,\,q_+(L))$ and hence finishes the proof of Theorem 1.4. □

Now we turn to the proof of Proposition 1.5.

Proof of Proposition 1.5. For any $(x,\,t)\in\mathbb{R}^{n+1}_+$, following [1, pp. 59-60, 3.2(c)], let

$$W^{m,\,2}_0(B(x,\,2t))$$

be the *Sobolev space over* $B(x,\,2t)$, defined as the completion of $C^\infty_c(B(x,\,2t))$ with respect to the *norm* $\|\cdot\|_{W^{m,\,2}(B(x,\,2t))}$, where, for all $\varphi\in C^\infty_c(B(x,\,2t))$,

$$\|\varphi\|_{W^{m,\,2}(B(x,\,2t))} := \left[\sum_{0\le|\alpha|\le m} \|\partial^\alpha\varphi\|^2_{L^2(B(x,\,2t))}\right]^{1/2}.$$

Since $f\in L^2(\mathbb{R}^n)$, it follows that $e^{-t^{2m}L}(f)\in W^{m,\,2}(\mathbb{R}^n)$. Thus, we have

$$\psi_{x,\,t}e^{-t^{2m}L}(f)\in W^{m,\,2}_0(B(x,\,2t))\subset W^{m-1,\,2}_0(B(x,\,2t)).$$

Recall the following Poincaré's inequality from [29, p. 69, Theorem 3.2.1]: For all $k\in\{0,\,\dots,\,m-1\}$ and $v\in W^{m-1,\,2}_0(B(x,\,2t))$, it holds true that

$$\int_{B(x,\,2t)} |\nabla^k v(y)|^2\,dy \le 2^{k-m+1}(2t)^{(m-1-k)2}\int_{B(x,\,2t)} \left|\nabla^{m-1}v(y)\right|^2\,dy.$$

Thus, by this and properties of ψ, we see that

$$\text{(3.36)}\qquad \left\{\frac{1}{t^n}\int_{B(x,\,t)} \left|e^{-t^{2m}L}(f)(z)\right|^2\,dz\right\}^{1/2}$$

$$\le \left\{\frac{1}{t^n}\int_{B(x,\,2t)} \left|\psi_{x,\,t}(z)e^{-t^{2m}L}(f)(z)\right|^2\,dz\right\}^{1/2}$$

$$\lesssim \left\{ \frac{1}{t^n} \int_{B(x,\,2t)} \left| (t\nabla)^{m-1} \left(\psi_{x,\,t} e^{-t^{2m}L}(f) \right)(z) \right|^2 \, dz \right\}^{1/2}$$

and

$$\left\{ \frac{1}{t^n} \int_{B(x,\,t)} \left| (t\nabla)^{m-1} \left(e^{-t^{2m}L}(f) \right)(z) \right|^2 \, dz \right\}^{1/2} \le \left\{ \frac{1}{t^n} \int_{B(x,\,2t)} \left| (t\nabla)^{m-1} \left(\psi_{x,\,t} e^{-t^{2m}L}(f) \right)(z) \right|^2 \, dz \right\}^{1/2},$$

which, combined with (3.36) and the interpolation inequality (3.9), further shows that, for all $x \in \mathbb{R}^n$,

$$\mathcal{N}_{h,\,L}(f)(x) \lesssim \mathcal{N}^2_{h,\,\psi,\,L}(f)(x). \tag{3.37}$$

As for the converse direction, by Leibnitz's rule and properties of ψ, we have

$$\left\{ \frac{1}{t^n} \int_{B(x,\,2t)} \left| (t\nabla)^{m-1} \left(\psi_{x,\,t} e^{-t^{2m}L}(f) \right)(z) \right|^2 \, dz \right\}^{1/2} \lesssim \left\{ \frac{1}{t^n} \int_{B(x,\,2t)} \sum_{k=0}^{m-1} \left| (t\nabla)^{k} \left(e^{-t^{2m}L}(f) \right)(z) \right|^2 \, dz \right\}^{1/2},$$

which implies that, for all $x \in \mathbb{R}^n$,

$$\mathcal{N}^2_{h,\,\psi,\,L}(f)(x) \lesssim \mathcal{N}^2_{h,\,L}(f)(x). \tag{3.38}$$

Combining (3.37), (3.38) and Theorem 1.4, we conclude that $H^p_L(\mathbb{R}^n) = H^p_{\mathcal{N}_{h,\,\psi,\,L}}(\mathbb{R}^n)$ with equivalent quasi-norms, which completes the proof of Proposition 1.5. $\square$

Now, we prove Theorem 1.7. To this end, we need the following proposition.

