Math 2263 Problem Sets

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1. Vectors and the Three-Dimensional Space

Problem 1.1. Determine if the given three points are co-linear (i.e. lie on one line).

(1) $A = (2, 0, -1)$, $B = (1, -1, -2)$ and $C = (-3, 1, 0)$
(2) $A = (-1, 4, 3)$, $B = (-2, 4, 1)$ and $C = (2, 0, 1)$

Problem 1.2. Describe and find the equation of the set of all points that are equidistant to the two points $A = (-1, 5, 3)$ and $B = (6, 2, -2)$.

Problem 1.3. For each of the vectors given below, find a unit vector that has the same direction.

$$\mathbf{v} = \langle 2, 1, -2 \rangle \qquad \mathbf{w} = \langle -4, 0, 3 \rangle$$

Further, find vectors of length 2 with the same direction.

Problem 1.4. In $\mathbb{R}^2$, $\mathbf{v}$ is a unit vector which lies in the first quadrant. Suppose the angle between $\mathbf{v}$ and the positive y-axis is $\pi/4$, find $\mathbf{v}$ in component form.

Problem 1.5. Let $\mathbf{a} = \langle 2, 1, 1\rangle$ and $\mathbf{b} = \langle -1, x, 3\rangle$. Find the value of x such that $\mathbf{a}$ is orthogonal to $\mathbf{b}$.

2. Cross Product, Lines and Planes

Problem 2.1. Find a non-zero vector that is orthogonal to the plane containing the three points

$$A = (2, -3, 4) \quad B = (-1, -2, 2) \quad C = (3, 1, -3)$$

Problem 2.2. Determine whether the following points are co-planer.

$$A = (1, 3, 2) \quad B = (3, -1, 6) \quad C = (5, 2, 0) \quad D = (3, 6, -4)$$

Problem 2.3. Use equations of lines to determine whether the following three points are colinear.

$$A = (2, 4, -3) \quad B = (3, -1, 1) \quad C = (1, 9, 1)$$

Hint: Find the equation of the line through AB and check if C is on the line.

Problem 2.4. Find the equation of the plane through $A = (2, 4, -3)$, $B = (3, -1, 1)$, and $C = (1, 9, 1)$.

Problem 2.5. Find the equation of the line through $(3, 2, -4)$ with direction $\langle -1, 2, 5 \rangle$. Find its intersection with the plane from Problem 2.4.

3. Multivariable Functions, Limits and Partial Derivatives

Problem 3.1. Find the domains and level curves of the functions

$$f(x,y) = \sqrt{4 - x^2 - y^2} \quad \text{and} \quad f(x,y) = x + \sqrt{y},$$

and sketch their graphs.

Problem 3.2. Find the following limits, or demonstrate if not exists.

(1) $\displaystyle \lim_{(x,y)\to(2,-1)} \frac{x^2y + xy^2}{x^2 - y^2}$

(2) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4 + y^4}$

(3) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5y^2 \cos^2 x}{x^2 + y^2}$

Problem 3.3. Determine the set of points where the function is continuous.

(1) $f(x, y) = \dfrac{2x^2 + y}{1 - x^2 - y^2}$

(2) $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2 + xy} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Problem 3.4. Evaluate the following second partial derivatives.

(1) $\dfrac{\partial^2}{\partial x \partial y} \ln(x + y)$

(2) $\dfrac{\partial^2}{\partial x \partial y} e^{xy} \sin(x)$

4. Chain Rule and Directional Derivatives

Problem 4.1. Find dz/dt for $z = \sqrt{xy+1}$, $x = \tan t$ and $y = \arctan(t)$.

Problem 4.2. Find $\partial u/\partial s$ and $\partial u/\partial t$ for

$$u = ze^{xy} \quad x = s+t \quad y = s-t \quad z = st$$

Problem 4.3. Find $\partial z/\partial x$ and $\partial z/\partial y$, where

$$x^2 + 4y^2 + z^2 - 2z = 6$$

Problem 4.4. For each function f, find the gradient ∇f and the directional derivative $D_{\mathbf{u}}f$.

(1) $f(x, y, z) = x^2 z + xyz + yz^2$, $\mathbf{u} = \langle 1, -1, 1\rangle$.
(2) $f(x, y) = e^x \sin(xy)$, $\mathbf{u} = \langle 2, 1\rangle$.
(3) $f(x, y, z) = xe^y - y^2 e^{xz}$, $\mathbf{u} = \langle -1, 0, 2\rangle$.

Problem 4.5. Find the maximal rate of change of $f(x, y, z) = xe^y - y^2 e^{xz}$ at the point $P(1, 0, -1)$. In what direction does that occur?

Problem 4.6. Find the tangent plane and normal line to $xy^2 = 2ze^{x+y} + 3$ at $(1, -1, -1)$.

A. Additional Problems I

Problem A.1. Show that the following limits do not exist.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\sin y}{y^2}$

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$

Problem A.2. Find the limit or show that it doesn't exist.

