Problem 1

Proof. 1. Choose \( w \in \mathcal{A} \). Then (46) implies
\[
0 = \int_U (-\Delta u - f)(u - w) \, dx.
\]
An integration by parts yields
\[
0 = \int_U Du \cdot D(u - w) - f(u - w) \, dx,
\]
and there is no boundary term since \( u - w = g - g \equiv 0 \) on \( \partial U \). Hence
\[
\int_U |Du|^2 - uf \, dx = \int_U Du \cdot Dw - wf \, dx
\]
\[
\leq \int_U \frac{1}{2}|Du|^2 \, dx + \int_U \frac{1}{2}|Dw|^2 - wf \, dx,
\]
where we employed the estimates
\[
|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,
\]
following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude
\[
(48) \quad I[u] \leq I[w] \quad (w \in \mathcal{A}).
\]
Since \( u \in \mathcal{A} \), (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any \( v \in C_c^\infty(U) \) and write
\[
i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).
\]
Since \( u + \tau v \in \mathcal{A} \) for each \( \tau \), the scalar function \( i(\cdot) \) has a minimum at zero, and thus
\[
i'(0) = 0 \quad \left( \tau = \frac{d}{d\tau} \right),
\]
provided this derivative exists. But
\[
i(\tau) = \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v)f \, dx
\]
\[
= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u + \tau v)f \, dx.
\]
Consequently
\[
0 = i'(0) = \int_U Du \cdot Dv - vf \, dx = \int_U (-\Delta u - f)v \, dx.
\]
This identity is valid for each function \( v \in C_c^\infty(U) \) and so \(-\Delta u = f\) in \( U \).

Dirichlet’s principle is an instance of the calculus of variations applied to Laplace’s equation. See Chapter 8 for more.
We have already employed the maximum principle in §2.2.3 to show uniqueness, but now set forth a simple alternative proof. Assume $U$ is open, bounded, and $\partial U$ is $C^1$.

**THEOREM 16** (Uniqueness). There exists at most one solution $u \in C^2(\bar{U})$ of (46).

**Proof.** Assume $\tilde{u}$ is another solution and set $w := u - \tilde{u}$. Then $\Delta w = 0$ in $U$, and so an integration by parts shows

$$0 = -\int_U w \Delta w \, dx = \int_U |Dw|^2 \, dx.$$

Thus $Dw \equiv 0$ in $U$, and, since $w = 0$ on $\partial U$, we deduce $w = u - \tilde{u} \equiv 0$ in $U$. \hfill \Box

b. Dirichlet's principle.

Next let us demonstrate that a solution of the boundary-value problem (46) for Poisson's equation can be characterized as the minimizer of an appropriate functional. For this, we define the energy functional

$$I[w] := \int_U \frac{1}{2} |Dw|^2 - wf \, dx,$$

$w$ belonging to the admissible set

$$\mathcal{A} := \{ w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U \}.$$ 

**THEOREM 17** (Dirichlet's principle). Assume $u \in C^2(\bar{U})$ solves (46). Then

$$I[u] = \min_{w \in \mathcal{A}} I[w]. \quad (47)$$

Conversely, if $u \in \mathcal{A}$ satisfies (47), then $u$ solves the boundary-value problem (46).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that $u$ minimizes the energy $I[\cdot]$. 
7.1.1f

\[
\frac{1}{\sqrt{2\pi} \left( -k^2 + 2i k + 2 \right)} = \frac{-k^2 - 2i k + 2}{\sqrt{2\pi} \left( k^4 + 4 \right)}.
\]

7.1.3a,b

(a) By the Shift Theorem 7.4, \( f(x) = i \sqrt{\frac{\pi}{2}} e^{-iax} \text{sign } x \).

(b) Using the Table, if \( b > 0 \), then \( f(x) = i \sqrt{2\pi} e^{bx} (\sigma(x) - 1) \), while if \( b < 0 \), then \( f(x) = i \sqrt{2\pi} e^{bx} \sigma(x) \). For \( b = 0 \), use part (a).

7.1.13

Use the change of variables \( \bar{x} = x - \xi \) in the integral:

\[
\mathcal{F}[f(x - \xi)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\bar{x}) e^{-ik(\bar{x} + \xi)} \, d\bar{x}
\]

\[
= \frac{e^{-ik\xi}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\bar{x}) e^{-ik\bar{x}} \, d\bar{x} = e^{-ik\xi} \mathcal{F}[f](k).
\]

To prove the second statement,

\[
\mathcal{F}[e^{i\kappa x} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k-\kappa)x} \, dx = \mathcal{F}[f](k - \kappa).
\]

7.1.20 a): (i), (iii), and b)

\[
\begin{align*}
\cdot (a) & \quad (i) \quad \frac{2}{\pi(k^2 + 1)(l^2 + 1)}, \quad \star (\text{iii}) \quad \frac{e^{-i(\xi \cdot k + \eta \cdot l)}}{2\pi}, \\
\end{align*}
\]

7.2.2a

\[
(a) - \frac{i}{k} \sqrt{\frac{2}{\pi}} e^{-k^2/4} + \sqrt{2\pi} \delta(k).
\]

7.2.3d
\[(d) \quad -\frac{d^2}{dx^2} \left[ \sqrt{2\pi} e^{-x} \sigma(x) \right] = \sqrt{2\pi} \left[ -e^{-x} \sigma(x) + \delta(x) - \delta'(x) \right].\]

7.2.12

(a) Indeed, applying the inverse Fourier transform:

\[f(x) \sim \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dk = \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(k - n) e^{ikx} \, dk = \sum_{n=-\infty}^{\infty} c_n e^{ikx}\]

recovers the complex Fourier series for \(f(x)\), proving the result.

(b) (i) \(\frac{1}{2} i \delta(x + 2) - \frac{1}{2} i \delta(x - 2)\), (iii) \(\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} \delta(k - n)\).

7.3.4

The Fourier transformed equation is \((k^2 + 4) \tilde{u}(k) = 1/\sqrt{2\pi}\), and hence a solution is \(u(x) = \frac{1}{4} e^{-2|x|}\).