Proposition 3.8. *Let $k \in \mathbb{N}$, L be as in* (1.1) *and satisfy the Ellipticity condition* ($\mathcal{E}_0$). *Let $S_{L,\,k}$ and $S_{h,\,L,\,k}$ be the same, respectively, as in* (1.15) *and* (1.16). *Then*

(i) *for all $q \in (0, \infty)$, there exists a positive constant $C_{(k,\,q)}$ such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\|S_{L,\,k}(f)\|_{L^q(\mathbb{R}^n)} \le C_{(k,\,q)} \|S_{L,\,1}(f)\|_{L^q(\mathbb{R}^n)};$$

(ii) *for all $q \in (0, \infty)$, there exists a positive constant $C_{(k,\,q)}$ such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\|S_{h,\,L,\,k}(f)\|_{L^q(\mathbb{R}^n)} \le C_{(k,\,q)} \|S_{L,\,1}(f)\|_{L^q(\mathbb{R}^n)}.$$

Proof. We prove Proposition 3.8 by mathematical induction.

If $k = 1$, (i) of Proposition 3.8 automatically holds true. To prove (ii) of Proposition 3.8, by (3.16), we know that, for all $\widetilde{\epsilon} \in (0, \infty)$, there exists a positive constant $\widetilde{C}_{(\epsilon)}$ such that, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\left[S_{h,L,1}^{\epsilon/2,2R,3/2}(f)(x)\right]^2 \le (\widetilde{\epsilon})^4 \left[S_{h,L,1}^{\epsilon/4,3R,2}(f)(x)\right]^2 + \left[C_{(\widetilde{\epsilon})}\right]^4 \left[S_{L,1}^{\epsilon/4,3R,2}(f)(x)\right]^2. \tag{3.39}$$

By this, together with letting $\epsilon \to 0$ and $R \to \infty$, (3.11) and (3.12), we further conclude that, for all $q \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$,

$$\|S_{h,L,1}(f)\|_{L^q(\mathbb{R}^n)} \lesssim \|S_{L,1}(f)\|_{L^q(\mathbb{R}^n)},$$

which, combined with (i) when $k = 1$ of Proposition 3.8, implies that (ii) of Proposition 3.8 holds true in this case.

If $k = 2$, we prove (i) by first establishing a desired estimate for $\|S_{L,2}(f)\|_{L^q(\mathbb{R}^n)}$ (see (3.46) below). To this end, for all $0 < \epsilon \ll R < \infty$, $\lambda \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $\Gamma^{\epsilon,R,\lambda}(x)$ be the truncated cone as in (3.10). Also, let $\eta \in C_c^\infty(\Gamma^{\epsilon/2,2R,3/2}(x))$ satisfy $\eta \equiv 1$ on $\Gamma^{\epsilon,R,1}(x)$, $0 \le \eta \le 1$ and, for all $l \in \mathbb{N}$ with $l \le m$ and $(y, t) \in \Gamma^{\epsilon/2,2R,3/2}(x)$,

$$\left|\nabla^l \eta(y, t)\right| \lesssim \frac{1}{t^l}.$$

From properties of η, the definition of L, Leibnitz's rule and Minkowski's inequality, we deduce that, for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
(3.40)\quad & S_{L,2}^{\epsilon,R,1}(f)(x) \\
&= \left[\iint_{\Gamma^{\epsilon,R,1}(x)} \left|\left(t^{2m}L\right)^2 e^{-t^{2m}L}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{\frac12} \\
&\le \left[\iint_{\Gamma^{\epsilon/2,2R,3/2}(x)} \left(t^{2m}L\right)^2 e^{-t^{2m}L}(f)(y)\overline{\left(t^{2m}L\right)^2 e^{-t^{2m}L}(f)(y)}\eta(y, t)\,\frac{dy\,dt}{t^{n+1}}\right]^{\frac12} \\
&= \left|\sum_{|\alpha|=m=|\beta|} \iint_{\Gamma^{\epsilon/2,2R,3/2}(x)} a_{\alpha,\beta}(y) t^m \partial^\beta \left(t^{2m} L e^{-t^{2m}L}(f)\right)(y)\right. \\
&\quad \left.\times \overline{t^m \partial^\alpha \left(\left(t^{2m}L\right)^2 e^{-t^{2m}L}(f)\eta\right)(y, t)}\,\frac{dy\,dt}{t^{n+1}}\right|^{\frac12} \\
&\lesssim \sum_{l=0}^m \left|\sum_{|\alpha|=m=|\beta|} \iint_{\Gamma^{\epsilon/2,2R,3/2}(x)} a_{\alpha,\beta}(y) t^m \partial^\beta \left(t^{2m} L e^{-t^{2m}L}(f)\right)(y)\right. \\
&\quad \left.\times \left[t^m \sum_{|\widetilde{\alpha}|=l,\,\widetilde{\alpha}\le\alpha} C_{(\alpha,\widetilde{\alpha})} \overline{\partial^{\widetilde{\alpha}} \left(\left(t^{2m}L\right)^2 e^{-t^{2m}L}(f)\right)(y)\,\partial^{\alpha-\widetilde{\alpha}}\eta(y, t)}\right] \frac{dy\,dt}{t^{n+1}}\right|^{\frac12}
\end{aligned}$$

