Layer Potentials and Boundary Value Problems for Laplacian in Lipschitz Domains with Data in Quasi-Banach Besov Spaces

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Abstract

We study the Dirichlet and Neumann boundary value problems for the Laplacian in a Lipschitz domain Ω, with boundary data in the Besov space $B_s^{p,p}(\partial\Omega)$. The novelty is to identify a way of measuring smoothness for the solution u which allows us to consider the case $p < 1$. This is accomplished by using a Besov-based non-tangential maximal function in place of the classical one. This builds on the works of D. Jerison and C. Kenig [**JFA-1995**] where the case $p > 1$ was treated.

1 Introduction and statement of main results

The well-posedness of the classical Dirichlet problem for the Laplacian in a Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, i.e.

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = f, \end{cases} \tag{1.1}$$

is well-understood at the moment for the datum f in various standard smoothness spaces. More specifically, the case $f \in L^p(\partial\Omega)$ has been treated by B. Dahlberg and C. Kenig in [**4**], [**6**], where they have established the estimate

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p(\partial\Omega)} \tag{1.2}$$

for the optimal range $2 - \varepsilon < p < \infty$, with $\varepsilon = \varepsilon(\partial\Omega)$ (for the case of C^1 domains see [**8**], [**5**]). Hereafter, $\mathcal{N}$ is the classical non-tangential maximal operator defined by

$$\mathcal{N}(u)(x) = \|u\|_{L^\infty(\gamma(x))}, \qquad x \in \partial\Omega, \tag{1.3}$$

where $\gamma(x) := \{y \in \Omega;\ |x - y| < \kappa \operatorname{dist}(y, \partial\Omega)\}$, with $\kappa = \kappa(\Omega) > 1$ fixed, is a cone-like non-tangential approach region with vertex at x.

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More smoothness for the boundary data translates into more smoothness for the solution and, in the case when f belongs to the Sobolev space $L^p_1(\partial\Omega)$, G. Verchota [**26**], and B. Dahlberg and C. Kenig [**6**] have shown that

$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p)\|f\|_{L^p_1(\partial\Omega)} \tag{1.4}$$

for the optimal (dual) range $1 < p < 2 + \varepsilon$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$.

The well-posedness of (1.1) for boundary data in spaces with a fractional amount of smoothness, such as functions in the Besov class $B^{p,p}_s(\partial\Omega)$, has been first studied by D. Jerison and C. Kenig [**14**], where they have proved the *a priori* estimate

$$\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \leq C(\partial\Omega, p, s)\|f\|_{B^{p,p}_s(\partial\Omega)} \tag{1.5}$$

for solutions of (1.1), whenever $0 < s < 1$ and $1 \leq p \leq \infty$ satisfy either of the following three conditions:

$$\begin{aligned} &\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1; \\ &1 \leq p < \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1; \\ &\tfrac{2}{1-\varepsilon} < p \leq \infty \quad \text{and} \quad 0 < s < \tfrac{2}{p} + \varepsilon. \end{aligned} \tag{1.6}$$

A more geometric way of understanding the conditions (1.6) is to identify the space $B^{p,p}_s(\partial\Omega)$ with the point in $\mathbb{R}^2$ with coordinates $(s, 1/p)$. In this scenario, the validity region for (1.5) becomes the hexagonal region below:

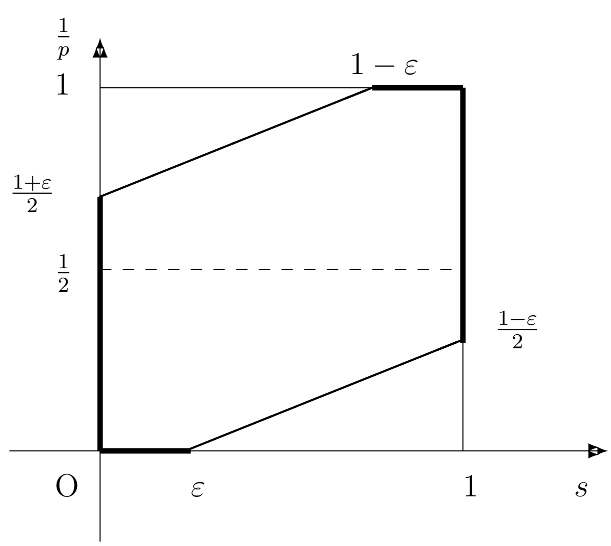

Figure 1.

Though exquisite in their elegance and sharpness, the main results in [**14**] are limited to the case $p \geq 1$ and to Dirichlet boundary conditions.

In this paper we are interested in measuring the smoothness of u in (1.1) when $f \in B_s^{p,p}(\partial\Omega)$ in a new way, by relying on the Besov-based non-tangential maximal function

$$\mathcal{N}_s^q(u)(x) := \|u\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))}, \qquad x \in \partial\Omega. \tag{1.7}$$

One of the main virtues of this approach is that it allows us to consider boundary data from Besov spaces with $p < 1$. The issue of studying elliptic boundary value problems in non-smooth domains with data in quasi-Banach spaces arose in C. Kenig's book **[16]** (cf. Problem 3.2.10 on p. 119 *loc. cit.*). As we shall see momentarily, the consideration of (1.7) also allows us to treat the case of Neumann boundary conditions.

To state our first main result, set $(a)_+ := \max\{a, 0\}$.

Theorem 1.1 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Consider the following boundary value problem:*

$$\begin{cases} \Delta u = 0 \ \text{ in } \Omega, \\ u\Big|_{\partial\Omega} = f \in B_s^{p,p}(\partial\Omega), \\ \mathcal{N}_s^q(u) \in L^p(\partial\Omega), \end{cases} \tag{1.8}$$

where it is assumed that

$$\tfrac{n-1}{n} < p \leq q, \quad \tfrac{n}{1-s} < q \leq \infty, \quad (n-1)(\tfrac{1}{p} - 1)_+ < s < 1. \tag{1.9}$$

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (1.8) is well-posed if, in addition to (1.9), the indices s, p also satisfy one of the following three conditions:

$$\begin{aligned} &\tfrac{2}{1+\varepsilon} < p < \tfrac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1; \\ &\tfrac{2}{2+\varepsilon} < p < \tfrac{2}{1+\varepsilon} \quad \text{and} \quad \tfrac{2}{p} - 1 - \varepsilon < s < 1; \\ &\tfrac{2}{1-\varepsilon} < p \leq \infty \quad \text{and} \ 0 < s < \tfrac{2}{p} + \varepsilon. \end{aligned} \tag{1.10}$$

Furthermore, the solution has the integral representation formula

$$u = \mathcal{D}[(\tfrac{1}{2}I + K)^{-1} f] \ \text{ in } \Omega, \tag{1.11}$$

and satisfies

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p, q, s)\|f\|_{B_s^{p,p}(\partial\Omega)}. \tag{1.12}$$

It is illuminating to point out that the conditions (1.10) amount to the membership of the point with coordinates $(s, 1/p)$ to the pentagon below:

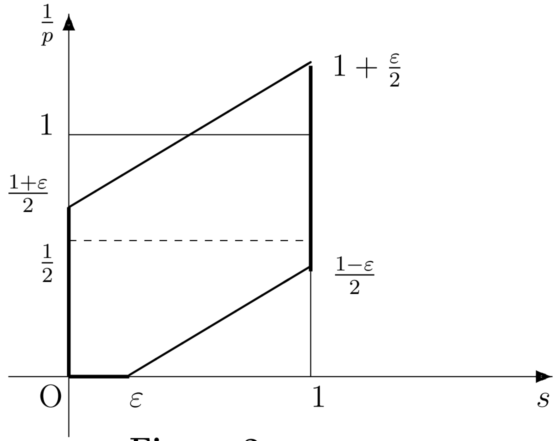

Figure 2.

A few remarks are in order here. For starters, we note that the accompanying estimate (1.12) formally agrees with (1.2) and (1.4). That is, the latter can be thought of as a limiting case of (1.12), corresponding to $q = \infty$ and $s = 0$, $s = 1$, respectively. In fact, it was this observation which gave us the original impetus for measuring the smoothness of the solution using the Besov-based non-tangential maximal function (1.7) when the data is in $B^{p,p}_s(\partial\Omega)$.

Second, the concept of trace $u|_{\partial\Omega}$ is well-defined for the problem under discussion. Indeed, since $\mathcal{N}^q_s(u) \in L^p(\partial\Omega)$ and $B^{q,q}_{s+\frac{n}{q}}(\gamma(x)) \hookrightarrow C^s(\gamma(x)) = C^s(\overline{\gamma(x)})$, the Hölder class, it follows that $u|_{\gamma(x)} \in C^s(\overline{\gamma(x)})$ for a.e. $x \in \partial\Omega$. In particular, it makes sense to talk about $u|_{\partial\Omega}$ as a non-tangential limit, i.e.

$$u(x) := \lim_{y \in \gamma(x)} u(y), \qquad \text{for } a.e. \ x \in \partial\Omega, \tag{1.13}$$

and the corresponding estimate $\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \leq C\|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)}$ holds.

Third, regular elliptic boundary problems in C^∞ domains with data in Besov spaces with $p < 1$ were first treated by H. Triebel in **[24]**. Subsequently, these results have been extended by J. Franke and T. Runst in **[10]**. A good account of these developments can be found in Chapter 3 of **[21]**.

As alluded to before, there is a version of Theorem 1.1 in the case when a Neumann boundary condition is considered. Specifically, we have the following:

Theorem 1.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider the following boundary value problem:*

$$\begin{cases} \Delta u = 0 \ \textit{in} \ \Omega, \\ \partial_\nu u\Big|_{\partial\Omega} = f \in B^{p,p}_{-s}(\partial\Omega), \quad \langle f, 1\rangle = 0, \\ \mathcal{N}^q_{1-s}(u) \in L^p(\partial\Omega), \end{cases} \tag{1.14}$$

where

$$\tfrac{n-1}{n} < p \leq q, \quad \tfrac{n}{s} < q \leq \infty, \quad (n-1)(\tfrac{1}{p}-1)_+ < 1-s < 1. \tag{1.15}$$

Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that (1.14) has a unique, modulo constants, solution if $(1-s, 1/p)$ belongs to the pentagonal region in Figure 2.

In addition, the solution has the integral representation formula

$$u = \mathcal{S}[(-\tfrac{1}{2}I + K^*)^{-1} f] \,+\, \text{const}, \tag{1.16}$$

and (after a normalization) satisfies the estimate

$$\|\mathcal{N}^q_{1-s}(u)\|_{L^p(\partial\Omega)} \leq C(\partial\Omega, p, q, s)\|f\|_{B^{p,p}_{-s}(\partial\Omega)}. \tag{1.17}$$

Although we do not fully pursue this point here, it is possible to show that the well-posedness regions described in Theorems 1.1-t1.2 are in the nature of best possible. This follows by combining our results in §4 (highlighting connections between smoothness measured via the modified non-tangential function (1.7) and membership to global Besov spaces in Ω), with the counterexamples in **[18]**.

One of the main ingredients in the proofs of Theorems 1.1-1.2 is establishing estimates for potential-like singular integral operators. These results are summarized in the next theorem. Throughout the paper, we shall follow the well-established practice of denoting by C generic constants, which may differ from one occurrence to another.

Theorem 1.3 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator*

$$Tf(x) = \int_{\partial\Omega} k(x,y) f(y) d\sigma(y), \qquad x \in \Omega. \tag{1.18}$$

Assume that

$$T1 = const \tag{1.19}$$

and

$$|\nabla_x k(x,y)| \leq C|x-y|^{-n}. \tag{1.20}$$

Then

$$\|\mathcal{N}^q_s(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega)} \tag{1.21}$$

whenever $\frac{n-1}{n} < p \leq q$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $\frac{n}{1-s} < q \leq \infty$.

It should be noted that the class of operators satisfying (1.19)-(1.20) contains the double layer potential as well as the Cauchy operator (in two and higher dimensions; cf. §4 for a discussion). A similar theorem also holds for single layer-like operators. Concretely, we have the following:

Theorem 1.4 *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ and consider the integral operator $Tf(x) = \int_{\partial\Omega} k(x,y)f(y)d\sigma(y)$. Suppose its integral kernel satisfies*

$$|\nabla_x \nabla_y^j k(x,y)| \leq C|x-y|^{-(n-1+j)} \quad \textit{for } j = 0, 1. \tag{1.22}$$

Then

$$\|\mathcal{N}^q_{1-s}(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B^{p,p}_{-s}(\partial\Omega)} \tag{1.23}$$

if $\frac{n-1}{n} < p \leq q$, $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$, and $\frac{n}{s} < q \leq \infty$.