(1) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2-2xy}{x^2-4y^2}$

(2) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{y-1}{x^2+y-1}$

(3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y+x^2y^2}{2x^6+y^3}$

5. Maxima and Minima

Problem 5.1. Find the local maxima/minima and saddle points of the function.

$$f(x,y) = x^2 + y - 2xy \quad \text{and} \quad f(x,y) = \frac{x^2+y^2}{e^x}$$

Problem 5.2. Find the shortest distance from the plane $x - 2y - z - 3 = 0$ to the origin.

Problem 5.3. Find the absolute minima of the function $f(x, y) = x^2 - 4xy + y^2 + 3y$ in the quadrilateral given by the four points $(0, 0)$, $(2, 0)$, $(0, 3)$ and $(2, 3)$.

Problem 5.4. Find the absolute maximum and minimum of the function $f(x, y) = x^2 + 2xy + y$ in the region bounded by $y = 1 - x^2$, $y = x - 1$, the y-axis and $x \geq 0$.

6. Lagrange Multipliers

Problem 6.1. Find the extreme values of $f(x, y, z) = e^{xyz}$ with constraint $2x^2 + y^2 + z^2 = 24$

Problem 6.2. Find the shortest distance from the plane $x - 2y - z - 3 = 0$ to the origin. Problem 5.2 once again, this time use Lagrange multiplier.

Problem 6.3. Find the extreme value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $x - y = 1$ and $y^2 - z^2 = 1$.

7. Basic Double Integrals

Problem 7.1. Evaluate the following integrals.

(1) $\int_0^\pi \int_0^1 2x + \sin(y)\ dx\ dy$

(2) $\int_1^3 \int_1^{\frac{1}{3}} \frac{\ln y}{xy}\ dy\ dx$

(3) $\iint_R \frac{2xy^2}{x^2+1}\ dA$, where $R = [0,1] \times [-3,3]$. (i.e. $0 \le x \le 1$, $-3 \le y \le 3$.)

Problem 7.2. Fill in the boxes so that the following equality holds

$$\int_0^2 \int_{-1}^{x^2-1} xy\ dy\ dx = \int_\square^\square \int_\square^\square xy\ dx\ dy.$$

Then evaluate the integral using one of the above.

8. More on Double Integrals

Problem 8.1. Evaluate the following double integrals.

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^x x \sin y \; dy \; dx$

(2) $\displaystyle\iint_D e^{y^2} \; dA$, where $D = \{(x, y) : 0 \leq y \leq 1, 0 \leq x \leq y\}$

Problem 8.2. Evaluate the following integrals.

(1) $\displaystyle\iint_D (x^2 + 2y) \; dA$, where D is bounded by $y = x, y = x^3, x \geq 0$.

(2) $\displaystyle\iint_D (2x - y) \; dA$, where D is the circle centered at the origin with radius 2.

Problem 8.3. Find the volume of the solid bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

9. Double Integral with Polar Coordinates

Problem 9.1 (Problems 8.2 (2))**.** Evaluate $\iint_D (2x - y)\, dA$, where D is the circle centered at the origin with radius 2.

Problem 9.2. Find the following integral using polar coordinates.

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} xy^2 \, dx \, dy$$

Problem 9.3. Find the $\iint_R (x^2 + y^2) \, dA$ where R is in the first quadrant bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 9$, $y = x$ and $y = 0$.

10. Triple integrals

Problem 10.1. Evaluate the integral $\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy \; dz \; dx \; dy$

Problem 10.2. Evaluate the integral $\iiint_E e^{z/y} \; dV$, where E is bounded by $E = \{(x, y, z) | 0 \leq y \leq 1, y \leq x \leq 1, 0 \leq z \leq xy\}$.

Problem 10.3. Evaluate $\iiint_E x^2 \, dV$ where E is the solid bounded by $x^2 + y^2 = 4$, $x + z = 2$, and $z = 0$. (Hint: You may use the fact that $\int_0^{2\pi} \cos^3(\theta) \, d\theta = 0$.)

Problem 10.4. Find the volume of the solid bounded by the cylinders $x^2 + y^2 = r^2$ and $x^2 + z^2 = r^2$.

11. Cylindrical, spherical coordinates, and change of variables.

Problem 11.1. Set up the integral to calculate the volume bounded by the sphere $x^2+y^2+z^2 = 16$ and the cone $z = \sqrt{3(x^2 + y^2)}$ using Cartesian coordinates, cylindrical coordinates and spherical coordinates respectively.

Problem 11.2. Rewrite the integral $\iiint_E xe^{x^2+y^2+z^2} dV$ where E is the portion of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Problem 11.3. Evaluate $\iint_R (4x+8y)dA$ where R is the parallelogram wit vertices $(-1,3),(1,-3),(3,-1)$ and $(1,5)$. Use the transformation $x = \frac{1}{4}(u+v)$ and $y = \frac{1}{4}(v-3c)$.

