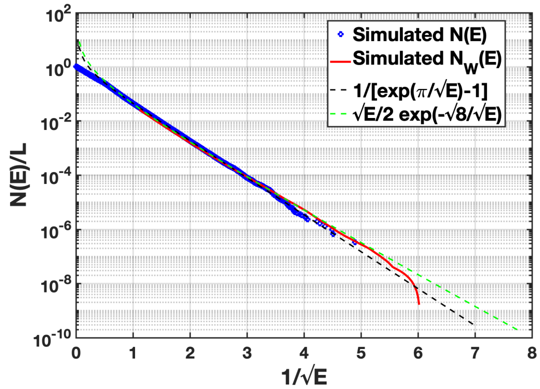

Figure 1. Counting functions $N(E)$ and $N_W(E)$ computed on an interval of length $L = 3.10^6$ for $\rho = 1$, superimposed with the corresponding predicted analytical formulas.

In their comment, Comtet and Texier propose two interesting test cases of the formula $N_{\mathrm{ADJMF}}(E) \approx N(E)$ introduced in [1]. Here $N(E)$ is the counting function (or IDOS) and $N_{\mathrm{ADJMF}}(E)$ (called hereafter N_W) is obtained by replacing in the asymptotic Weyl formula the original potential V by the effective potential $W \equiv 1/u$, u being the localization landscape. Although this effective potential brings out a "classical" interpretation of the disorder-induced quantum confinement, consistent with the use of Weyl's law even at low energies, we did not expect the above formula to be a universal. Nevertheless, the efficiency of this formula has been tested in [1] and successfully applied to disordered semiconductors [2, 3].

More recent works of our team help us to answer the authors' comment. It was observed in [4] that the minimum of W inside a localization region, $W_{\min,i}$, often offers a very good approximation of the fundamental eigenvalue $E_{0,i}$ of the same region through the relation $E_{0,i} = (1 + d/4)\, W_{\min,i}$, d being the dimension. In the case of a one-dimensional infinite square well of width a, this prediction gives $(\hbar^2/2m)(10/a^2)$, remarkably close to the exact value $(\hbar^2/2m)(\pi^2/a^2)$.

The potential of the "pieces" model is a sum of Dirac functions of infinite weights, which amount to partitioning the domain into many infinite square wells of various sizes. The resulting IDOS is a superposition of the IDOS of these wells. Its low energy asymptotics is dominated by the lower eigenvalues of the larger wells. This explains why the factor found in the asymptotics of N_W is $\exp\big(-\sqrt{8}\rho/k\big)$ instead of $\exp(-\pi\rho/k)$. Accounting for the aforementioned factor $(1 + d/4)$ would lead instead to an asymptotic factor $\exp\big(-\sqrt{10}\rho/k\big)$, much closer to the real one ($\sqrt{10} \approx 3.16 \approx \pi$). In addition, despite the difference between the analytic formulas for $N(E)$ and $N_W(E)$, Fig. 1 shows that they are remarkably close on a wide range of values of E.

It has to be underlined that recent developments on the landscape theory introduced a new approximation $N_u(E)$, called "landscape law" [5], which provides bounds to $N(E)$ at all energies in the form

$$C_1 N_u(\alpha E) \le N(E) \le C_2 N_u(E)\,. \qquad (1)$$

This rigorous estimate confirms that the landscape-based formula $N_u(E)$ accurately captures the scaling of the counting function.

In the second example (called "supersymmetric") the distribution of values of the landscape u is found to follow a power law, $P(u) \propto u^{-|\mu|-1}$, leading to $N_W(E) = \frac{L}{\pi}\int_{1/E}^{+\infty} \sqrt{(E - 1/u)}\; P(u)\, du \quad \propto\; E^{|\mu|+\frac{1}{2}}$ which differs from the theoretical behavior $N(E) \propto E^{|\mu|}$ found in the literature [6]. Interestingly in this case, we can evaluate the aforementioned $N_u(E)$ with a back-of-the envelope calculation. $N_u(E)$ is defined as the number of sub-intervals of length $1/\sqrt{E}$ where the maximum of u is larger than $1/E$. The probability of u being larger than $1/E$ is $\int_{1/E}^{+\infty} P(u)\, du$ and the total number of sub-intervals is about $L\sqrt{E}$ which, assuming independence of the sub-intervals, means that $N_u(E) \approx L\sqrt{E}\int_{1/E}^{+\infty} P(u)\, du \;\propto E^{|\mu|+\frac{1}{2}}$. According to Eq. (1), the actual IDOS $N(E)$ should follow the same behavior. The W-based formula $N_W(E)$ is then consistent with the scaling of $N(E)$.

We can imagine several reasons for the discrepancy between N_W and the scaling of N found in [6]. Among others, the independence assumption above may fail due to the possible clustering of the minima of $1/u$. Also, the potentials presented in both examples are much more singular than those considered in [1, 5].

More generally, the comment raises the question of the domain of validity of $N_W(E)$ which has proved to be very efficient in surprisingly many cases. We reiterate, however, that the new landscape law [5], rigorously proven for all potentials bounded from below provides the correct scaling independently of energy. On the other hand, what are the precise prefactors, which approximation gives a better asymptotics in concrete examples, and why remains to be seen.

M.F. is supported by Simons Foundation grant 601944, MF. D.N.A. is supported by the NSF grant DMS-1719694 and Simons Foundation grant 601937, DNA. G.D. is supported by the H2020 grant GHAIA 777822 and Simons Foundation grant 601941, GD. D.J. is supported by NSF grants DMS-1069225 and DMS-1500771, a Simons Fellowship, a Guggenheim Fellowship and Simons Foundation grant 601948, DJ. S.M. is supported by the NSF DMS 1344235, DMS 1839077, and Simons Foundation 563916, SM.

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