As evidenced by the integral representation formulas (1.11), (1.16), one final major ingredient has to do with the invertibility of (principal value) boundary layer potential operators.

Theorem 1.5 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. Then*

$$\tfrac{1}{2}I + K : B^{p,p}_s(\partial\Omega) \longrightarrow B^{p,p}_s(\partial\Omega) \tag{1.24}$$

is an isomorphism provided $(s, 1/p)$ belongs to the pentagonal region depicted in Figure 2.

Moreover,

$$-\tfrac{1}{2}I + K^* : \{f \in B^{p,p}_{-s}(\partial\Omega);\ \langle f, 1\rangle = 0\} \longrightarrow \{f \in B^{p,p}_{-s}(\partial\Omega);\ \langle f, 1\rangle = 0\} \tag{1.25}$$

is also an isomorphism granted that $(1-s, 1/p)$ belongs to the pentagonal region depicted in Figure 2.

Our methods are quite general and flexible and, in principle, can be adapted to other situations of interest. For example, appropriate versions of Theorems 1.1-1.2 hold for the Lamé system of elastostatics in Lipschitz domains of dimension ≤ 3. Indeed, in this situation, the analogues of (1.2), (1.4) have been established in **[7]** via atomic estimates, which are also the main ingredients in the proof of Theorem 1.5. Also, the Theorems 1.3-1.4 readily extend to the vector-valued case.

Finally, in an upcoming paper, we consider the case of parabolic PDE's in Lipschitz cylinders with lateral data in quasi-Banach Besov spaces (with parabolic anisotropy).

The layout of the paper is as follows. In §2 we recall standard definitions and collect some basic prerequisites. Mapping properties of singular integral operators are established in §3. The class of operators treated is rather general and is modeled upon the behavior of the harmonic double and single layer potential operators. The estimates established here involve our Besov-based nontangential maximal function (1.7). In §4 we explore various connections between this operator and other, more classical, ways of measuring smoothness. The relevant invertibility properties of the boundary versions of the classical harmonic layer potentials are established in §5. Finally, in §6, the proofs of Theorems 1.1-1.2 are carried out.

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2 Basic definitions and notation

Let us start by recalling the Littlewood-Paley definition of Triebel-Lizorkin ($F^{p,q}_s$) and Besov ($B^{p,q}_s$) spaces. Let Φ be the collection of all systems $\{\phi_j\}_{j=0}^{\infty} \subset \mathcal{S}$, the Schwartz class, such that

(i) there exist positive constants A, B, C such that

$$\begin{cases} \mathit{supp}\ \phi_0 \subset \{x;\ |x| \leq A\}, \\ \mathit{supp}\ \phi_j \subset \{x;\ B2^{j-1} \leq |x| \leq C2^{j+1}\} \quad \text{if } j = 1, 2, 3..., \end{cases} \tag{2.1}$$

(ii) for every multi-index α there exists a positive number c_α such that

$$\sup_{x\in\mathbb{R}^n} \sup_{j\in\mathbb{N}} 2^{j|\alpha|} |\partial^\alpha \phi_j(x)| \leq c_\alpha, \tag{2.2}$$

(iii)

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \text{ for every } x \in \mathbb{R}^n. \tag{2.3}$$

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$ and fix some family $\{\phi_j\}_{j=0}^{\infty} \in \Phi$. Also, let $\mathcal{F}$ denote the Fourier transform in $\mathbb{R}^n$.

(i) If $0 < p < \infty$ then the Triebel-Lizorkin scale is defined by

$$F^{p,q}_s(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}';\quad \|f\|_{F^{p,q}_s(\mathbb{R}^n)} := \Big\| \Big(\sum_{j=0}^{\infty} |2^{sj} \mathcal{F}^{-1}(\phi_j \mathcal{F} f)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\}. \tag{2.4}$$

(ii) If $0 < p \leq \infty$ then the Besov scale is

$$B^{p,q}_s(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}';\quad \|f\|_{B^{p,q}_s(\mathbb{R}^n)} := \Big(\sum_{j=0}^{\infty} \|2^{sj} \mathcal{F}^{-1}(\phi_j \mathcal{F} f)\|^q_{L^p(\mathbb{R}^n)} \Big)^{1/q} < \infty \Big\}. \tag{2.5}$$

As is well-known, a different choice of the system $\{\phi_j\}_{j=0}^{\infty} \in \Phi$ yields the same spaces (2.4)-(2.5), albeit equipped with equivalent norms.

For the range of indices $n/(n+1) < p, q < \infty$ and $n(1/p - 1)_+ < s < 1$, an intrinsic definition for membership to $B^{p,q}_s(\mathbb{R}^n)$ is obtained by requiring that

$$\|f\|_{B^{p,q}_s(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^n} \frac{\|f(\cdot + t) - f(\cdot)\|^q_{L^p(\mathbb{R}^n)}}{|t|^{n+sq}}\, dt \right)^{1/q} < +\infty. \tag{2.6}$$

It has been long known that many classical smoothness spaces are encompassed by the above two scales. For example,

$$
\begin{array}{lll}
C^s(\mathbb{R}^n) = B_s^{\infty,\infty}(\mathbb{R}^n), & 0 < s \notin \mathbf{Z}, & \\
L^p(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), & 1 < p < \infty, & \\
L^p_k(\mathbb{R}^n) = F_k^{p,2}(\mathbb{R}^n), & 1 < p < \infty, & k = 1, 2..., \\
H^p_{at}(\mathbb{R}^n) = F_0^{p,2}(\mathbb{R}^n), & 0 < p \leq 1, & \\
H^{1,p}_{at}(\mathbb{R}^n) = F_1^{p,2}(\mathbb{R}^n), & 0 < p \leq 1. &
\end{array}
$$

Next, recall that $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz if there exists a finite constant $C > 0$ such that $|\varphi(x) - \varphi(y)| \leq C|x - y|$, for every $x, y \in \mathbb{R}^{n-1}$. According to a fundamental theorem of Rademacher, $\nabla\varphi$ exists a.e. and the best constant in the previous inequality is $\|\nabla\varphi\|_{L^\infty(\mathbb{R}^{n-1})}$.

Going further, a bounded domain $\Omega \subset \mathbb{R}^n$, with connected boundary, is called *Lipschitz* if:

i) the boundary $\partial\Omega$ can be covered by a finite family of open (appropriately rotated) cylinders $\{Z_i\}_{i=1}^m$ in $\mathbb{R}^n$;

ii) for each i, there exists a Lipschitz function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ so that $2\|\varphi_i\|_{L^\infty}$ is less than the height of Z_i and, if $2Z_i$ denotes the concentric double of Z_i, in the rectangular coordinate system defined by Z_i one has

$$
\begin{aligned}
\Omega \cap 2Z_i &= \{x = (x', x_n);\, \varphi_i(x') < x_n\} \cap 2Z_i, \\
\partial\Omega \cap 2Z_i &= \{x = (x', x_n);\, \varphi_i(x') = x_n\} \cap 2Z_i.
\end{aligned}
$$

In the sequel, we shall say that a constant depends on the Lipschitz character of Ω if its size is controlled in terms of m, the number of cylinders $\{Z_i\}_i$, the size of these cylinders, $\sup\{\|\nabla\varphi_i\|_{L^\infty};\, 1 \leq i \leq m\}$. See, e.g., **[26]** for more details.

We let $d\sigma$ denote the canonical surface measure on $\partial\Omega$ so that ν, the outward unit normal to $\partial\Omega$, is well-defined $d\sigma$-a.e.

The scales of Besov and Triebel-Lizorkin spaces can be naturally transported from $\mathbb{R}^{n-1}$ to the boundary of a (bounded) Lipschitz domain Ω via pull-back and a partition of unity. We denote them by $B_s^{p,q}(\partial\Omega)$ and by $F_s^{p,q}(\partial\Omega)$, respectively. More specifically, if $(n-1)/n < p, q < \infty$, $(n-1)(1/p - 1)_+ < s < 1$, when Ω is the region from $\mathbb{R}^n$ lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, we define $B_s^{p,q}(\partial\Omega)$ as the space of functions f for which the assignment $x \mapsto f(x, \phi(x))$ belongs to $B_s^{p,q}(\mathbb{R}^{n-1})$. This definition then readily extends to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a standard partition of unity argument. The case when $p = q = \infty$ corresponds to the usual (non-homogeneous) Hölder spaces $C^s(\partial\Omega)$. Similar considerations apply to $F_s^{p,q}(\partial\Omega)$.

Denote by $L^p_1(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ whose tangential gradients are in $L^p(\partial\Omega)$, $1 < p < \infty$. Sobolev and Besov spaces with positive, fractional smoothness can then be defined via complex and real interpolation methods, respectively, i.e.

$$L^p_\theta(\partial\Omega) := [L^p(\partial\Omega), L^p_1(\partial\Omega)]_\theta,\ 0<\theta<1,\ 1<p<\infty,$$
$$B^{p,q}_\theta(\partial\Omega) := (L^p(\partial\Omega), L^p_1(\partial\Omega))_{\theta,q}\,, \quad \text{with } 0<\theta<1,\ 1<p,q<\infty.$$

Next, for a Lipschitz domain $\Omega\subset\mathbb{R}^n$, we let $B^{p,q}_s(\Omega)$, $1\leq p,q\leq\infty$, $s>0$, consist of restrictions to Ω of functions from $B^{p,q}_s(\mathbb{R}^n)$. Recall that the trace operator

$$\mathrm{Tr}: B^{p,p}_s(\Omega)\longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{2.7}$$

is well defined, bounded and onto if $1\leq p\leq\infty$ and $\frac{1}{p}<s<1+\frac{1}{p}$. This also has a bounded right inverse whose operator norm is controlled exclusively in terms of p, s and the Lipschitz character of Ω.

Regarding Besov spaces with a negative amount of smoothness, if Ω is the domain in $\mathbb{R}^n$ above the graph of a Lipschitz function $\phi:\mathbb{R}^{n-1}\to\mathbb{R}$, we agree that

$$f\in B^{p,q}_{-s}(\partial\Omega)\Longleftrightarrow f(x,\phi(x))\sqrt{1+|\nabla\phi(x)|^2}\in B^{p,q}_{-s}(\mathbb{R}^{n-1}),$$

whenever $(n-1)/n<p,q<\infty$, $(n-1)(1/p-1)_+<1-s<1$. As before, this definition is then extended to the case of (bounded) Lipschitz domains in $\mathbb{R}^n$ via a simple partition of unity argument. In particular, for $-1<s<0$ and $1<p,q<\infty$,

$$B^{p,q}_s(\partial\Omega)=(B^{p',q'}_{-s}(\partial\Omega))^*, \qquad 1/p+1/p'=1,\ 1/q+1/q'=1, \tag{2.8}$$

where the duality pairing between $f\in B^{p,q}_s(\partial\Omega)$ and $g\in B^{p',q'}_{-s}(\partial\Omega)$ is (a natural extension of) $\int_{\partial\Omega}fg\,d\sigma$. We also refer to **[21]**, **[24]**, **[1]**, **[14]**, **[20]**, for a more detailed exposition of these and other related matters, such as embedding theorems.

