The Robinson-Schensted Correspondence Locally

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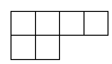
Abstract. This worksheet is about the local aspects of the Robinson-Schensted bijection via Fomin's reinterpretations of the RS.

The Robinson-Schensted (RS) Correspondence is arguably the most important bijection in modern combinatorics. This worksheet aims to give a crash course on it through several different lenses. We start with some basic definitions.

A partition of n is a weakly decreasing sequence of integers whose sum is n, denoted $\lambda \vdash n$.

We will represent a partition by Young diagrams, e.g.  for $\lambda = (4, 2)$.

A column strict tableau (aka. SSYT) of shape λ is a filling of the Young diagram of λ with integers such that every row is weakly increasing and column strictly increasing.

A standard Young tableau of shape λ is a CST of shape λ with numbers from 1 to n ($\lambda \vdash n$).

1. The Original Schensted Insertion Algorithm

Start with a permutation $w \in \mathfrak{S}_n$, write w in one-line notation as $w_1, w_2, \cdots, w_n$. Our goal is to produce two standard Young tableaux (SYT) $P(w), Q(w)$ with the same shape.

Start with $i = 1$ and $P, Q = \varnothing$.

(1) Insert w_i to the first row of P.

(1.1) Let x be the number being inserted. If x is bigger than everything else in this row, append it to this row and move to (2).

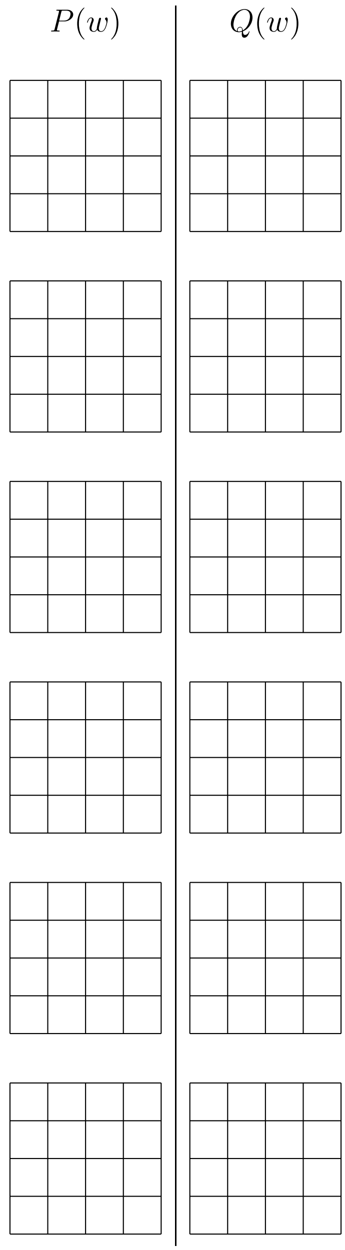
(1.2) otherwise, let y be the smallest number that is bigger than x in this row. Replace y with x. Insert y to the next row and repeat (1.1)

(2) Append i to Q so that the shape of Q remains the same as that of P. Increase i by 1 and repeat (1), until $i = n$.

Here $P(w)$ and $Q(w)$ are called *insertion tableau* and *recording tableau* respectively.

Theorem 1.1. *This is a bijection.*

Exercise 1.2. Calculate $P(w), Q(w)$ for $w = 412563$.



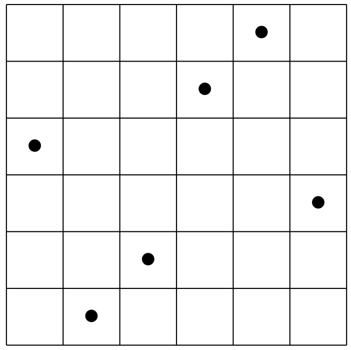
Remark 1.3. RS-correspondence is later generalized by Knuth [Knu70] to a bijection between words and pairs of (CST,SYT); and further to a bijection between integer matrices and pairs of (CST,CST). The later is know and RSK correspondence and is more useful for Schur functions. We will focus on RS in this worksheet, most of the materials here extend to RSK easily but with a bit more technicality.

This insertion algorithm, however, is not very intuitive. We shall next turn to Fomin's alternative definition of the RS, which reveals more unobvious symmetries.

2. Fomin's Growth diagram

For w a permutation in $\mathfrak{S}_n$ and an $n \times n$ checker board, put a $\bullet$ at the $w(i)$-th row (counting from bottom to top), and i-th column. This is sometimes called a *rock diagram*.