$$=: \sum_{l=0}^{m} \mathrm{I}_l,$$

where, for all multi-indices α and $\widetilde{\alpha}$, $C_{(\alpha,\,\widetilde{\alpha})}$ denotes a positive constant depending on α and $\widetilde{\alpha}$.

For I_0, by Hölder's inequality, the size condition of η and the Ellipticity condition $(\mathcal{E}_0)$, we see that

$$\begin{aligned}(3.41)\qquad \mathrm{I}_0 &\lesssim \left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left|t^m\nabla^m\left(t^{2m}Le^{-t^{2m}L}(f)\right)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\ &\quad\times\left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)} \left|\left(t^{2m}L\right)^2 e^{-t^{2m}L}(f)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\ &\sim\left[S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12}\left[S_{L,\,2}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12},\end{aligned}$$

where $S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)$ and $S_{L,\,2}^{\epsilon/2,\,2R,\,3/2}(f)$ are defined, respectively, similar to $S_{h,\,L,\,1}(f)$ in (1.16) and $S_{L,\,2}(f)$ in (1.15), with $\Gamma(x)$ in (1.5) replaced by $\Gamma^{\epsilon/2,\,2R,\,3/2}(x)$ in (3.10).

For I_m, using an argument similar to that used in the proof of (3.18), we conclude that

$$(3.42)\qquad \mathrm{I}_m \lesssim \left[S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12}\left[S_{h,\,L,\,2}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12}.$$

Moreover, by an argument similar to that used in the proof of (3.16) (see also (3.39)), we find that, for all $\widetilde{\widetilde{\epsilon}}\in(0,\,\infty)$, there exists a positive constant $C_{(\widetilde{\widetilde{\epsilon}})}$ such that

$$(3.43)\quad S_{h,\,L,\,2}^{\epsilon/2,\,2R,\,3/2}(f)(x)\le\left(\widetilde{\widetilde{\epsilon}}\right)^4\left[S_{h,\,L,\,2}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2+\left[C_{(\widetilde{\widetilde{\epsilon}})}\right]^4\left[S_{L,\,2}^{\epsilon/4,\,3R,\,2}(f)(x)\right]^2.$$

By this, together with letting $\epsilon\to0$ and $R\to\infty$, (3.11) and (3.12), we further conclude that, for all $q\in(0,\,\infty)$ and $f\in L^2(\mathbb{R}^n)$,

$$(3.44)\qquad \|S_{h,\,L,\,2}(f)\|_{L^q(\mathbb{R}^n)}\lesssim\|S_{L,\,2}(f)\|_{L^q(\mathbb{R}^n)}.$$

Also, similar to (3.19), we know that, for all $l\in\{1,\,\dots,\,m-1\}$,

$$\begin{aligned}(3.45)\ \mathrm{I}_l &\lesssim\left[S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12}\left\{\iint_{\Gamma^{\epsilon/2,\,2R,\,3/2}(x)}\left|t^l\nabla^l\left(\left[t^{2m}L\right]^2e^{-t^{2m}L}(f)\right)(y)\right|^2\frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\ &\lesssim\left[S_{h,\,L,\,1}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{\frac12}\left[S_{h,\,L,\,2}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{l/(2m)}\left[S_{L,\,2}^{\epsilon/2,\,2R,\,3/2}(f)(x)\right]^{(m-l)/(2m)}.\end{aligned}$$

By combining (3.40) through (3.45) and then letting $\epsilon\to0$ and $R\to\infty$, together with (3.11) and (3.12), we conclude that, for all $q\in(0,\,\infty)$,

$$(3.46)\qquad \|S_{L,\,2}(f)\|_{L^q(\mathbb{R}^n)}\lesssim\|S_{L,\,1}(f)\|_{L^p(\mathbb{R}^n)},$$

which implies that (i) of Proposition 3.8 holds true for $k = 2$. Moreover, (ii) of Proposition 3.8 when $k = 2$ follows from (3.44) and (i) of Proposition 3.8 when $k = 2$.