In the sequel, we shall also use atomic characterization of the Besov spaces $B^{p,p}_s(\partial\Omega)$. These are a straightforward adaptation of the Euclidean results from **[11]**, given the definitions we adopt in this paper. Specifically, let us assume that $(n-1)/n<p<\infty$ and that $(n-1)(\frac{1}{p}-1)_+<s<1$. A function a is called an atom for $B^{p,p}_s(\partial\Omega)$ if

$$\begin{aligned}&(1)\ \exists S_r - \text{surface ball}: supp(a)\subseteq S_r,\\ &(2)\ \|a\|_{L^\infty(\partial\Omega)}\leq r^{s-\frac{n-1}{p}},\\ &(3)\ \|\nabla_{tan}a\|_{L^\infty(\partial\Omega)}\leq r^{s-\frac{n-1}{p}-1}.\end{aligned}$$

Here and elsewhere, $\nabla_{tan}=\nabla-\nu\partial_\nu$ will denote the tangential gradient on $\partial\Omega$. Also, a surface ball $S_r(x)$ is any set of the form $B_r(x)\cap\partial\Omega$, with $x\in\partial\Omega$ and $0<r<\operatorname{diam}\Omega$. Parenthetically, let us note that (1) & (3) $\Rightarrow$ (2). Then

$$\|f\|_{B^{p,p}_s(\partial\Omega)}\approx\inf\Big\{\Big(\sum_j|\mu_j|^p\Big)^{1/p};\ f=\sum_j\mu_ja_j,\ a_j \text{ are } B^{p,p}_s(\partial\Omega) \text{ atoms, } \{\mu_j\}_j\in\ell^p\Big\}. \tag{2.9}$$

Similarly, there are atomic decompositions for $B^{p,p}_{-s}(\partial\Omega)$. Concretely, let us assume that $(n-1)/n < p < \infty$ and $(n-1)(\frac{1}{p}-1)_+ < 1-s < 1$, and call a an atom for $B^{p,p}_{-s}(\partial\Omega)$ if

$$
\begin{aligned}
&(1)\ \exists S_r - \text{surface ball} : supp(a) \subseteq S_r,\\
&(2)\ \|a\|_{L^\infty(\partial\Omega)} \le r^{-s-\frac{n-1}{p}},\\
&(3)\ \int_{\partial\Omega} a\, d\sigma = 0.
\end{aligned}
$$

Then the Euclidean results from **[11]** lifted to $\partial\Omega$ give

$$
\|f\|_{B^{p,p}_{-s}(\partial\Omega)} \approx \inf\Big\{ \Big(\sum_j |\mu_j|^p\Big)^{1/p};\ f = \sum_j \mu_j a_j,\ a_j \text{ are } B^{p,p}_{-s}(\partial\Omega) \text{ atoms, } \{\mu_j\}_j \in \ell^p \Big\}. \tag{2.10}
$$

As far as the Hardy spaces $H^p_{at}(\partial\Omega)$, $\frac{n-1}{n} < p \le 1$, are concerned, call a an atom for $H^p_{at}(\partial\Omega)$, if

$$
\begin{aligned}
&(i)\ \exists S_r - \text{surface ball} : supp(a) \subseteq S_r,\\
&(ii)\ \|a\|_{L^\infty(\partial\Omega)} \le r^{-\frac{n-1}{p}},\\
&(iii)\ \int_{\partial\Omega} a\, d\sigma = 0.
\end{aligned}
$$

Then

$$
H^p_{at}(\partial\Omega) := \Big\{ \sum_j \lambda_j a_j;\ a_j \text{ is an } H^p_{at}(\partial\Omega) \text{ atom, } \{\lambda_j\}_j \in \ell^p \Big\}. \tag{2.11}
$$

See, e.g., **[3]** for the more general setting of homogeneous spaces.

For $H^{1,p}_{at}(\partial\Omega)$, where $\frac{n-1}{n} < p \le 1$, a molecule is a function $B : \partial\Omega \to \mathbb{R}$ such that $\exists\, x_0 \in \partial\Omega$, $\exists\, \eta > 0$, $\exists\, r > 0$ and $\exists\, C < \infty$ for which:

(i) there holds

$$
\|\nabla_{tan} B\|_{L^q(S_r(x_0))} + r^{-1}\|B\|_{L_q(S_r(x_0))} \le C r^{(n-1)(\frac{1}{q}-\frac{1}{p})}, \tag{2.12}
$$

(ii) $\forall \rho > 0$, $\forall x^* \in \partial\Omega$ such that $dist(S_\rho(x^*), S_r(x_0)) \ge \rho$, there holds

$$
\|\nabla_{tan} B\|_{L^q(S_\rho(x^*))} + \rho^{-1}\|B\|_{L_q(S_\rho(x^*))} \le C \rho^{(n-1)(\frac{1}{q}-\frac{1}{p})} \left(\frac{\rho}{r}\right)^{-\eta}, \tag{2.13}
$$

where $q > max\{1, p\}$.

It is not too difficult to show that $B \in H^{1,p}_{at}(\partial\Omega)$, with norm controlled in terms of C. In fact, it can be proved that a molecular decomposition of Hardy spaces similar in spirit to the above atomic decomposition holds; we leave the details to the interested reader. Here we only want to record a useful connection between $H^{1,p}_{at}(\partial\Omega)$ and $H^p_{at}(\partial\Omega)$.

Theorem 2.1 *(cf.* **[19]***) For every Lipschitz domain* $\Omega \subset \mathbb{R}^n$,

$$H^{1,p}_{at}(\partial\Omega) = \{f \in B^{1,1}_{1-(n-1)(\frac{1}{p}-1)}(\partial\Omega);\ \nabla_{tan} f \in H^p_{at}(\partial\Omega)\}. \tag{2.14}$$

We now proceed to record some important interpolation results. By $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,q}$ we will denote the complex and real methods of interpolation, respectively.

Theorem 2.2 *(cf.* **[12]***) Let* μ *be a fixed Borel measure and denote by* $L^p(\mu)$ *the corresponding Lebesgue spaces. Also, let* X_0, X_1 *be a couple of quasi-Banach spaces and let* $X_\theta = [X_0, X_1]_\theta$, $0 < \theta < 1$, *be the complex interpolation intermediate spaces.*

Finally, let $\mathcal{T}$ *be a sublinear operator from* $X_0 + X_1$ *into* $L^{p_0}(\mu) + L^{p_1}(\mu)$, $1 \leq p_0, p_1 \leq \infty$, *such that*

$$\|\mathcal{T}x\|_{L^{p_0}(\mu)} \leq A_0 \|x\|_{X_0}, \quad \textit{for } x \in X_0, \tag{2.15}$$

and

$$\|\mathcal{T}x\|_{L^{p_1}(\mu)} \leq A_1 \|x\|_{X_1}, \quad \textit{for } x \in X_1, \tag{2.16}$$

for some finite, positive constants A_0 *and* A_1. *Then*

$$\|\mathcal{T}x\|_{L^{p_\theta}(\mu)} \leq A_0^{1-\theta} A_1^\theta \|x\|_{X_\theta}, \quad \textit{for } \ x \in X_\theta, \ \textit{with } \tfrac{1}{p_\theta} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}. \tag{2.17}$$

Theorem 2.3 *(cf.* **[19]***) Let* $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1, q_0, q_1 \leq \infty$, $p_0 + q_0 < \infty$, $p_1 + q_1 < \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ *and* $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. *Then*

$$[F^{p_0,q_0}_{\alpha_0}(\mathbb{R}^n), F^{p_1,q_1}_{\alpha_1}(\mathbb{R}^n)]_\theta = F^{p,q}_\alpha(\mathbb{R}^n). \tag{2.18}$$

Theorem 2.4 *(cf.* **[24]***) Let* $\alpha_0, \alpha_1 \in \mathbb{R}$, $\alpha_0 \neq \alpha_1$, $0 < p < \infty$, $0 < q_0, q_1 \leq \infty$, $0 < \theta < 1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. *Then*

$$(F^{p,q_0}_{\alpha_0}(\mathbb{R}^n), F^{p,q_1}_{\alpha_1}(\mathbb{R}^n))_{\theta,q} = B^{p,q}_\alpha(\mathbb{R}^n). \tag{2.19}$$

An important remark is that appropriate versions of the last two theorems above continue to hold in the case when the underlying Euclidean space is replaced by the boundary of a Lipschitz domain –this is more or less immediate from the various definitions adopted in this paper. In the sequence, we shall tacitly use this without any special mention. The interested reader is further referred to **[27]**, **[25]**, for a more detailed discussion in this regard.

We conclude our review of interpolation with (a version of) the celebrated theorem of E. Stein for analytic families of operators.

Theorem 2.5 *(cf.* **[2]***) Let* (A_0, A_1) *be an interpolation pair of complex Banach spaces and set* $\mathcal{X} = A_0 \bigcap A_1$, $\mathcal{X}_s = [A_0, A_1]_s$, *for* $0 \leq s \leq 1$. *Analogously, let* (B_0, B_1) *be another interpolation pair of complex Banach spaces; set* $\mathcal{Y} = B_0 \bigcap B_1$ *and* $\mathcal{Y}_s = [B_0, B_1]_s$, *for* $0 \leq s \leq 1$.

Next, let L_z be a family of linear operators defined in $\mathcal{X}$, with values in $\mathcal{Y}$, indexed by a complex parameter z, with $0 \leq \Re ez \leq 1$. Assume that $l(L_z f)$ is continuous and bounded in $0 \leq \Re ez \leq 1$, and analytic in $0 < \Re ez < 1$ for every $f \in \mathcal{X}$ and every continuous linear functional l on $\mathcal{Y}$. Assume further that for $\Re ez = 0$ and $f \in \mathcal{X}$

$$\|L_z f\|_{\mathcal{Y}_0} \leq c_0 \|f\|_{\mathcal{X}_0}, \tag{2.20}$$

and for $\Re ez = 1$ and $f \in \mathcal{X}$

$$\|L_z f\|_{\mathcal{Y}_1} \leq c_1 \|f\|_{\mathcal{X}_1}. \tag{2.21}$$

Then for $0 < \Re ez = s < 1$ there exists $c = c(s, q_0, q_1, c_0, c_1)$ such that

$$\|L_z f\|_{\mathcal{Y}_s} \leq c \|f\|_{\mathcal{X}_s}, \tag{2.22}$$

uniformly for $f \in \mathcal{X}$.

Two results dealing with the membership of (derivatives of) harmonic functions to weighted $L^p(\Omega)$ spaces – when the weight is a power of δ, the distance function to $\partial\Omega$ – are going to be of importance for us. These are recorded below.

Theorem 2.6 *(cf. **[14]**) Suppose u is a function defined in the Lipschitz domain Ω. For $0 < \alpha < 1$, a nonnegative integer k, and $1 \leq p \leq \infty$, consider the following statements:*
(a) u belongs to $B^p_{k+\alpha}(\Omega)$;
(b) $\delta^{1-\alpha}|\nabla^{k+1}u| + |\nabla^k u| + |u|$ belongs to $L^p(\Omega)$.
Then
(i) (b) $\Rightarrow$ (a), for any $u \in L^1_{loc}(\Omega)$ such that $\nabla^j u \in L^1_{loc}(\Omega)$, $1 \leq j \leq k+1$;
(ii) (b) $\Leftrightarrow$ (a), if u is harmonic in Ω.

Above, $\nabla^j u$ stands for the collection of all partial derivatives $\partial^\alpha u$ with $|\alpha| = j$.

Theorem 2.7 *(cf. **[22]**) Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain in $\mathbb{R}^n$, $\mathcal{O} \subset \Omega$ a relatively compact subset, $0 < p < \infty$, $q > -1$, and $k \in \mathbf{N}$. Then there is a constant C such that for every harmonic function u in Ω we have*

$$C^{-1} \int_\Omega \delta^q |\delta^k \nabla^k u|^p \, dx \leq C \int_\Omega \delta^q |u|^p \, dx \leq C \int_\Omega \delta^q |\delta^k \nabla^k u|^p \, dx + C \sup\{|\nabla^j u(x)|^p;\ x \in \mathcal{O},\ 0 \leq j \leq k-1\}. \tag{2.23}$$

We conclude this section by recalling the definitions and some of the most basic properties of the classical harmonic layer potentials for a Lipschitz domain $\Omega \subset \mathbb{R}^n$. With $E(x)$ denoting the canonical radial fundamental solution for the Laplace operator Δ in $\mathbb{R}^n$, we define the single and double layer potential operators by

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x-y)\, f(y)\, d\sigma(y), \qquad x \in \Omega, \tag{2.24}$$

and, with $\partial_\nu := \nu \cdot \nabla$,

$$\mathcal{D}f(x) := \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)]\, f(y)\, d\sigma(y), \qquad x \in \Omega, \tag{2.25}$$

respectively. As is well-known (cf., e.g., **[6]**, **[26]**),

$$\partial_\nu \mathcal{S}\Big|_{\partial\Omega} = -\tfrac{1}{2}I + K^* \quad \text{and} \quad \mathcal{D}\Big|_{\partial\Omega} = \tfrac{1}{2}I + K, \tag{2.26}$$

where

$$Kf(x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_y}[E(x-y)]\, f(y)\, d\sigma(y), \qquad x \in \partial\Omega, \tag{2.27}$$

and K^* is the formal adjoint of K. Here *p.v.* indicates that the integral is taken in the principal value sense, i.e. removing balls of radius ε and passing to the limit, $\varepsilon \to 0$.