For example, the rock diagram of 412563 is



A *growth diagram* is, loosely speaking, a way to put Young diagrams (partitions) to each corner of the rock diagram, following a set of rules.

First, place $\varnothing$'s at the left column and the bottom row (see Figure 1). Then locally in any square, given the bottom left three entries, the top-right entry is uniquely determined by the following rules, which generate the entire diagrams.

Definition 2.1 (Local Rules)**.** For a square
$$\begin{tikzcd}[arrows=dash] \lambda_{i,j+1} \arrow[r] \arrow[d] & \lambda_{i+1,j+1} \arrow[d] \\ \lambda_{i,j} \arrow[r] & \lambda_{i+1,j} \end{tikzcd}$$
without a dot in the middle, we have

(1) If $\lambda_{i,j+1} \neq \lambda_{i+1,j}$, then $\lambda_{i+1,j+1} = \lambda_{i,j+1} \cup \lambda_{i+1,j}$, i.e.
$$\begin{tikzcd}[arrows=dash] \mu_2 \arrow[r] \arrow[d] & \mu_1 \cup \mu_2 \arrow[d] \\ \lambda \arrow[r] & \mu_1 \end{tikzcd}$$

As a special case, if $\lambda_{ij} = \lambda_{i+1,j}$, then $\lambda_{i+1,j+1} = \lambda_{i,j+1}$, i.e.
$$\begin{tikzcd}[arrows=dash] \mu \arrow[r] \arrow[d] & \mu \arrow[d] \\ \lambda \arrow[r] & \lambda \end{tikzcd}$$

(2) If $\lambda_{i,j+1} = \lambda_{i+1,j} \neq \lambda_{i,j}$, then
$$\begin{tikzcd}[arrows=dash] \mu \arrow[r] \arrow[d] & \lambda' \arrow[d] \\ \lambda \arrow[r] & \mu \end{tikzcd}$$
, where λ' is obtained by adding one box to λ to the row immediately below $\mu - \lambda$[1].

For a square with a dot in the middle, we have

(3)
$$\begin{tikzcd}[arrows=dash] \lambda \arrow[r] \arrow[d] & \lambda' \arrow[d] \\ \lambda \arrow[r] \arrow[ur, phantom, "\bullet"] & \lambda \end{tikzcd}$$
, where λ' is obtained by adding one box to the first row of λ[2].

[1]In this case, μ will always have exactly one more box than λ.
[2]In this case we must have $\lambda_{ij} = \lambda_{i,j+1} = \lambda_{i+1,j}$. (why?)

Exercise 2.2. A few parts in the growth diagram for 412563 is shown in Figure 1, finish the rest of the diagram.

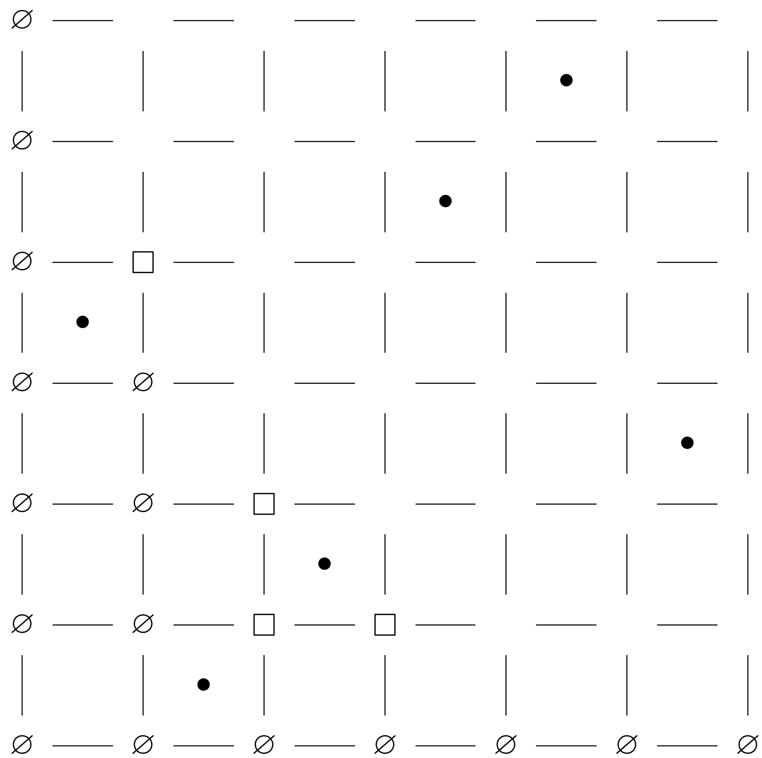


FIGURE 1. Unfinished growth diagram for 412563.