Now, let $\widetilde{k} \in \mathbb{N} \cap [3, \infty)$. Assume that Proposition 3.8 holds true for all $k \in \{1, \dots, \widetilde{k}\}$. Thus, by mathematical induction, to finish the proof of Proposition 3.8, it suffices to show that Proposition 3.8 also holds true for $\widetilde{k} + 1$.

Similar to (3.40), for all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

$$
\begin{aligned}
(3.47)\quad & S^{\epsilon, R, 1}_{L, \widetilde{k}+1}(f)(x)\\
&= \left[\iint_{\Gamma^{\epsilon, R, 1}(x)} \left|\left(t^{2m}L\right)^{\widetilde{k}+1} e^{-t^{2m}L}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right]^{\frac12}\\
&\le \left[\iint_{\Gamma^{\epsilon/2, 2R, 3/2}(x)} \left(t^{2m}L\right)^{\widetilde{k}+1} e^{-t^{2m}L}(f)(y)\overline{\left(t^{2m}L\right)^{\widetilde{k}+1} e^{-t^{2m}L}(f)(y)}\right.\\
&\quad \left.\times \eta(y, t)\,\frac{dy\,dt}{t^{n+1}}\right]^{\frac12}\\
&= \left|\sum_{|\alpha|=m=|\beta|} \iint_{\Gamma^{\epsilon/2, 2R, 3/2}(x)} a_{\alpha,\beta}(y) t^m \partial^\beta \left(\left[t^{2m}L\right]^{\widetilde{k}} e^{-t^{2m}L}(f)\right)(y)\right.\\
&\quad \left.\times \overline{t^m \partial^\alpha \left(\left(t^{2m}L\right)^{\widetilde{k}+1} e^{-t^{2m}L}(f)\eta\right)(y, t)}\,\frac{dy\,dt}{t^{n+1}}\right|^{\frac12}\\
&\lesssim \sum_{l=0}^m \left|\sum_{|\alpha|=m=|\beta|} \iint_{\Gamma^{\epsilon/2, 2R, 3/2}(x)} a_{\alpha,\beta}(y) t^m \partial^\beta \left(\left[t^{2m}L\right]^{\widetilde{k}} e^{-t^{2m}L}(f)\right)(y)\right.\\
&\quad \left.\times \left[t^m \sum_{|\widetilde{\alpha}|=l,\, \widetilde{\alpha}\le\alpha} C_{(\alpha,\widetilde{\alpha})}\overline{\partial^{\widetilde{\alpha}}\left(\left(t^{2m}L\right)^{\widetilde{k}+1} e^{-t^{2m}L}(f)\right)(y)\,\partial^{\alpha-\widetilde{\alpha}}\eta(y, t)}\right] \frac{dy\,dt}{t^{n+1}}\right|^{\frac12}\\
&=: \sum_{l=0}^m \widetilde{\mathrm{I}}_l.
\end{aligned}
$$

For $\widetilde{\mathrm{I}}_0$, by Hölder's inequality, the size condition of η and the Ellipticity condition $(\mathcal{E}_0)$, we see that

$$
\begin{aligned}
(3.48)\quad \widetilde{\mathrm{I}}_0 &\lesssim \left\{\iint_{\Gamma^{\epsilon/2, 2R, 3/2}(x)} \left|t^m \nabla^m \left(\left[t^{2m}L\right]^{\widetilde{k}} e^{-t^{2m}L}(f)\right)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\
&\quad \times \left\{\iint_{\Gamma^{\epsilon/2, 2R, 3/2}(x)} \left|\left(t^{2m}L\right)^{\widetilde{k}+1} e^{-t^{2m}L}(f)(y)\right|^2 \frac{dy\,dt}{t^{n+1}}\right\}^{\frac14}\\
&\sim \left[S^{\epsilon/2, 2R, 3/2}_{h, L, \widetilde{k}}(f)(x)\right]^{\frac12} \left[S^{\epsilon/2, 2R, 3/2}_{L, \widetilde{k}+1}(f)(x)\right]^{\frac12},
\end{aligned}
$$