3 Mapping properties of singular integral operators

We debut with the

Proof of Theorem 1.3. The basic idea of the proof is to consider the sublinear operator

$$\mathcal{T}f = \mathcal{N}_s^q(Tf) : B_s^{p,p}(\partial\Omega) \to L^p(\partial\Omega) \tag{3.1}$$

and interpolate −using Theorem 2.2− between $(n-1)/n < p \leq 1$ and $p = q \in [1, \infty]$. We proceed by analyzing several cases starting with:

Case 1. $1 \leq p = q \leq \infty$**.** The estimate (1.21) will be a consequence of the following sequence of inequalities

$$\|\mathcal{N}_s^p(Tf)\|_{L^p(\partial\Omega)} \leq C\|\delta^{1-(s+\frac{1}{p})}|\nabla Tf|\|_{L^p(\Omega)}, \tag{3.2}$$

$$\|\delta^{1-(s+\frac{1}{p})}|\nabla Tf|\|_{L^p(\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{3.3}$$

each of which will be proved separately (recall that δ denotes the distance to $\partial\Omega$). In fact, the inequality (3.2) is proved later, in (*ii*) of Corollary 4.3.

The estimate (3.3) will be proved in three steps. The idea is to obtain the result for $p = q = 1$, then for $p = q = \infty$ and, finally, use Stein's interpolation theorem for analytic families of operators (Theorem 2.5) to cover the range in between.

Consider first the case $p = q = 1$. We shall prove that if

$$|\nabla_x^k k(x,y)| \leq C|x-y|^{-(n+k-1)}, \quad k = 1, 2, ..., N, \tag{3.4}$$

for some positive integer N, then

$$\|\delta^{k-1-s}|\nabla^k T f|\,\|_{L^1(\Omega)} \leq C\|f\|_{B_s^{1,1}(\partial\Omega)}, \tag{3.5}$$

(this result is stated in a slightly more general form than the one we need, which would correspond to the case $N = 1$).

Turning to the actual proof of (3.5), first recall the intrinsic characterization of the Besov space $B_s^{1,1}(\partial\Omega)$

$$\|f\|_{B_s^{1,1}(\partial\Omega)} = \|f\|_{L^1(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|}{|x-y|^{n-1+s}} d\sigma(x)d\sigma(y). \tag{3.6}$$

The estimate we seek has local character. Thus, using a partition of unity, we may assume that the support of f is included in a coordinate patch where $\partial\Omega$ is represented by the graph of the Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$. Assuming that this is the case, we make a change of variables and set $\tilde{f}(x) := f(x, \phi(x))$, extended by zero outside of the support. In particular, $\tilde{f} \in B_s^{1,1}(\mathbb{R}^{n-1})$.

Thanks to (1.19) we have that $\nabla^k T$ annihilates constants. In concert with the assumption (3.4) on the kernel, this implies that

$$\int_\Omega \delta^{k-1-s}(x)|\nabla^k T f(x)|dx \tag{3.7}$$

can be controlled by a multiple of

$$\int_0^\infty t^{k-1-s} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|\tilde{f}(x)-\tilde{f}(y)|}{(|x-y|^2+t^2)^{\frac{n+k-1}{2}}}\, dxdy. \tag{3.8}$$

In turn, this can be majorized by

$$C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} |\tilde{f}(x)-\tilde{f}(y)|\Big(\int_0^\infty \frac{t^{k-1-s}}{(|x-y|+t)^{n+k-1}}dt\Big)\, dxdy. \tag{3.9}$$

Making the change of variables $r = \frac{t}{|x-y|}$, $r \in (0,\infty)$, $dt = |x-y|dr$, we can further bound the innermost integral above by

$$\int_0^\infty \frac{t^{k-1-s}}{(|x-y|+t)^{n+k-1}}dt = |x-y|^{-s+1-n}\int_0^\infty \frac{r^{k-1-s}}{(1+r)^{n+k-1}}dr \leq C|x-y|^{-s+1-n} \tag{3.10}$$

for some finite constant $C = C(k, s, n)$. Thus,

$$\begin{aligned}\int_\Omega \delta^{k-1-s}(x)|\nabla^k T f(x)|\,dx &\leq C\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|\tilde{f}(x)-\tilde{f}(y)|}{|x-y|^{n+s-1}}\, dxdy \\ &\leq C\|\tilde{f}\|_{B_s^{1,1}(\mathbb{R}^{n-1})} \leq C\|f\|_{B_s^{1,1}(\partial\Omega)}\end{aligned} \tag{3.11}$$

as desired; this completes the proof of (3.5) in the case $p = q = 1$.

Next we turn our attention to the case $p = q = \infty$. The goal is to show that $\delta^{k-s}|\nabla^k T f| \in L^\infty(\Omega)$ for $f \in B_s^{\infty,\infty}(\partial\Omega)$, with appropriate control of the norms. To this end, let x^* denote the point on $\partial\Omega$ such that $|x - x^*| = \delta(x)$. Then we may write

$$\nabla^k T f(x) = \int_{\partial\Omega} \nabla_x^k k(x,y)(f(y) - f(x^*))\, d\sigma(y). \tag{3.12}$$

Since $f \in B_s^{\infty,\infty}(\partial\Omega) = C^s(\partial\Omega)$ we have $|f(y) - f(x^*)| \leq \|f\|_{B_s^{\infty,\infty}(\partial\Omega)} |y - x^*|^s$, $y \in \partial\Omega$. With this in mind and recalling the assumptions (3.4) on the kernel, we split the last integral into two parts, I_1, I_2, corresponding to $y \in S_{100r}(x^*)$ and $y \in \partial\Omega \setminus S_{100r}(x^*)$, respectively, where $r := |x - x^*|$. We have

$$\begin{aligned} |I_1| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{S_{100r}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k}}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{S_{100r}(x^*)} |x-y|^{s-n+1-k}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{S_{100r}(x^*)} r^{s-n+1-k} d\sigma(y) \leq C\, r^{s-k}\, \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \end{aligned} \tag{3.13}$$

On the other hand,

$$\begin{aligned} |I_2| &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{100r}(x^*)} \frac{|y-x^*|^s}{|x-y|^{n-1+k}}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{\partial\Omega\setminus S_{100r}(x^*)} |x^*-y|^{s-n+1-k}\, d\sigma(y) \\ &\leq C\|f\|_{B_s^{\infty,\infty}(\partial\Omega)} \int_{100r}^{\infty} \rho^{s-1-k} d\rho \leq C\, r^{s-k}\, \|f\|_{B_s^{\infty,\infty}(\partial\Omega)}. \end{aligned} \tag{3.14}$$

These inequalities complete the argument for the case $p = q = \infty$.

Now the estimate

$$\begin{gathered} \delta^{k-\frac{1}{p}-s}|\nabla^k T f| \in L^p(\Omega) \quad \text{for} \quad f \in B_s^{p,p}(\partial\Omega), \\ k = 1, 2, ..., N, \quad s \in (0,1), \quad p \in [1, \infty], \end{gathered} \tag{3.15}$$

will follow from Theorem 2.5. The details are as follows. Consider the family of operators

$$L_z f := \delta^{k-1+z-[(1-z)s_0+zs_1]}|\nabla^k T f|, \tag{3.16}$$

so that

$$\begin{aligned} \Re e z = 0 &\Rightarrow |L_0 f| = \delta^{k-1-s_0}|\nabla^k T f|, \\ \Re e z = 1 &\Rightarrow |L_1 f| = \delta^{k-s_1}|\nabla^k T f|. \end{aligned}$$

Our results for $p = q = 1$ and $p = q = \infty$ lead to the conclusion that the operators

$$L_0 : B_{s_0}^{1,1}(\partial\Omega) \to L^1(\Omega),$$
$$L_1 : B_{s_1}^{\infty,\infty}(\partial\Omega) \to L^\infty(\Omega),$$

are well-defined and bounded. In turn, thanks to Theorem 2.5, and standard complex interpolation results for the Besov scale (cf. **[1]**; here $s_0 \neq s_1$ is needed), we may conclude that

$$\delta^{k-\frac{1}{p}-s}|\nabla^k Tf| : B_s^{p,p}(\partial\Omega) \to L^p(\Omega) \tag{3.17}$$

is also well defined and bounded granted that $s \in (0,1)$ and $p \in [1,\infty]$.

Case 2. $(n-1)/n < p \leq 1$**.** In this situation we use the atomic characterization (2.9) of Besov spaces. As is well-known, the desired estimate, i.e.

$$\|\mathcal{N}_s^q(Tf)\|_{L^p(\partial\Omega)} \leq C\|f\|_{B_s^{p,p}(\partial\Omega)}, \tag{3.18}$$

is a direct consequence of

$$\|\mathcal{N}_s^q(Ta)\|_{L^p(\partial\Omega)} \leq C, \tag{3.19}$$

for every $B_s^{p,p}(\partial\Omega)$-atom a, with a finite constant $C > 0$ independent of the atom. Turning our attention to (3.19), we will split the discussion into two parts: near and away from $\operatorname{supp} a$.

Near the support of a we employ a rescaling technique. Specifically,

$$\mathcal{N}_s^q(Ta)(x) = \|Ta\|_{B^{q,q}_{s+\frac{n}{q}}(\gamma(x))}$$

can be bounded by

$$r^{-\frac{n-1}{p}}\|T\tilde{a}\|_{B^{q,q}_{s+\frac{n}{q}}(\Omega)},$$

where $\tilde{a} := r^{\frac{n-1}{p}}a$ is a $B^{q,q}_{s+\frac{n-1}{q}}(\partial\Omega)$ atom.

Upon noticing that $s + \frac{n-1}{q} \in (0,1)$ and recalling that $q > 1$, we may now invoke what we have already proved in the case $1 \leq p \leq q$, i.e. (3.17), along with Theorem 2.6, in order to conclude that

$$r^{-\frac{n-1}{p}}\|T\tilde{a}\|_{B^{q,q}_{s+\frac{n}{q}}(\Omega)} \leq Cr^{-\frac{n-1}{p}}\|\tilde{a}\|_{B^{q,q}_{s+\frac{n-1}{q}}(\partial\Omega)} < Cr^{-\frac{n-1}{p}}.$$

The constant $C = C(\partial\Omega, n, s, p, q)$ above does not depend on the particular atom a. Then $\mathcal{N}_s^q(Ta) \leq Cr^{-\frac{n-1}{p}}$ pointwise on S_{10r} (where $\operatorname{supp} a \subseteq S_r$). Consequently,

$$\|\mathcal{N}^q_s(Ta)\|_{L^p(S_{10r})} \leq C, \tag{3.20}$$

which is of the right order.

Next, let us estimate the contribution away from supp a. Without any loss of generality we may assume that $0 \in \partial\Omega$ and supp $a \subset S_r(0)$. For every $y, z \in \Omega$ we have

$$\begin{aligned} |Ta(y) - Ta(z)| &= \Big|\int_{\partial\Omega} (k(y,w) - k(z,w))a(w)\,d\sigma(w)\Big| \\ &= \Big|\int_{\partial\Omega}\int_0^1 \frac{d}{dt}k((1-t)y+tz,w)dt\ a(w)\,d\sigma(w)\Big|. \end{aligned}$$

Keeping in mind the hypotheses (1.20) we make on the kernel, the last integral above is bounded by

$$C|z-y|\int_{\partial\Omega}\int_0^1 \frac{dt}{[((1-t)y+tz)^2+w^2]^{n/2}}\ |a(w)|\,d\sigma(w).$$

This, in turn, can be majorized by

$$C|z-y|r^{s+(1-\frac{1}{p})(n-1)}\int_0^1 \frac{dt}{[(1-t)y+tz]^n}, \tag{3.21}$$

where r is the radius of the minimal surface ball containing the support of the atom a. Relying on the intrinsic characterization of Besov spaces, we may write

$$\Big(\mathcal{N}^q_s(Ta)(x)\Big)^q = \int_{\gamma(x)}\int_{\gamma(x)} \frac{|Ta(y)-Ta(z)|^q}{|y-z|^{2n+sq}}\,dydz + \|Ta\|^q_{L^q(\gamma(x))}. \tag{3.22}$$

The last term above is residual and we leave the details of estimating it to the interested reader. As regards the next-to-the-last one, using (3.21) we estimate it by

$$Cr^{q(s+(1-\frac{1}{p})(n-1))}\int_{\gamma(x)}\int_{\gamma(x)}\Big(|y-z|^{1-s-\frac{2n}{q}}\int_0^1 \frac{dt}{((1-t)y+tz)^n}\Big)^q dydz. \tag{3.23}$$

Thus, matters have been reduced to estimating the above integral. A general result to that effect (whose proof is postponed for a little while) is recorded below.