Going back to RS, we have:

Theorem 2.3. *The right-most column (resp. top-most row) of the growth diagram is a flag of partitions which is equivalent to the insertion tableau $P(w)$ (resp. recording tableau $Q(w)$).*

Question 2.4. *How does the local rules relate to insertion algorithm?*

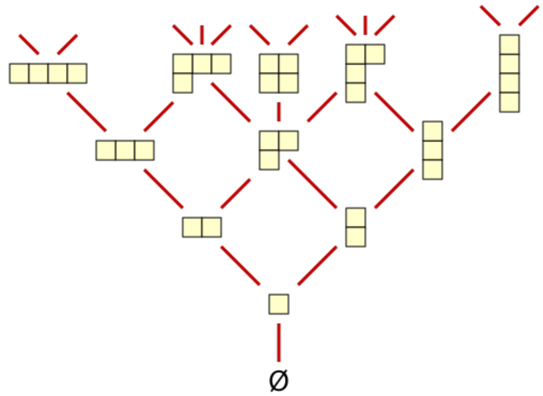


FIGURE 2. Young's lattice.

3. Dual Graphs

For any poset $\mathcal{P}$ we define two (formal) operators $\mathbf{U}$ and $\mathbf{D}$, where $\mathbf{U}(v) = \sum_{v \lessdot w} w$ and $\mathbf{D}(v) = \sum_{w \lessdot v} w$. Stanley [Sta88] called posets satisfying $\mathbf{DU} - \mathbf{UD} = r\mathbf{I}$ *differential posets.*

Exercise 3.1. Convince yourself that the Young's lattice $\mathbb{Y}$ satisfies

$$\mathbf{DU} - \mathbf{UD} = \mathbf{I}. \tag{$*$}$$

Meanwhile, Fomin defined a family of similar objects called *dual graded graphs.*

Definition 3.2. Let G_1 and G_2 be two directed graphs with the same vertex set, and are both graded (having a nice rank function). Let $U(v) = \sum_{v \xrightarrow{G_1} w} w$ and $D(v) = \sum_{v \xrightarrow{G_2} w} w$. If they satisfy $UD - DU = I$, then (G_1, G_2) are called a pair of dual graded graphs.

Note that if Γ_1 is the graph of $\mathbb{Y}$ and Γ_2 is the reverse graph of $\mathbb{Y}$, then the U, D in Definition 2.2 is the same as the $\mathbf{U}, \mathbf{D}$ for $\mathbb{Y}$. Thus (Γ_1, Γ_2) forms a pair of dual graded graphs.

Fomin showed that, for every such dual graded graphs, there exists a RS-like bijection which sends a permutation in $\mathfrak{S}_n$ to a pair of (P_1, P_2), where P_1 (resp. P_2) is an length-n path in G_1 (resp. G_2) sharing the same end points.

Now how does this work for Young's lattice, i.e. how can one get RS from Equation ($*$)?

<u>RS via $\mathbf{DU} - \mathbf{UD} = \mathbf{I}$</u>

A pair of SYT's (T_1, T_2) can be thought of as a sequence of n up-moves followed by a sequence of n down-moves in $\mathbb{Y}$ that start sand ends at $\varnothing$. Consider the formal product

$$\mathbf{D}^n\mathbf{U}^n = \underbrace{\mathbf{DD}\cdots\mathbf{D}}_{n\text{ times}}\underbrace{\mathbf{UU}\cdots\mathbf{U}}_{n\text{ times}}$$

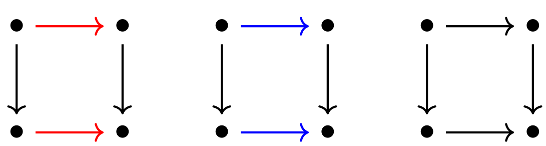
Recall that $\mathbf{U}$ is the formal sum of all possible up-moves in $\mathbb{Y}$ (and similarly for $\mathbf{D}$), thus the expansion of $\mathbf{D}^n\mathbf{U}^n$ will include all such paths with n up followed by n down, not all of which return to the origin. Thus, precisely, we want to find $\mathbf{I}$'s (constant term) in the expansion of $\mathbf{D}^n\mathbf{U}^n$.