where $S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}}(f)$ and $S^{\epsilon/2,\,2R,\,3/2}_{L,\,\widetilde{k}+1}(f)$ are defined, respectively, similar to $S_{h,\,L,\,\widetilde{k}}(f)$ in (1.16) and $S_{L,\,\widetilde{k}+1}(f)$ in (1.15), with $\Gamma(x)$ in (1.5) replaced by $\Gamma^{\epsilon/2,\,2R,\,3/2}(x)$ in (3.10).

For $\widetilde{\mathrm{I}}_m$, using an argument similar to that used in the proof of (3.42), we conclude that

$$\widetilde{\mathrm{I}}_m \lesssim \left[S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}}(f)(x)\right]^{\frac12}\left[S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}+1}(f)(x)\right]^{\frac12}. \tag{3.49}$$

By an argument similar to that of (3.39), we know that, for all $\widetilde{\epsilon}\in(0,\infty)$, there exists a positive constant $C_{(\widetilde{\epsilon},\,\widetilde{k})}$ such that

$$\left[S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}+1}(f)\right]^2 \le (\widetilde{\epsilon})^4\left[S^{\epsilon/4,\,3R,\,2}_{h,\,L,\,\widetilde{k}+1}(f)(x)\right]^2+\left[C_{(\widetilde{\epsilon},\,\widetilde{k})}\right]^4\left[S^{\epsilon/4,\,3R,\,2}_{L,\,\widetilde{k}+1}(f)(x)\right]^2.$$

By this, together with letting $\epsilon\to0$ and $R\to\infty$, (3.11) and (3.12), we further conclude that, for all $q\in(0,\,\infty)$ and $f\in L^2(\mathbb{R}^n)$,

$$\left\|S_{h,\,L,\,\widetilde{k}+1}(f)\right\|_{L^q(\mathbb{R}^n)} \lesssim \left\|S_{L,\,\widetilde{k}+1}(f)\right\|_{L^q(\mathbb{R}^n)}. \tag{3.50}$$

Also, similar to (3.45), we know that, for all $l\in\{1,\,\dots,\,m-1\}$,

$$\widetilde{\mathrm{I}}_l \le \left[S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}}(f)(x)\right]^{\frac12}\left[S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}+1}(f)(x)\right]^{l/(2m)}\left[S^{\epsilon/2,\,2R,\,3/2}_{L,\,\widetilde{k}+1}(f)(x)\right]^{(m-l)/(2m)}. \tag{3.51}$$

Thus, combining (3.47) through (3.51), we conclude that, for all $q\in(0,\,\infty)$,

$$\begin{aligned}\left\|S^{\epsilon,\,R,\,1}_{L,\,\widetilde{k}+1}(f)\right\|_{L^q(\mathbb{R}^n)} &\lesssim \sum_{l=0}^m\left\|\widetilde{\mathrm{I}}_l\right\|_{L^q(\mathbb{R}^n)}\\ &\lesssim \left\|S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}}(f)\right\|^{1/2}_{L^q(\mathbb{R}^n)}\left\|S^{\epsilon/2,\,2R,\,3/2}_{L,\,\widetilde{k}+1}(f)\right\|^{1/2}_{L^q(\mathbb{R}^n)}\\ &\quad+\left\|S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}}(f)\right\|^{1/2}_{L^q(\mathbb{R}^n)}\left\|S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}+1}(f)\right\|^{1/2}_{L^q(\mathbb{R}^n)}\\ &\quad+\sum_{l=1}^{m-1}\left\|S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}}(f)\right\|^{1/2}_{L^q(\mathbb{R}^n)}\left\|S^{\epsilon/2,\,2R,\,3/2}_{h,\,L,\,\widetilde{k}+1}(f)\right\|^{l/(2m)}_{L^q(\mathbb{R}^n)}\\ &\quad\times\left\|S^{\epsilon/2,\,2R,\,3/2}_{L,\,\widetilde{k}+1}(f)\right\|^{(m-l)/(2m)}_{L^q(\mathbb{R}^n)}.\end{aligned} \tag{3.52}$$

By letting $\epsilon\to0$ and $R\to\infty$, this, together with Lemma 3.4, (3.11), (3.12), the assumption that Proposition 3.8 holds true for all $k\in\{1,\,\dots,\,\widetilde{k}\}$ and (3.50), implies that (i) of Proposition 3.8 holds true in the case $\widetilde{k}+1$.