Lemma 3.1 *For $0<s<1$ there holds*

$$\int_{\gamma(x)}\int_{\gamma(x)}\Big(|y-z|^{1-s-\frac{2n}{q}}\int_0^1 \frac{dt}{[(1-t)y+tz]^n}\Big)^q dydz \leq C\frac{1}{|x|^{sq+(n-1)q}}, \tag{3.24}$$

where the constant C is independent of x and $q > \frac{n}{1-s}$.

Returning to the main stream discussion, i.e. the sequence of inequalities (3.22)-(3.23), we may therefore bound

$$\int_{x\in\partial\Omega\setminus S_{10r}} (\mathcal{N}_s^q(Ta)(x))^p\, d\sigma(x) \tag{3.25}$$

by

$$C\, r^{(s+(1-\frac{1}{p})(n-1))p} \int_{|x|\geq 10r} \frac{dx}{|x|^{(s+n-1)p}} = C\, r^{[s+(1-\frac{1}{p})(n-1)]p} \int_{10r}^{\infty} \frac{\rho^{n-2}d\rho}{\rho^{(s+n-1)p}} = const, \tag{3.26}$$

provided the last integral converges. This is indeed the case, granted that $s > (n-1)(\frac{1}{p}-1)$, which is part of our assumptions. This observation finishes the proof in the case when $(n-1)/n < p \leq 1$, modulo the

Proof of Lemma 3.1. Pulling everything back to the Euclidean model, it suffices to show (after a translation and a rotation) that

$$\int_{\gamma(0)}\int_{\gamma(0)} \Big(\int_0^1 \frac{|y-z|^{1-s-\frac{2n}{q}}\, dt}{(1-t)^n y_n^n + t^n z_n^n + 1}\Big)^q\, dydz \leq C\frac{1}{|x|^{sq+(n-1)q}}. \tag{3.27}$$

By symmetry, there is no loss of generality in assuming that $y_n \leq z_n$, which allows us to bound the corresponding piece by

$$\int_{\gamma(0)}\int_{\gamma(0)} \Big(\int_0^1 \frac{|y-z|^{1-s-\frac{2n}{q}}\, dt}{t^n z_n^n + 1}\Big)^q\, dydz. \tag{3.28}$$

Switching to polar coordinates $y - z = r\omega$ where $r \in (0,\infty)$ and $\omega \in S_{n-1}$, we may rewrite (3.28) as

$$\int_{\gamma(0)} \Big[\int_0^1 \frac{dt}{t^n z_n^n + 1}\Big]^q \int_0^{cz_n} r^{n-1+q-sq-2n} drdz \tag{3.29}$$

which, for $q > \frac{n}{1-s}$, is controlled by

$$C\int_0^{\infty} z_n^{q-sq-n}\Big[\int_0^1 \frac{dt}{t^n z_n^n + 1}\Big]^q\, dz_n. \tag{3.30}$$

After the change of variables $\tau = tz_n$, (3.30) becomes

$$C\int_0^{\infty} z_n^{-sq-n}\Big[\int_0^{z_n} \frac{d\tau}{\tau^n + 1}\Big]^q\, dz_n.$$

Upon noticing that the inner integral is $\mathcal{O}(1)$ as $z_n \to \infty$ and $\mathcal{O}(z_n)$ as $z_n \to 0$, we see that the entire expression amounts to a finite constant as long as $q > (n-1)/(1-s)$. This is, however, covered by the current assumptions, so the proof of Lemma 3.1 is finished. □

Case 3. $1 < p < q$**.** The idea is to use Theorem 2.2 for the operator $\mathcal{T}$ defined in (3.1), which is obviously sublinear and satisfies the conditions (2.15)-(2.16) for $p_0 = 1$, $p_1 = q$, $X_0 := B_s^{1,1}(\partial\Omega)$, and $X_1 := B_s^{q,q}(\partial\Omega)$.

Invoking a complex interpolation result (cf. **[24]**, or the diagonal case of Theorem 2.3) it follows that

$$\mathcal{T} f = \mathcal{N}_s^q(Tf) : B_s^{p,p}(\partial\Omega) \to L^p(\partial\Omega) \tag{3.31}$$

is bounded for every $p \in [1, q]$, whenever (1.9) holds. The proof of Theorem 1.3 is therefore completed. □

Remark. It is illuminating to sketch an alternative approach, most easily described in the case when Ω is the (unbounded) domain in $\mathbb{R}^n$ lying above the graph of a real-valued Lipschitz function. In this situation, fix an upright, circular cone Γ with vertex at the origin so that $\gamma(x) = x + \Gamma$, for each $x \in \partial\Omega$, and introduce the linear operator

$$\Big(\mathcal{Q}f(x)\Big)(z) := (Tf)(x+z), \qquad x \in \partial\Omega,\ z \in \Gamma.$$

The estimate (1.21) is then equivalent to the boundedness of

$$\mathcal{Q} : B_s^{p,p}(\partial\Omega) \longrightarrow L^p(\partial\Omega, B^{q,q}_{s+\frac{n}{q}}(\Gamma))$$

whenever $\frac{n-1}{n} < p \le q$, $(n-1)(\frac{1}{p}-1)_+ < s < 1$, and $\frac{n}{1-s} < q \le \infty$. Note that the reasonings in Cases 1 and 2 ensure the boundedness of the above operator when $0 \le (n-1)(\frac{1}{p}-1) < s < 1$, $\frac{n}{1-s} < q \le \infty$, and when $p = q = \infty$, $0 < s < 1$. The remaining cases are covered by interpolating by the real method between $p = 1$ and $p = q = \infty$. In the process, the classical result

$$\Big(L^{p_0}(A_0)\,,\, L^{p_1}(A_1)\Big)_{\theta,q} = L^q((A_0, A_1)_{\theta,q})$$

if $p_0, p_1 \in [1, \infty]$, $1/q = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$ is used; cf. **[17]**.

We continue with the

Proof of Theorem 1.4. Again, we proceed in a sequence of steps.

Case 1. $1 \le p = q \le \infty$**.** Using the estimates for $p = q = 1$ and $p = q = \infty$ which have been proved in **[20]**, Lemma 7.2, p.35 and Lemma 7.3, p.37 respectively, and Theorem 2.5, we conclude that

$$\delta^{k-\frac{1}{p}+s-1}|\nabla^k Tf| : B_{-s}^{p,p}(\partial\Omega) \to L^p(\Omega), \tag{3.32}$$

whenever

$$|\nabla_x^k \nabla_y^j k(x,y)| \le C|x-y|^{-(n+k+j-2)}, \quad k = 1, 2, ..., N, \quad j = 0, 1, \tag{3.33}$$

for some positive integer N.

Going further, the estimate

$$\|\mathcal{N}_{1-s}^p(Tf)\|_{L^p(\partial\Omega)} \le C\|\delta^{s-\frac{1}{p}}|\nabla Tf|\|_{L^p(\Omega)} \tag{3.34}$$

which follows from (3.2) by simply replacing s with $1-s$, then implies (1.23) in the case when $1 \le p = q \le \infty$.

Case 2. $(n-1)/n < p \leq 1$**.** Similarly to Theorem 1.3, it is sufficient to show that

$$\|\mathcal{N}^q_{1-s}(Ta)\|_{L^p(\partial\Omega)} \leq C \tag{3.35}$$

for every $B^{p,p}_{-s}(\partial\Omega)$-atom a, for a finite constant C independent of the particular atom.

Near the support of a we use the same technique as in Theorem 1.3, i.e. re-scale a, originally an atom in $B^{p,p}_{-s}(\partial\Omega)$, to $\tilde{a} := r^{(n-1)/p}a$, an atom in $B^{q,q}_{-s-\frac{1-n}{q}}(\partial\Omega)$. Invoking the mapping properties of single layer-like operators proved in Case 1 and, much as before, Theorem 2.6, we can show that the analogue of (3.20) holds in this case.

Away from $\operatorname{supp} a$ we proceed, in broad outline, as in Theorem 1.3 although the analytical details are different (most notably, atoms have a vanishing moment condition in this case). To get started, for every $y, z \in \Omega$ we write

$$|Ta(y) - Ta(z)| = \Big|\int_{supp\ a}\int_0^1 \frac{d}{dt}k((1-t)y + tz, w)dt\ a(w)\, d\sigma(w)\Big|.$$

Since atoms in $B^{p,p}_{-s}(\partial\Omega)$ have a vanishing moment condition, this implies that the expression in the right side can be further written as

$$\Big|\int_{supp\ a}\int_0^1 (z-y)[(\nabla_x k((1-t)y+tz, w) - \nabla_x k((1-t)y+tz, 0))]dt\ a(w)d\sigma(w)\Big|$$

where it is assumed that $0 \in \partial\Omega$ and $\operatorname{supp} a \subseteq S_r(0)$. Then the last integral above is equal to

$$\Big|\int_{supp\ a}\int_0^1 (z-y)\Big[\int_0^1 \frac{d}{d\theta}(\nabla_x k((1-t)y+tz, (1-\theta)w))d\theta\Big]\, dt\ a(w)d\sigma(w)\Big|.$$

Using our assumptions on the kernel, this expression can be further bounded by

$$\max_{\theta\in[0,1]}\Big|\int_{supp\ a}\int_0^1 \frac{|z-y||w|}{|((1-t)y+tz)-(1-\theta)w|^n}dt\ a(w)d\sigma(w)\Big|.$$

If we now observe that $|((1-t)y+tz)-(1-\theta)w| \approx |(1-t)y+tz|$ and $|w| < r$, we may invoke the size conditions on atoms to eventually arrive at

$$|Ta(y) - Ta(z)| \leq Cr^{1-s-(n-1)(\frac{1}{p}-1)}\int_0^1 \frac{|z-y|}{|(1-t)y+tz|^n}dt. \tag{3.36}$$

Using the estimate (3.36), the proof for the case $(n-1)/n < p \leq 1$ can be finished in the same way as it was done in Theorem 1.3. We omit the details.

Case 3. $1 < p < q$**.** With the help of Theorem 2.2, this follows from what we have proved so far via interpolation between the cases $p = 1$ and $p = q$. This completes the proof of the theorem. □

4 Connections with classical smoothness spaces in Ω

The goal of this section is to highlight connections between our Besov-based nontangential maximal function and the standard Besov scale in a Lipschitz domain Ω.

Theorem 4.1 *Assume that* $1 < p < q \leq \infty$, $\frac{n}{1-s} < q \leq \infty$ *and* $0 < s < 1$. *Then for every Lipschitz domain* Ω *there exists* $C = C(\Omega, p, q, s) > 0$ *such that*

$$\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \leq C\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \tag{4.1}$$

for every harmonic function u *in* Ω.

The estimate (4.1) shows that the solution of (1.1) taken in the sense of Jerison and Kenig [**14**], i.e. $u \in B^{p,p}_{s+1/p}(\Omega)$, is automatically a solution in the sense of Theorem 1.1 for the range of indices specified above. This also serves as evidence of the appropriateness of our choice of the Besov-based non-tangential maximal function as a way of measuring smoothness.

Proof: The proof is based on ideas from complex analysis in higher dimensions, mostly inspired by the Clifford algebra formalism. In order to be able to follow the main steps more easily, we shall first carry out the proof in the somewhat simpler geometrical context of a domain $\Omega \subset \mathbb{R}^n$ lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$.