Exercise 3.3. Rewrite $\mathbf{DDDUUU}$ by iteratively applying $\mathbf{DU} = \mathbf{UD} + \mathbf{I}$ and FOIL. What's the constant coefficient at the end?

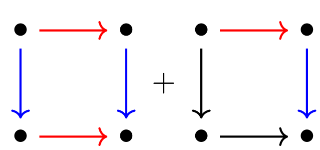
This can be also counted as follows.

Let's represent the relations among $\mathbf{U}, \mathbf{D}$ and $\mathbf{I}$ using small tiles. To better illustrate we will use red edge to indicate $\mathbf{U}$ and blue for $\mathbf{D}$, and finally black for $\mathbf{I}$.

First we have the easy relations $\mathbf{IU} = \mathbf{UI}$, $\mathbf{ID} = \mathbf{DI}$ and $\mathbf{I}^2 = \mathbf{I}^2$, which constitutes the following tiles



The next one $\mathbf{DU} = \mathbf{UD} + \mathbf{I}$ is captured by two types of tiles.



There are of course more relations, but these 5 are the only ones that we will need.

Consider a $n \times n$ board with boundary edges given as in Figure 3. The number of tilings using the tiles given above, are exactly the number of such $\mathbf{D}^n\mathbf{U}^n$-paths.

Exercise 3.4. Try to fill out the diagram in Figure 3, show that there are exactly $n!$ of them, and speculate a bijection between tilings and permutations.

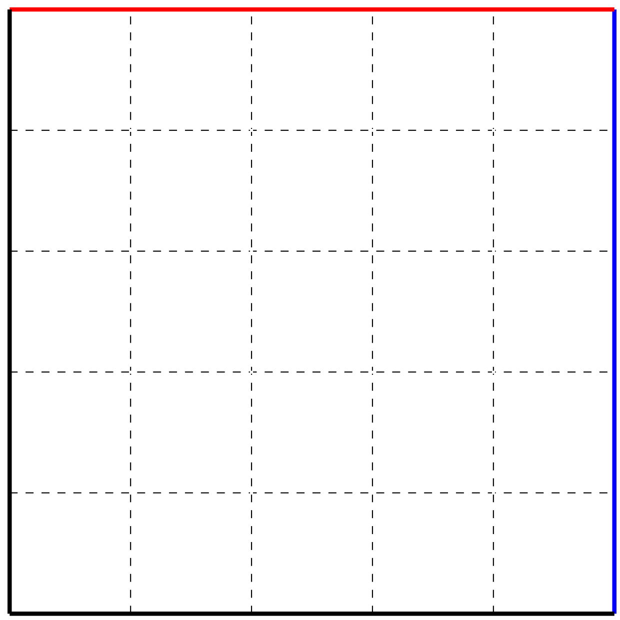
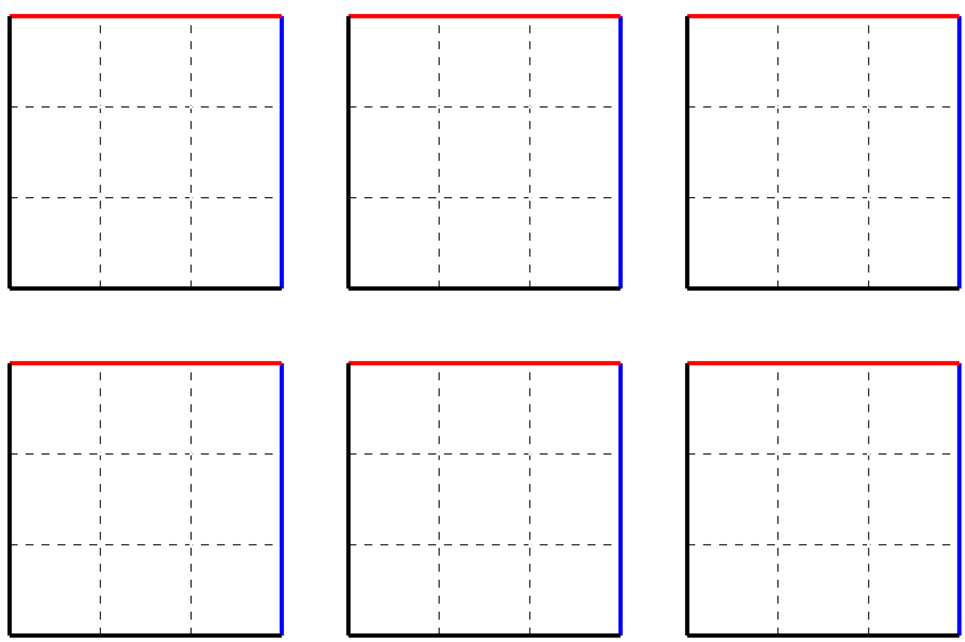


FIGURE 3. Unfilled commutative diagram for **U** and **D**'s, with arrows omitted.