12. Vector Fields and Line Integral

Problem 12.1. Find the gradient vector fields of the following functions and sketch them.

$$f(x,y) = \frac{1}{2}(x^2 - y^2), \quad f(x,y) = (x+y)^2$$

Problem 12.2. Find the gradient vector fields of

$$f(x,y,z) = x^2 y e^{\frac{y}{z}}, \quad f(x,y,z) = z^2 \mathbf{e}^{x^2+4y} + \ln\left(\frac{xy}{z}\right)$$

Problem 12.3. Compute the line integral $\int_C e^x \, dx$ where C is the arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$.

Problem 12.4. Compute the line integral $\int_C y^2 z\, ds$ where C is the line segment from $(3, 1, 2)$ to $(1, 2, 5)$.

Problem 12.5. Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (x^2 + y)\ \mathbf{i} + xz\ \mathbf{j} + (y + z)\ \mathbf{k}$, and C is given by the function $\mathbf{r}(t) = t^2\ \mathbf{i} + t^3\ \mathbf{j} - 2t\ \mathbf{k}$, $0 \leq t \leq 2$.

13. Conservative vector fields and fundamental theorem of path integrals.

Problem 13.1. Determine whether or not $\mathbf{F}$ is a conservative vector field, and if so, find the function f such that $\mathbf{F} = \nabla f$.

(1) $\mathbf{F}(x, y) = (y^2 - 2x)\mathbf{i} + 2xy\mathbf{j}$
(2) $\mathbf{F}(x, y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$

Problem 13.2. Evaluate the following line integrals $\int_C \nabla f \; d\mathbf{r}$.

(1) $f(x, y) = x^3(3 - y^2) + 4y$ and C is given by $\mathbf{r}(t) = \langle 3 - t^2, 5 - t\rangle$ with $-2 \leq t \leq 3$
(2) $f(x, y) = ye^{x^2-1} + 4x\sqrt{y}$ and C is given by $\mathbf{r}(t) = \langle 1 - t, 2t^2 - 2t\rangle$ with $0 \leq t \leq 2$.

Problem 13.3. Evaluate $\int_C \mathbf{F}\ d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$ and C is given by $\langle \sqrt{t}, t+1, t^2 \rangle$ with $0 \leq t \leq 1$.

14. Green's Theorem

Problem 14.1. Evaluate the integral $\int_C y^4\ dx + 2xy^3\ dy$ where C is the ellipse $x^2 + 2y^2 = 2$ oriented positively.

Problem 14.2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (x^2 + y)\ \mathbf{i} + (2x - y^2)\ \mathbf{j}$ and C is a positively oriented circle given by $(x-2)^2 + (y-7)^2 = 4$.

Problem 14.3. Find the area of the polar curve $r = 1 - \cos\theta$. (Use calculator.)

15. Curl and Divergence

Problem 15.1. Find the curl and divergence of the vector fields.

(1) $\mathbf{F}(x, y, z) = \sin(yz)\ \mathbf{i} + \sin(xz)\ \mathbf{j} + \sin(xy)\ \mathbf{k}$

(2) $\mathbf{F}(x, y, z) = xyz^4\ \mathbf{i} + x^2z^4\ \mathbf{j} + 4x^2yz^3\ \mathbf{k}$

Problem 15.2. Show that $\mathbf{F} = \langle ye^{xy} + yz + z, x(e^{xy} + z) - z\sin(yz), xy + x - y\sin(yz)\rangle$ is a conservative vector field and find the function f such that $\mathbf{F} = \nabla f$.

16. Parametric surface and surface integrals

Problem 16.1. Find a parametrization for the following surfaces.

(1) The plane that passes through the point $(0, -1, 5)$ and contains the vectors $\langle 2, 1, 4\rangle$ and $\langle -3, 2, 1\rangle$.
(2) The part of the ellipsoid $x^2 + 4y^2 + 9z^2 = 1$ which lies to the left of xz-plane.
(3) The parts of the plane $x + 2y + z = 1$ which lies inside the cylinder $x^2 + y^2 = 1$.

Problem 16.2. Find the tangent plane to surfaces $\mathbf{r}(u, v) = (u^2 + 1)\mathbf{i} + (v^3 + 1)\mathbf{j} + (u + v)\mathbf{k}$ at $(5, 2, 3)$.

Problem 16.3. Evaluate the surface integral $\iint_S (x^2 + y^2)\, dS$, where S is given by $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$, $u^2 + v^2 \leq 1$.

Problem 16.4. Find the surface area of part of the sphere $x^2 + y^2 + z^2 = 4$ which lies inside the cylinder $x^2 + y^2 = 2x$.

Problem 16.5. Evaluate the surface integral $\iint_S z^2 \, dS$ where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ which lies inside the cone $z = \sqrt{x^2 + y^2}$.

17. Flux integral

Problem 17.1. Find $\iint \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F}(x, y, z) = \langle y, -x, 2z \rangle$, where S is the hemisphere $x^2 + y^2 + z^2 = 4$ $(z \geq 0)$ oriented downward.

Problem 17.2. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = -x\,\mathbf{i} + 2y\,\mathbf{j} - z\,\mathbf{k}$ and S is the portion of $y = 2x^2 + 2z^2$ that lies behind $y = 8$ oriented in the positive y-axis direction.

18. Stoke's theorem

References

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