Let $\{u_j\}_{1\leq j\leq n}$ be a system of conjugate harmonic functions in Ω with $u = u_n$. For example, we may take

$$u_j(x) := -\int_0^\infty (\partial_j u)(x + te_n)\, dt, \quad x \in \Omega, \tag{4.2}$$

where $e_n = (0, ..., 0, 1) \in \mathbb{R}^n$. Our first claim is that if

$$u \in B^{p,p}_{\alpha+k}(\Omega), \quad k \in \mathbf{N} \cup \{0\}, \quad \alpha \in (0, 1), \tag{4.3}$$

then

$$u_j \in B^{p,p}_{\alpha+k}(\Omega) \quad \text{and} \quad \sum_j \|u_j\|_{B^{p,p}_{\alpha+k}(\Omega)} \leq C\|u\|_{B^{p,p}_{\alpha+k}(\Omega)}. \tag{4.4}$$

Indeed, using Theorem 2.6, it is enough to control, for each $1 \leq j \leq n$,

$$\int_\Omega \Big(\delta(x)^{1-\alpha}|\nabla^{k+1}u_j(x)|\Big)^p\, dx \tag{4.5}$$

which, after the change of variables $x = x' + re_n$, $r > 0$, $x' \in \partial\Omega$, is bounded by

$$C\int_{\partial\Omega}\int_0^\infty (r^{1-\alpha}|\nabla^{k+1}u_j(x' + re_n)|)^p\, drdx' \tag{4.6}$$

$$\leq C\int_{\partial\Omega}\int_0^\infty r^{(1-\alpha)p}\Big[\int_0^\infty |\nabla^{k+2}u(x' + (r+t)e_n)|\, dt\Big]^p\, drdx'. \tag{4.7}$$

The last inequality above utilizes the fact that

$$|\nabla^{k+1}u_j(x)| \leq C \int_0^\infty |\nabla^{k+2}u(x+te_n)|\,dt. \tag{4.8}$$

If we now put $\lambda = r + t$, $\lambda > r$, $d\lambda = dt$, the last integral becomes

$$\int_{\partial\Omega}\int_0^\infty r^{(1-\alpha)p}\Big[\int_r^\infty |\nabla^{k+2}u(x'+\lambda e_n)|\,d\lambda\Big]^p\,drdx'. \tag{4.9}$$

This, using Hardy's inequality, can be further bounded by

$$C\int_{\partial\Omega}\int_0^\infty |\nabla^{k+2}u(x'+\lambda e_n)|^p\lambda^{(1-\alpha)p}d\lambda dx' \tag{4.10}$$

and, invoking Theorem 2.7, by

$$C\int_\Omega \delta(x)^{(1-\alpha)p}|\nabla^{k+1}u(x)|^p\,dx. \tag{4.11}$$

The sequence of inequalities (4.5)-(4.11) along with Theorem 2.6 complete the proof of (4.3)-(4.4).

Next, define

$$\{E_j\}_{1\leq j\leq n} \subset M_{2^n\times 2^n}(\mathbb{R}^n) \tag{4.12}$$

to be a set of matrices (of type $2^n \times 2^n$ with real entries) such that the following properties hold

$$E_jE_k = -E_kE_j, \quad \text{if} \quad j\neq k, \tag{4.13}$$

$$E_j^2 = -I_{2^n}, \quad \forall j = 1,..,n, \tag{4.14}$$

where I_N is the $N\times N$ identity matrix. For each $k = 1,2,...,n$, $\{E_j^k\}_{j=1}^k$ are inductively defined by setting

$$E_1^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and for $1\leq k\leq n-1$, $1\leq j\leq k$

$$E_j^{k+1} := \begin{pmatrix} E_j^k & 0 \\ 0 & -E_j^k \end{pmatrix},$$

$$E_{k+1}^{k+1} := \begin{pmatrix} 0 & -I_{2^k} \\ I_{2^k} & 0 \end{pmatrix}.$$

Finally, take E_j to be E_j^n, for $j = 1,2,...,n$. It is easy to see that the E_j's just constructed satisfy the properties (4.13)-(4.14).

An appropriate *Dirac*-type operator in the current context is

$$D := \sum_{j=1}^{n} E_j \partial_j. \tag{4.15}$$

Consider the matrix-valued function

$$F := \sum_{j=1}^{n} u_j E_j : \Omega \longrightarrow M_{2^n \times 2^n}(\mathbb{R}^n). \tag{4.16}$$

A direct calculation shows that $D^2 = -\Delta$ and $DF = 0$, i.e. F is *analytic* (relative to D).

The next order of business is to define a suitable *Cauchy*-type operator. With this goal in mind, recall that the fundamental solution for Laplacian is

$$\Gamma_{Lap}(x) = \frac{-1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, \tag{4.17}$$

if $n \geq 3$, where ω_n is the surface area of unit sphere (the modifications for the case $n = 2$ are minor). A fundamental solution for the Dirac operator D is

$$\Gamma_{Dir}(x) = -D\Gamma_{Lap}(x) = -\frac{1}{\omega_n} \sum_{j=1}^{n} \frac{x_j}{|x|^n} E_j. \tag{4.18}$$

If we denote $\overline{x} := \sum_{j=1}^{n} x_j E_j$ we arrive at

$$\Gamma_{Dir}(x) = -\frac{1}{n\omega_n} \frac{\overline{x}}{|x|^n}. \tag{4.19}$$

By methods similar to those in classic complex analysis, one may derive Cauchy's reproducing formula

$$F(x) = \mathcal{C}(F|_{\partial\Omega})(x), \qquad x \in \Omega, \tag{4.20}$$

granted that $DF = 0$ in Ω. Here

$$\mathcal{C}f(x) := -\frac{1}{n\omega_n} \int_{\partial\Omega} \frac{\overline{x} - \overline{y}}{|x-y|^n} \cdot \overline{\nu}(y) \cdot f(y)\, d\sigma(y), \quad x \in \Omega, \tag{4.21}$$

where 'dot' denotes matrix multiplication, and f is a matrix-valued function defined on $\partial\Omega$.

Note that the Cauchy operator (4.21) satisfies the conditions (1.19), (1.20) and, therefore, obeys (1.21). This leads to the following estimate

$$\|\mathcal{N}_s^q(F)\|_{L^p(\partial\Omega)} \leq C \|\mathrm{Tr}\, F\|_{B_s^{p,p}(\partial\Omega)}, \tag{4.22}$$

where Tr the trace operator, defined in (2.7). In turn, the last expression above is bounded by

$$\|F\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \leq C \sum_{j=1}^{n} \|u_j\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \tag{4.23}$$

which, using (4.4), can be controlled by a multiple of

$$\|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)}. \tag{4.24}$$

Therefore it remains to show that

$$\|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)} \leq C\|\mathcal{N}^q_s(F)\|_{L^p(\partial\Omega)} \tag{4.25}$$

which, in concert with the sequence of inequalities (4.22)-(4.25), would complete the argument.

To 'extract' $u = u_n$ out of F and obtain (4.25) we observe that

$$\sum_{k=1}^{n-1} E_k F E_k = \sum_{k=1}^{n-1}(\sum_{j=1}^{n} u_j E_j - u_k E_k) = (n-1)F - 2(F - u_n E_n). \tag{4.26}$$

Therefore,

$$u = u_n = -\frac{1}{2}E_n\Big[\sum_{k=1}^{n-1} E_k F E_k - (n-3)F\Big] \tag{4.27}$$

which obviously implies (4.25).

After this preamble, we are ready to tackle the technically more involved case of a general (bounded) Lipschitz domain. In fact, since the estimate we seek is local in character, there is no loss of generality in assuming that $\Omega \subset \mathbb{R}$ is a *star-like* Lipschitz domain with respect to the origin. Fix a harmonic function $u \in B^{p,p}_{k+\alpha}(\Omega)$, $k \in \mathbf{N} \cap \{0\}$, $\alpha \in (0,1)$, and set

$$v := \sum_{j<k} v_{jk} dx_j \wedge dx_k, \qquad v_{jk}(x) := \int_0^1 t^{n-2}[x_k(\partial_j u)(tx) - x_j(\partial_k u)(tx)]\, dt. \tag{4.28}$$

Hereafter, 'wedge' will denote the exterior product of forms. Also, $d := \sum_j \partial_j dx_j \wedge$ will stand for the standard exterior derivative operator in $\mathbb{R}^n$, with formal adjoint d^*. It is easy to show that

$$v(x) = \frac{1}{2}\int_0^1 t^{n-2} d(\|x\|^2) \wedge du(tx) dt$$

which implies $dv = 0$.

Going further, with δ_{jk} denoting the usual Kronecker symbol, we have

$$\begin{aligned} d^*v(x) &= \sum_{j,k} \partial_k v_{kj}(x)\, dx_j \\ &= \sum_{j,k} \int_0^1 t^{n-2}\Big[(\partial_j u)(tx) + tx_k(\partial_k\partial_j u)(tx) - \delta_{jk}(\partial_k u)(tx) - tx_j(\partial_k^2 u)(tx)\Big]\, dt dx_j \end{aligned}$$

$$= \sum_j \int_0^1 \Big[t^{n-2}(n-1)(\partial_j u)(tx) + t\nabla(\partial_j u)(tx)\cdot x \Big]\, dt dx_j$$

$$= \sum_j \partial_j u(x)\, dx_j$$

where, in the last step, we integrate by parts in t. Therefore, $d^*v = du$. In particular, $-(dd^* + d^*d)v = 0$ so that each v_{jk} is harmonic.

Our first goal is to show that $v \in B^{p,p}_{k+\alpha}(\Omega, \Lambda^2\mathbb{R}^n)$ with norm controlled by that of u. By Theorem 2.6 it suffices to show that

$$\delta^{\alpha-1}|\nabla^{k+1}v_{jk}| \in L^p(\Omega), \quad \forall j,k. \tag{4.29}$$

In order to prove (4.29) we assume that, in polar coordinates, Ω can be described as the set $\{x = \omega\rho;\ \omega \in S^{n-1},\ 0 < \rho < \phi(\omega)\}$ for some Lipschitz function $\phi : S^{n-1} \to \mathbb{R}$ with $\inf \phi > 0$. Dropping lower order terms, we can write

$$\sum_{j,k} \int_\Omega (\delta^{1-\alpha}(x)|\nabla^{k+1}v_{jk}(x)|)^p dx$$

$$\leq C \int_{S^{n-1}} \int_0^{\phi(\omega)} (\phi(\omega)-\rho)^{p(1-\alpha)} \Big(\int_0^1 |\nabla^{k+2}u(t\rho\omega)|\, dt \Big)^p \rho^{n-1}\, d\rho d\omega. \tag{4.30}$$

At this stage, making the change of variables $\rho = \phi(\omega)e^{-\eta}$, $\eta \in (0,\infty)$ and $t = e^{-r}$, $r \in (0,\infty)$, we can re-write the last integral as

$$\int_{S^{n-1}} \int_0^\infty (\phi(\omega)(1-e^{-\eta}))^{p(1-\alpha)} \Big(\int_0^\infty |\nabla^{k+2}u(\phi(\omega)\omega e^{-r-\eta})|\, dr \Big)^p (\phi(\omega)e^{-\eta})^n\, d\eta d\omega. \tag{4.31}$$

If we set $\lambda = r + \eta$, $\lambda \in (\eta, \infty)$, we can bound the above expression by

$$C \int_{S^{n-1}} \int_0^\infty (1-e^{-\eta})^{p(1-\alpha)} \Big(\int_\eta^\infty |\nabla^{k+2}u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p e^{-n\eta}\, d\eta d\omega. \tag{4.32}$$

If we now split the integral in η into two integrals over the sets $[0,M]$ and $[M,\infty)$, for some large M, matters are reduced to controlling

$$I := \int_{S^{n-1}} \int_0^M \eta^{p(1-\alpha)} \Big(\int_\eta^M |\nabla^{k+2}u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p d\eta d\omega \tag{4.33}$$

$$II := \int_{S^{n-1}} \int_M^\infty (1-e^{-\eta})^{p(1-\alpha)} \Big(\int_\eta^\infty |\nabla^{k+2}u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p e^{-n\eta}\, d\eta d\omega. \tag{4.34}$$

Note that, by interior estimates, $|II| \leq C\|u\|_{L^p(\Omega)}$ since the argument of $\nabla^{k+2}u$ stays away from $\partial\Omega$. As for I,

$$|I| \leq C \int_{S^{n-1}} \int_0^\infty \eta^{p(1-\alpha)} \Big(\int_\eta^M |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|\, d\lambda \Big)^p \, d\eta d\omega. \tag{4.35}$$

Now an application of Hardy's inequality (cf., e.g., [**23**]) leads to

$$\begin{aligned} |I| &\leq \int_{S^{n-1}} \int_0^M |\nabla^{k+2} u(\phi(\omega)\omega e^{-\lambda})|^p \lambda^{p(2-\alpha)} \, d\lambda d\omega \\ &\leq C \int_{\Omega \setminus \mathcal{O}} \delta^{(2-\alpha)p}(x) |\nabla^{k+2} u(x)|^p \, dx, \end{aligned} \tag{4.36}$$

for a suitable compact subset $\mathcal{O}$ of Ω. In the last step we have changed variables back to the original $x \in \Omega$ and, in the process, have noticed that the Jacobian is bounded given that $0 < \lambda < M$. This proves that the norm of v in $B^{p,p}_{k+\alpha}(\Omega)$ can be controlled by that of u in any star-like Lipschitz domains. The case of a general (bounded) Lipschitz domain follows from what we have proved so far via a standard localization argument.