Here's a smaller one, there should be 6 different ways to fill out:



Exercise 3.5. Now go back to Figure 1. For each horizontal edge, color it red if there's a 'growth' of partitions, and uncolored (black) otherwise. For each vertical edge, color it blue if there's a 'growth' and uncolored otherwise. Does it look familiar?

4. Schur Operators

In the preceding exercise, we see that the colored diagrams generated by $\mathbf{DU} - \mathbf{UD} = \mathbf{I}$ are the same growth diagrams, which on the other hand, are generated by the local rules (see Definition 2.1).

This is not a surprise. In fact, we can "localize" $\mathbf{DU} - \mathbf{UD} = \mathbf{I}$ by breaking down the operators $\mathbf{U}$ and $\mathbf{D}$ to more refined pieces of up and down operators called *Schur operators*.

Let

$$u_i(\lambda) = \begin{cases} \lambda \cup \{\text{a box in row } i\} & \text{if the result is partition,} \\ \lambda & \text{otherwise.} \end{cases}$$

and

$$d_i(\lambda) = \begin{cases} \lambda - \{\text{a box in row } i\} & \text{if the result is partition,} \\ \lambda & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{U} = \sum_{i=1}^{\infty} u_i, \quad \mathbf{D} = \sum_{i=1}^{\infty} d_i$$

Proposition 4.1. *The u_i's and d_i's satisfy the following relations.*

$$\begin{aligned} d_1 u_1 &= \mathrm{id} \\ d_j u_i &= u_i d_j \quad i \neq j \\ d_{i+1} u_{i+1} &= u_i d_i \end{aligned}$$

Exercise 4.2. Derive $\mathbf{DU} - \mathbf{UD} = \mathbf{I}$ from the above relations.

Exercise 4.3. Derive local rules of growth diagrams (Definition 2.1) from the relations in Proposition 4.1.

These Schur operators have useful applications, e.g. they can prove Cauchy identities.

Define $A(x) = \cdots(1+xu_3)(1+xu_2)(1+xu_1)$ and $B(x) = (1+xd_1)(1+xd_2)(1+xd_3)\cdots$, and more generally

$$\begin{aligned}\mathfrak{U}(\mathbf{x}) &= A(x_1)A(x_2)A(x_3)\cdots\\ \mathfrak{D}(\mathbf{x}) &= B(x_1)B(x_2)B(x_3)\cdots\end{aligned}$$

which are formal power series in infinitely many variables $x_1x_2\cdots$.

Using Proposition 4.1, one can show that

$$A(x)B(y)(1-xy)^{-1} = B(y)A(x)$$

and more generally

$$\mathfrak{U}(x)\mathfrak{D}(y)\prod_{i,j}(1-x_iy_j)^{-1} = \mathfrak{D}(y)\mathfrak{U}(x)$$

Since Schur functions are coefficients in the power series expansions of $\mathfrak{U}$ and $\mathfrak{D}$, i.e.

$$\mathfrak{U}(\mathbf{x})\lambda = \sum_{\mu} s_{\lambda/\mu}(\mathbf{x})\mu \quad \mathfrak{D}(x)\mu = \sum_{\lambda} s_{\lambda/\mu}(\mathbf{x})\lambda$$

we can rewrite as follows, fix some partitions α and β,

$$\sum_{\mu} s_{\alpha/\mu}(\mathbf{x})s_{\beta/\mu}(\mathbf{y})\prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda} s_{\lambda/\alpha}(\mathbf{x})s_{\lambda/\beta}(\mathbf{y})$$

This the *generalized Cauchy identity* for Schur functions, first discovered by A. Zelevinsky. Setting $\alpha=\beta=\varnothing$ recovers the classical Cauchy's identity.

5. Notes

Section 2 (growth diagram) is based on results of Fomin [Fom88] and can be found in Stanley's EC2 [Sta]. Section 3 (dual graded graphs) is based on Fomin's [Fom94] and [Fom95a]. The colored tiling is essentially a version of Viennot's cellular anstaz [Vie09]. Section 4 (Schur operators) is based on [Fom95b].

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