Topics in Combinatorics: Kazhdan-Lusztig Theory

ABSTRACT. Topics course in combinatorics at University of Minnesota (MATH 8680) taught by Prof. Pavlo Pylyavskyy.

CONTENTS

1. Introduction

The *Hecke algebra* $H_W(q)$ associated to a Coxeter group W is, loosely speaking, a q-deformation of the group algebra of W. It is defined via a set of generators T_w for each $w \in W$, with relations inherited from the Coxeter group W. The seminal work of Kazhdan and Lusztig [KL79] showed that, $H_W(q)$ admits a different basis $\{C_w\}$ which better controls the representation theory of $H_W(q)$.

The Schur Weyl duality between a Weyl group W and a Lie algebra $\mathfrak{g}$, extend to a duality between the Hecke algebra $H_W(q)$ and the q-deformed universal enveloping algebra $U_q(\mathfrak{g})$ (quantum group). The KL basis $\{C_w\}$, under the Schur-Weyl duality, is exactly Lusztig's canonical basis for quantum groups. In other words, the KL basis is the "canonical basis" for $H_W(q)$. The canonical basis was independently discovered (dually?) by Kashiwara under the name of global basis. Taking modulo q, it specializes to the *crystal basis*, which has rich combinatorial properties.

Alternatively, considering vector space dual of the quantum group, we have the *dual canonical basis* of the quantized coordinate ring. To study elements of dual canonical basis, Fomin and Zelevinsky introduced *cluster algebras*[1], which has now grown to a important area of research of its own.

A *Coxeter system* is a pair (W,S), where W is the *Coxeter group* and S is the set of simple generators, satisfying relations like

$$s_i^2 = 1, \quad (s_is_j)^{m_{ij}} = 1$$

for some positive integers m_{ij}. Our main example will be the symmetric group $\mathfrak{S}_n$. The *reduced expression* of $w \in W$ is the minimal way to write w in terms of simple generators: $w = s_{a_1} \cdots s_{a_l}$. This leads to the notion of *length* of w, denoted $\ell(w)$, that is the number of s_i's in the reduced expression of w. For example, in $\mathfrak{S}_3$, we have $321 = s_1s_2s_1 = s_2s_1s_2$, and $\ell(321) = 3$. The subword

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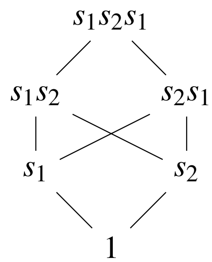
[1] a combinatorial "machine" designed to produce elements of dual canonical basis

order on reduced expressions is called the *Bruhat order* on W. For example, the Hasse diagram of the Bruhat order of $\mathfrak{S}_3$ is the follows.



The Hecke algebra $H_W(q)$ of a Coxeter group W is generated by a set of generators $\{T_w\}$, for each $w \in W$. They satisfy the following relations.

$$\begin{cases} T_sT_w = T_{sw} & \text{if } \ell(sw) > \ell(w) \\ T_s^2 = (q-1)T_s + qT_1 & \end{cases}$$

where q is a formal parameter. It can been seen that $H_W(q)$ is a q-deformation of the group algebra of W, by setting $q \mapsto 0$.

One may ask what are the inverses of these T_w's. The answer of this question leads to the following theorem/definition.

Theorem 1.1 (KL?). *There exists a family of polynomials $R_{v,w}(q)$ for every $v,w \in W$, such that*

$$(T_w)^{-1} = q \sum_{x \le w} R_{xw}(q) T_x$$

These polynomials $R_{v,w}(q)$ are called the R-polynomials.

The Hecke algebra $H_W(q)$ has an automorphism η (an involution) defined as follows.

$$q \mapsto q^{-1}$$
$$T_w \mapsto (T_{w^{-1}})^{-1}$$

We want a new basis $\{C_w\}$ for $H_W(q)$ such that

(1) C_w is a linear combination of T_x for $x \le w$.
(2) $\eta(C_w) = C_w$
(3) coefficients of C_w (as in (1)) are "as simple[2] as possible".

It turns out that the KL basis is the unique one satisfying these properties, which makes it "canonical". [Ben asked the geometric meaning of canonicity, fill in later.]

Theorem 1.2 (KL). *There exists a unique family polynomials $P_{x,w}(q)$ for each $x \le w \in W$ such that*

$$C'_w = q^{\cdots} \sum_{x \le w} P_{x,w}(q) T_x$$

These polynomials are called Kazhdan-Lusztig polynomials *and can be computed recursively.*

[2] being simple means it's a polynomial whose degree is not too big.

It is promised that Kazhdan-Lusztig polynomials give rise to beautiful combinatorics. The first evidence is that, they break Coxeter groups into "cells".

The KL basis defines a preorder[3] on W as follows. We say that $x \prec_L w$ if any left ideal spanned by KL basis containing C_w also contains C_x, called the KL left preorder. Similarly we may define the right preorder $\prec_*$ by looking at right ideals. The Hasse diagram of $\prec_*$ is constructed by drawing an arrow $w \to v$ for $x \prec_* w$, and since a preorder doesn't have to be antisymmetric, the graph may contain double arrows. An example of $\mathfrak{S}_3$ is given in Figure 1.

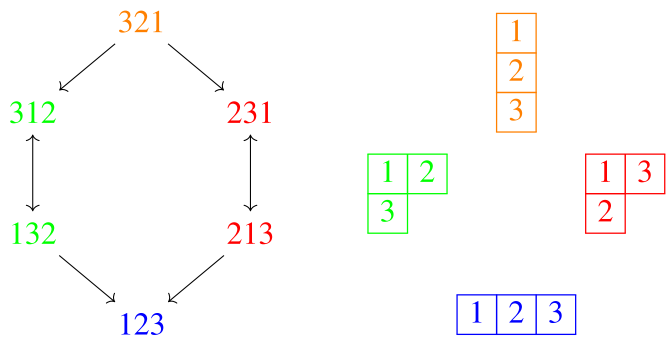


FIGURE 1. KL right preorder and cells. The colors represent the (right) cells.

Now if we ignore the single arrows, the 'doubly' connected components are called the KL left (right) cells. Each cell induces a representation of W, known as the (KL) cell representation. In the case of $\mathfrak{S}_n$, each cells are indexed by a standard Young tableau T, and the corresponding cellular representation is the same as the irreducible representation of $\lambda = \text{shape}(T)$. Moreover, the left (resp. right) cells contain those permutations which have the same recording (resp. insertion) tableaux under the Robinson-Schensted correspondence.

2. Basics of Coxeter Groups and Hecke Algebras

References

[KL79] David Kazhdan and George Lusztig. Representations of coxeter groups and hecke algebras. *Inventiones mathematicae*, 53(2):165–184, 1979.

[3] a preorder is a partial order without antisymmetry. In other words, it is possible that $a \leq b, b \leq a$ while $a \neq b$.