Now we use an argument similar to that employed in the case of the unbounded domain lying above the graph of a real-valued Lipschitz function to finish the proof of (4.1) in the current context. This time, we choose to work with the Dirac operator $D := d + d^*$ (so that $D^2 = -\Delta$) and the 'analytic' function $F := u - \sum_{j<k} v_{jk} dx_j \wedge dx_k$ (in fact, a differential form of inhomogeneous degree two).

Once again, there is a Cauchy integral operator $\mathcal{C}$ (which falls under the scope of the analysis in §3), a Cauchy reproducing formula, etc. We leave the details to the interested reader. □

Theorem 4.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that $0 < s < 1$, $s + \frac{n}{q} \notin \mathbf{N}$. Then*

$$\|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)} \leq C \|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)} \quad \textit{provided } 1 \leq q \leq p < \infty, \tag{4.37}$$

$$\|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)} \geq C \|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)} \quad \textit{provided } 1 \leq p \leq q < \infty, \tag{4.38}$$

uniformly for u harmonic in Ω and $s + \frac{n}{q} > \frac{n-1}{p}$.

Proof. Consider first the case when $p \geq q$ and set $k := [s + \frac{n}{q}]$, $\varepsilon := s + \frac{n}{q} - k$. We have

$$\begin{aligned} \|\mathcal{N}^q_s(u)\|_{L^p(\partial\Omega)} &= \Big[\int_{\partial\Omega} \Big(\int_{\gamma(x)} \mathrm{dist}\,(y, \partial\gamma(x))^{(1-\varepsilon)q} |\nabla^{k+1} u(y)|^q \, dy \Big)^{\frac{p}{q}} \, d\sigma(x) \Big]^{\frac{1}{p}} \\ &\leq C \Big[\int_{\partial\Omega} \Big(\int_\Omega \chi_{\gamma(x)}(y) \delta(y)^{(1-\varepsilon)q} |\nabla^{k+1} u(y)|^q \, dy \Big)^{\frac{p}{q}} d\sigma(x) \Big]^{\frac{1}{p}} \\ &\approx \Big\| \int_\Omega \chi_{\gamma(x)}(y) \delta(y)^{(1-\varepsilon)q} |\nabla^{k+1} u(y)|^q \, dy \Big\|^{\frac{1}{q}}_{L^{\frac{p}{q}}_x(\partial\Omega)}. \end{aligned} \tag{4.39}$$

Since we are currently assuming that $p \geq q$, this last expression can be bounded by

$$
\begin{aligned}
& C\Big(\int_\Omega \Big\| \chi_{\gamma(x)}(y)\delta(y)^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^q\Big\|_{L_x^{\frac{p}{q}}(\partial\Omega)}\,dy\Big)^{\frac{1}{q}} \\
& \approx \Big(\int_\Omega \delta(y)^{(1-\varepsilon+\frac{n-1}{p})q}|\nabla^{k+1}u(y)|^q\,dy\Big)^{\frac{1}{q}}. \qquad (4.40)
\end{aligned}
$$

By Theorem 2.7 the last integral above can be controlled by

$$
C\Big(\int_\Omega \delta(y)^{(1-\varepsilon+\frac{n-1}{p}-j)q}|\nabla^{k+1-j}u(y)|^q\,dy\Big)^{\frac{1}{q}}, \qquad (4.41)
$$

where

$$
j := \begin{cases} [\frac{n-1}{p}], & \text{if } \varepsilon > \{\frac{n-1}{p}\}, \\ [\frac{n-1}{p}]+1, & \text{if } \varepsilon < \{\frac{n-1}{p}\}. \end{cases} \qquad (4.42)
$$

Using Theorem 2.6, the expression (4.41) is further controlled by

$$
C\|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)}, \qquad (4.43)
$$

as desired.

We now turn to the analysis of the remaining case, i.e. $p \leq q$. Since u is harmonic, we may write

$$
\|\mathcal{N}_s^q(u)\|_{L^p(\partial\Omega)} \geq C\Big[\int_{\partial\Omega}\Big(\int_{\gamma_\theta(x)} \mathrm{dist}\,(y,\partial\gamma_\theta(x))^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^q\,dy\Big)^{\frac{p}{q}}\,d\sigma(x)\Big]^{\frac{1}{p}}. \qquad (4.44)
$$

Changing the aperture of non-tangential cones θ to $\theta/2$, we may bound the last expression from below by

$$
\begin{aligned}
& \geq \; C\Big[\int_{\partial\Omega}\Big(\int_{\gamma_{\theta/2}(x)} \delta(y)^{(1-\varepsilon)q}|\nabla^{k+1}u(y)|^q\,dy\Big)^{\frac{p}{q}}\,d\sigma(x)\Big]^{\frac{1}{p}} \\
& \approx \; \Big[\Big\|\int_{\partial\Omega} \chi_{\gamma_{\theta/2}(x)}(y)\delta(y)^{(1-\varepsilon)p}|\nabla^{k+1}u(y)|^p\Big\|_{L_y^{\frac{q}{p}}(\Omega)}\,d\sigma(x)\Big]^{\frac{1}{p}}. \qquad (4.45)
\end{aligned}
$$

Since $p \leq q$, this can be further bounded by

$$
\begin{aligned}
& C\Big\|\int_{\partial\Omega} \chi_{\gamma_{\theta/2}(x)}(y)\delta(y)^{(1-\varepsilon)p}|\nabla^{k+1}u(y)|^p\,d\sigma(x)\Big\|^{\frac{1}{p}}_{L_y^{\frac{q}{p}}(\Omega)} \\
& \approx \Big(\int_\Omega \delta(y)^{(1-\varepsilon+\frac{n-1}{p})q}|\nabla^{k+1}u(y)|^q\,dy\Big)^{\frac{1}{q}}. \qquad (4.46)
\end{aligned}
$$

By Theorem 2.7 the last integral is controlled (modulo lower order terms) by a multiple of

$$\Big(\int_\Omega \delta(y)^{(1-\varepsilon+\frac{n-1}{p}-j)q}|\nabla^{k+1-j}u(y)|^q\,dy\Big)^{\frac{1}{q}}, \tag{4.47}$$

where j is defined as in (4.42). Finally, we may bound the above expression from below by

$$C\|u\|_{B^{q,q}_{s+\frac{n}{q}-\frac{n-1}{p}}(\Omega)}, \tag{4.48}$$

as wanted. This finishes the proof of Theorem 4.2. □

Corollary 4.3 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$.*
(i) If $1 \leq p \leq \infty$, $0 < s < 1$, and $s + n/p \notin \mathbf{N}$, then

$$\|\mathcal{N}^p_s(u)\|_{L^p(\partial\Omega)} \approx \|u\|_{B^{p,p}_{s+\frac{1}{p}}(\Omega)} \tag{4.49}$$

uniformly for u harmonic function in Ω.
(ii) If $0 < s < 1$ and $n/(1-s) < p \leq \infty$, then

$$\|\mathcal{N}^p_s(u)\|_{L^p(\partial\Omega)} \leq C\|\delta^{1-(s+\frac{1}{p})}|\nabla u|\|_{L^p(\Omega)} \tag{4.50}$$

for any C^1 function u in Ω (not necessarily harmonic).

Proof. The first claim follows directly from Theorem 4.2 by making $p = q$. The second claim is a consequence of the (first half of the) proof of Theorem 4.2, in which we take $p = q$ (note that our current assumptions force $k = 0$). □

Remark. The range of indices for which the estimate (4.1) holds (uniformly for harmonic functions in Ω) also includes triplets s, p, q such that $0 < s < 1$, $1 \leq p \leq q \leq \infty$ and $s + n/p \notin \mathbf{N}$. Indeed, owing to classical embedding results for Besov spaces in Lipschitz domains, we have the pointwise estimate

$$\mathcal{N}^q_s(u)(x) \leq C\mathcal{N}^p_s(u)(x), \qquad \text{uniformly for } x \in \partial\Omega, \tag{4.51}$$

valid whenever $0 < p \leq q \leq \infty$. Then, under the assumptions on s, p, q mentioned above, (4.1) follows from this and (4.49).

Theorem 4.4 *For any Lipschitz domain $\Omega \subset \mathbb{R}^n$ and any $0 < s < 1$, $0 < p \leq \infty$, there exists $C > 0$ so that*

$$\|\delta^{1-s}|\nabla u|\|_{L^{\frac{pn}{n-1}}(\Omega)} \leq C\|\mathcal{N}^\infty_s(u)\|_{L^p(\partial\Omega)} \tag{4.52}$$

for any C^1 function u in Ω. In particular,

$$\mathcal{N}^\infty_s(u) \in L^p(\partial\Omega) \Longrightarrow u \in B^{\frac{pn}{n-1},\frac{pn}{n-1}}_s(\Omega) \tag{4.53}$$

for the range of indices as in (1.9).

A comment is in order here. While it is not clear whether the implication

$$\mathcal{N}_s^\infty(u) \in L^p(\partial\Omega) \Longrightarrow u \in B^{p,p}_{s+\frac{1}{p}}(\Omega)$$

holds for general indices, the above theorem shows that this is nonetheless true after an embedding, since $B^{p,p}_{s+\frac{1}{p}}(\Omega) \hookrightarrow B_s^{\frac{pn}{n-1},\frac{pn}{n-1}}(\Omega)$.

Proof of Theorem 4.4. For $\lambda \in (0,\infty)$ set

$$E_\lambda := \{x \in \partial\Omega;\ \mathcal{N}_s^\infty(u)(x) > \lambda\}, \tag{4.54}$$

$$F_\lambda := \{x \in \Omega;\ \delta(x)^{1-s}|\nabla u(x)| > \lambda\}. \tag{4.55}$$

The idea is to show that there exist two constants $C, c > 0$ such that

$$|F_\lambda| \leq C|E_{c\lambda}|^{\frac{n}{n-1}}, \quad \forall\, \lambda > 0, \tag{4.56}$$

where $|E|$ is the (surface) measure of the set E. Then, since for a general measure μ

$$\|f\|^p_{L^p_\mu} = \int_0^\infty p\lambda^{p-1}\mu(\{|f| > \lambda\})\, d\lambda, \tag{4.57}$$

we can write

$$\begin{aligned} \|u\|_{B_s^{\frac{pn}{n-1},\frac{pn}{n-1}}(\Omega)} &\leq C\|\delta^{1-s}|\nabla u|\|_{L^{\frac{np}{n-1}}(\Omega)} \\ &= \Big(\int_0^\infty \frac{np}{n-1}\lambda^{\frac{np}{n-1}}|F_\lambda|\, d\lambda\Big)^{\frac{n-1}{np}} \\ &\leq C\Big(\int_0^\infty \lambda^{\frac{np}{n-1}-1}|E_\lambda|^{\frac{n}{n-1}}\, d\lambda\Big)^{\frac{n-1}{np}}. \end{aligned} \tag{4.58}$$

Note that

$$\lambda^p|E_\lambda| \leq \int_{E_\lambda} (\mathcal{N}_s^\infty(u)(x))^p\, d\sigma(x) \leq \|\mathcal{N}_s^\infty(u)\|^p_{L^p(\partial\Omega)}, \tag{4.59}$$

so that

$$|E_\lambda| \leq \lambda^{-p}\|\mathcal{N}_s^\infty(u)\|^p_{L^p(\partial\Omega)}. \tag{4.60}$$

Therefore, the last expression in (4.58) can be further bounded by

$$\Big(\int_0^\infty \lambda^{\frac{np}{n-1}-1-\frac{p}{n-1}}|E_\lambda|\, d\lambda\Big)^{\frac{n-1}{np}} \|\mathcal{N}_s^\infty(u)\|^{\frac{1}{n}}_{L^p(\partial\Omega)} = \|\mathcal{N}_s^\infty(u)\|_{L^p(\partial\Omega)}, \tag{4.61}$$

as desired.