Finally, we see that (ii) of Proposition 3.8 in the case $\widetilde{k}+1$ follows from (3.50) and (i) of Proposition 3.8 in the case $\widetilde{k}+1$, which completes the proof of Proposition 3.8. □

From the proof of Proposition 3.8, we immediately deduce the following conclusions.

Corollary 3.9. *Let $k \in \mathbb{N}$, L be as in* (1.1) *and satisfy the Ellipticity condition* $(\mathcal{E}_0)$. *Let $S_{L,k}$ and $S_{h,L,k}$ be the same, respectively, as in* (1.15) *and* (1.16). *Then, for all $p \in (0, \infty)$, there exists a positive constant $C_{(p,k)}$ such that, for all $f \in L^2(\mathbb{R}^n)$,*

$$\frac{1}{C_{(p,k)}} \|S_{h,L,k}(f)\|_{L^p(\mathbb{R}^n)} \le \|S_{L,k}(f)\|_{L^p(\mathbb{R}^n)} \le C_{(p,k)} \|S_{h,L,k-1}(f)\|_{L^p(\mathbb{R}^n)}. \tag{3.53}$$

Proof. The first inequality of (3.53) follows immediately from (3.50) with $\widetilde{k}+1$ replaced by k in the proof of Proposition 3.8, while the second inequality of (3.53) follows from (3.52) with $\widetilde{k}+1$ replaced by k and the first inequality of (3.53), which completes the proof of Corollary 3.9. □

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. We first prove (i) of Theorem 1.7. If $p \in (0, 2]$, by Proposition 2.15, we know that $H^p_{S_{L,k}}(\mathbb{R}^n) = H^p_L(\mathbb{R}^n)$. Thus, to finish the proof of (i) of Theorem 1.7, it suffices to consider the case $p \in (2, p_+(L))$.

Moreover, by Lemma 3.4(i) and a density argument, we see that, for all $p \in (2, p_+(L))$,

$$H^p_L(\mathbb{R}^n) \subset H^p_{S_{L,k}}(\mathbb{R}^n).$$

On the other hand, for all $p \in (2, \infty)$, by an argument similar to that used in the proof of (3.22), we see that, for all $k \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|S_{L,k}(f)\|_{L^p(\mathbb{R}^n)},$$

which, together with Lemma 2.12 and a density argument, implies that, for all $p \in (2, p_+(L))$,

$$H^p_{S_{L,k}}(\mathbb{R}^n) \subset H^p_L(\mathbb{R}^n).$$

This shows that (i) of Theorem 1.7 holds true.

We now prove (ii) of Theorem 1.7. To prove the inclusion that $H^p_{S_{h,L,k}}(\mathbb{R}^n) \subset H^p_L(\mathbb{R}^n)$, for all $p \in (0, q_+(L))$, by (i) of Theorem 1.7 and Corollary 3.9, we conclude that, for all $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{H^p_L(\mathbb{R}^n)} \sim \|S_{L,k+1}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|S_{h,L,k}(f)\|_{L^p(\mathbb{R}^n)},$$

which, together with a density argument, implies that $H^p_{S_{h,L,k}}(\mathbb{R}^n) \subset H^p_L(\mathbb{R}^n)$.

For the converse inclusion, if $p \in (p_-(L), q_+(L))$, by Lemma 3.4(ii) and a density argument, we immediately see that $H^p_L(\mathbb{R}^n) \subset H^p_{S_{h,L,k}}(\mathbb{R}^n)$ holds true in the range $p \in (p_-(L), q_+(L))$.

If $p \in (0, 1]$, by considering the action of $S_{h,L,k}$ on each $(p, 2, M, \epsilon)_L$-molecule (see, for example, [9, (4.4)] for a proof of a similar result) and Theorem 2.11, we know that, for all $p \in (0, 1]$ and $f \in H^p_L(\mathbb{R}^n)$,

$$\|S_{h,L,k}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

which, together with case $p \in (p_-(L), q_+(L))$, Lemma 2.12, the interpolation and a density argument, implies that $H^p_L(\mathbb{R}^n) \subset H^p_{S_{h,L,k}}(\mathbb{R}^n)$ holds true for all $p \in (0, q_+(L))$. This finishes the proof of Theorem 1.7. □

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