In order to prove (4.56), we shall work with 'tent' regions

$$T(\mathcal{O}) := \Omega \setminus \Big[\bigcup_{x\in\partial\Omega\setminus\mathcal{O}} \gamma(x) \Big] \tag{4.62}$$

associated with arbitrary open subsets $\mathcal{O}$ of $\partial\Omega$. The idea is that (4.56) is going to be a consequence of the simple inclusion

$$F_\lambda \subseteq T(E_{c\lambda}), \quad \forall\, \lambda > 0, \tag{4.63}$$

used in concert with a general fact, to the effect that

$$|T(\mathcal{O})| \leq C\, |\mathcal{O}|^{n/(n-1)}, \quad \forall\, \mathcal{O} \text{ open subset of } \partial\Omega. \tag{4.64}$$

In turn, (4.64) is seen by decomposing $\mathcal{O}$ into a disjoint union of Whitney cubes $\{Q_k\}_k$ (considering $\partial\Omega$ as a space of homogeneous type), so that $T(\mathcal{O}) \subset \cup_k T(c\,Q_k)$ for some constant $c = c(\partial\Omega) > 0$, and then writing

$$\begin{aligned} |T(\mathcal{O})| &\leq \sum_k |T(c\,Q_k)| \leq C \sum_k |Q_k|^{n/(n-1)} \\ &\leq C \Big[\sum_k |Q_k|\Big]^{n/(n-1)} = C\, |\mathcal{O}|^{n/(n-1)}. \end{aligned} \tag{4.65}$$

This proves (4.52) and finishes the proof of the Theorem 4.4. □

5 Invertibility of boundary operators

Proof of Theorem 1.5. The operator $\frac{1}{2}I + K$ is invertible on $L^p(\partial\Omega)$ and $L^p_1(\partial\Omega)$ for $2-\varepsilon < p < \infty$ and $1 < p < 2+\varepsilon$, respectively, for some $\varepsilon > 0$ (**[26]**). By complex interpolation this further entails the invertibility of the operator on $L^p_s(\partial\Omega)$, for an appropriate range of indices s, p.

Using the description (2.12)-(2.13) of molecules in $H^{1,p}_{at}(\partial\Omega)$ it can be shown that K is a bounded mapping of $H^{1,p}_{at}(\partial\Omega)$ for each $(n-1)/n < p \leq 1$. Next, based on the invertibility of $\frac{1}{2}I + K$ on $H^{1,1}_{at}(\partial\Omega)$ (established in **[6]**) and the perturbation argument developed in **[15]**, it follows that this operator is also invertible on $H^{1,p}_{at}(\partial\Omega)$ for $1-\varepsilon < p \leq 1$, for some small $\varepsilon = \varepsilon(\partial\Omega) > 0$.

Observe next that, in terms of the Triebel-Lizorkin scale, the results mentioned above amount to the invertibility of $\frac{1}{2}I + K$ on $F^{p,2}_s(\partial\Omega)$ if $|p-2|$ is small, $0 < s < 1$, and on $F^{p,2}_1(\partial\Omega)$ provided $p \in (1-\varepsilon, 1]$. Now we can use complex interpolation (Theorem 2.3) to extend this result to all spaces $F^{p,2}_s(\partial\Omega)$ with

$$\tfrac{2}{2+\varepsilon} < p < \tfrac{2}{1+\varepsilon}, \qquad \tfrac{2}{p} - 1 - \varepsilon < s < 1 \tag{5.1}$$

(by eventually taking a smaller $\varepsilon > 0$). Note that the conditions in (5.1) entail the membership of the point $(s, 1/p)$ to the portion of the pentagon in Figure 2 lying above the line $1/p = (1+\varepsilon)/2$, provided $\varepsilon > 0$ is small enough.

At this stage, the real method of interpolation (Theorem 2.4) allows us to convert all the above Triebel-Lizorkin spaces into Besov spaces, for the same range of indices. The remaining portion of the pentagon is covered by the results in **[9]** and this completes the proof of the theorem. □

In the case of the operator $-\frac{1}{2}I + K^*$, an interpolation argument very similar in spirit to that utilized above works; the only difference is that, this time, we start with the invertibility of $-\frac{1}{2}I + K^*$ on

$$L^p(\partial\Omega)_z = F_0^{p,2}(\partial\Omega)_z, \qquad 1 < p < 2+\varepsilon,$$
$$L^p_{-1}(\partial\Omega)_z := \Big(L_1^{p'}(\partial\Omega)/\mathbb{R}\Big)^* = F_{-1}^{p,2}(\partial\Omega)_z, \qquad 2-\varepsilon < p < \infty,\ 1/p + 1/p' = 1,$$
$$H^p_{at}(\partial\Omega)_z = F_0^{p,2}(\partial\Omega)_z, \qquad 1-\varepsilon < p \leq 1.$$

Above, for a space $X \subset (\mathrm{Lip}(\partial\Omega))^*$, se set $X_z := \{f \in X;\ \langle f, 1\rangle = 0\}$. We omit the routine details. □

6 End of proofs of the main results

Existence, integral representation formulas and estimates in Theorem 1.1 and Theorem 1.2 follow from what we have proved so far. There remains to establish uniqueness, which we address next.

Theorem 6.1 *If u is a solution of (1.8) with $f = 0$, then $u \equiv 0$ in Ω.*

Proof. To begin with, from standard embedding results for Besov spaces in Lipschitz domain, we may infer that

$$\mathcal{N}_s^\infty(u) \in L^p(\partial\Omega). \tag{6.1}$$

Consider next a sequence of domains Ω_j such that for every j

$$\Omega_j \subset \Omega, \quad \Omega_j \subset \Omega_{j+1}, \quad \partial\Omega_j \in C^\infty, \quad \Omega_j \nearrow \Omega, \tag{6.2}$$

and so that the Lipschitz character of $\partial\Omega_j$ is bounded by some constant M, which is independent of j. The details of the construction of such a family of domains can be found in **[26]**. We adopt the following notational convention: objects labeled by j are in relation to Ω_j much as those originally constructed in connection with Ω. In particular,

$$\|\mathcal{N}_{s,j}^\infty(u)\|_{L^p(\partial\Omega_j)} \leq C\|\mathcal{N}_{B_s^{\infty,\infty}}(u)\|_{L^p(\partial\Omega)}, \tag{6.3}$$

with constant C independent of j.

Next, we make the claim that, for the range of indices specified in the statement of the theorem, for every $\varepsilon > 0$ there exists a finite constant $C_\varepsilon > 0$ such that

$$\|Tr_j u\|_{B^{p,p}_{s-\varepsilon}(\partial\Omega_j)} \leq C_\varepsilon \|\mathcal{N}^\infty_{s,j}(u)\|_{L^p(\partial\Omega_j)}, \tag{6.4}$$

for $j = 1, 2, ...$

To see this, using (6.3), we write

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(z)| + |u(z) - u(y)| \\ &\leq C|x-z|^s \|u\|_{B^{\infty,\infty}_s(\gamma(x))} + |z-y|^s \|u\|_{B^{\infty,\infty}_s(\gamma(y))}. \end{aligned} \tag{6.5}$$

It is easy to show that $|x - z| \leq C|x - y|$ as well as $|z - y| \leq C|x - y|$, where $C = C(\partial\Omega)$. Therefore, the last expression in (6.5) can be bounded by

$$C|x-y|^s [\|u\|_{B^{\infty,\infty}_s(\gamma(x))} + \|u\|_{B^{\infty,\infty}_s(\gamma(y))}]. \tag{6.6}$$

Then,

$$\begin{aligned} &\Big[\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+\alpha p}}\, d\sigma(x) d\sigma(y)\Big]^{\frac{1}{p}} \\ &\leq C\Big[\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\mathcal{N}^\infty_{s,j}(u)(x)|^p}{|x-y|^{n-1+(\alpha-s)p}}\, d\sigma(x) d\sigma(y)\Big]^{\frac{1}{p}}. \end{aligned} \tag{6.7}$$

Since Ω is bounded, the last integral above is majorized by $C\|\mathcal{N}^\infty_{s,j}(u)\|_{L^p(\partial\Omega)}$ for every $\alpha < s$. The proof of (6.4) is completed by taking $\alpha := s - \varepsilon$ and recalling (6.3).

By classical embedding results, for every $\delta < s$ such that $1 - \frac{\delta}{n-1} > \frac{1}{p} - \frac{s}{n-1}$ we have

$$\|Tr_j u\|_{B^{1,1}_\delta(\partial\Omega_j)} \leq C \|Tr_j u\|_{B^{p,p}_{s-\varepsilon}(\partial\Omega_j)}, \tag{6.8}$$

where the constant C is independent of j. Based on (6.3)-(6.4), the right side of (6.8) is controlled by

$$\|\mathcal{N}^\infty_s(u)\|_{L^p(\partial\Omega)} < +\infty. \tag{6.9}$$

On the other hand, we may write, thanks to the estimates in **[14]**, that

$$\|u\|_{B^{1,1}_{1+\delta}(\Omega_j)} \leq C \|Tr_j u\|_{B^{1,1}_\delta(\partial\Omega_j)}, \tag{6.10}$$

where, once again, the constant C is independent of j. Since the expression in the left side converges to $\|u\|_{B^{1,1}_{1+\delta}(\Omega)}$ as $j \to \infty$, we eventually arrive at the conclusion that

$$\|u\|_{B^{1,1}_{1+\delta}(\Omega)} < +\infty. \tag{6.11}$$

However, by **[14]**, a harmonic function $u \in B^{1,1}_{1+\delta}(\Omega)$, with δ sufficiently close to 1, is necessarily zero if $\mathrm{Tr}\, u \equiv 0$ on $\partial\Omega$. This proves uniqueness for (1.8) in the case $p \leq 1$.

In the case $p \geq 1$, since $u|_{\partial\Omega} \in B^{p,p}_{s-\varepsilon}(\partial\Omega)$, it follows that

$$u = \mathcal{D}[(\tfrac{1}{2}I + K)^{-1}(u|_{\partial\Omega})] \in B^{p,p}_{s-\varepsilon+\frac{1}{p}}(\Omega),$$

for every ε sufficiently small, whenever $(s, 1/p)$ lies in the hexagon depicted in Figure 1. In this case, the desired conclusion about the uniqueness of the solution follows even more directly from **[14]**. Theorem 6.1 is therefore proved. □

Theorem 6.2 *Let Ω be a Lipschitz domain in $\mathbb{R}^n$. If a function u solves (1.14) with $f = 0$, then, necessarily, u is constant in Ω.*

Proof. Using the same argument as in Theorem 6.1 for $p < 1$, with smoothness $1-s$ instead of s, it can be shown that the size restriction (1.17) imposed on solutions of (1.14) implies

$$u \in B^{1,1}_{1+\delta}(\Omega) \tag{6.12}$$

for some δ close to 1. Matters can also be arranged so that δ satisfies $\delta < 1-s$ and $1 - \frac{\delta}{n-1} > \frac{1}{p} - \frac{1-s}{n-1}$. By the work in **[20]**, this entails

$$\partial_\nu u \in B^{1,1}_{\delta-1}(\partial\Omega) \tag{6.13}$$

and the desired uniqueness (modulo constant) result for $p < 1$ will follow from **[20]** as soon as we show that

$$B^{p,p}_{-s}(\partial\Omega) \hookrightarrow B^{1,1}_{\delta-1}(\partial\Omega). \tag{6.14}$$

In turn, this latter embedding can be checked at the level of atoms; we omit the straightforward details.

For $p > 1$ the same argument as in Theorem 6.1 can be used to show that $u \in B^{p,p}_{1-s-\varepsilon+\frac{1}{p}}(\Omega)$ for $(1-s, 1/p)$ in the hexagon in Figure 1. However, this entails $\partial_\nu u \in B^{p,p}_{-s-\varepsilon}(\partial\Omega)$ and, further, $u = \mathcal{S}[(-\frac{1}{2}I + K^*)^{-1}(\partial_\nu u|_{\partial\Omega})] \in B^{p,p}_{1-s-\varepsilon+\frac{1}{p}}(\Omega)$ for every ε sufficiently small and every $(s, 1/p')$ in the hexagon in Figure 1. Observing that the hexagon in Figure 1 is invariant under the affine transformation $(s, 1/p') \mapsto (1-s, 1/p)$ allows us to finish the proof. □

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