Math 2374
Spring 2008
Exam 3 solutions

1. By Stokes' Theorem, the surface integral is equal to $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$. But ∂S lies in the three coordinate planes, where at least one of x, y, and z are 0, so $\mathbf{F} = \mathbf{0}$ on ∂S and hence $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 0$.

2. (a) Parametrize S by $\Phi(u, v) = (u, v, 4 - u^2)$, with $0 \leq u, v \leq 2$. Then $T_u = (1, 0, -2u)$, $T_v = (0, 1, 0)$, and $T_u \times T_v = (2u, 0, 1)$, which points up and has magnitude $\sqrt{1 + 4u^2}$. Then

$$\iint_S f\, dS = \int_0^2 \int_0^2 uv\sqrt{1 + 4u^2}\, du\, dv = \frac{1}{6}(17^{3/2} - 1).$$

(b) If we re-parametrize so that the normal vector is $(-2u, 0, -1)$, its magnitude is the same and hence the value of the surface integral is the same.

3. (a) The map $T(u, v) = (2, 1)u + (-1, -2)v = (2u - v, u - 2v)$ transforms the square $0 \leq u, v \leq 1$ to D. Then $J = 3$, and we have

$$\iint_D e^{x+y}\, dA = 3\int_0^1 \int_0^1 e^{2u-v+u-2v}\, du\, dv$$

$$= 3\int_0^1 e^{3u}\, du \cdot \int_0^1 e^{-3v}\, dv = -\frac{1}{3}(e^3 - 1)(e^{-3} - 1).$$

4. (a) $\Phi(\theta, \phi) = (2\cos\theta\sin\phi, 3\sin\theta\sin\phi, 6\cos\phi)$, where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

(b) $(\theta, \phi) = (\frac{\pi}{4}, \frac{\pi}{4})$ corresponds to the point $(1, \frac{3}{2}, \frac{6}{\sqrt{2}})$ on the ellipsoid. Thinking of the ellipsoid as the level surface $f = 36$ for the function $f(x, y, z) = 9x^2 + 4y^2 + z^2$, we have $\nabla f = (18x, 8y, 2z)$. At the point $(1, \frac{3}{2}, \frac{6}{\sqrt{2}})$, we have $\nabla f = (18, 12, 6\sqrt{2})$. The tangent plane to the ellipsoid there has equation $(18, 12, 6\sqrt{2}) \cdot (x - 1, y - \frac{3}{2}, z - \frac{6}{\sqrt{2}}) = 0$.

5. (a) Solving the two equations for z, we find $z = \sqrt{\frac{1}{5}}$, and then $x^2 + y^2 = \frac{4}{5}$. This circle of radius $\sqrt{\frac{4}{5}}$ in the plane $z = \sqrt{\frac{1}{5}}$ can be parametrized by $\mathbf{c}(t) = (\sqrt{\frac{4}{5}}\cos t, \sqrt{\frac{4}{5}}\sin t, \sqrt{\frac{1}{5}})$, for $0 \leq t \leq 2\pi$.

(b) Integrate in spherical coordinates

$$(x, y, z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi).$$

Then

$$\iiint z\, dz\, dy\, dx = \int_0^{\tan^{-1} 2} \int_0^{2\pi} \int_0^1 \rho^3 \cos\phi \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \frac{\pi}{2} \int_0^{\tan^{-1} 2} \cos\phi \sin\phi \, d\phi = \frac{\pi}{5}.$$

To see that the upper ϕ limit of integration is $\tan^{-1}2$, make a right triangle with legs $\sqrt{\frac{1}{5}}$, the distance from the origin to the center of the circle of intersection, and $\sqrt{\frac{4}{5}}$, the radius of the circle.

6. Since $\operatorname{div} \mathbf{F} = 0$, we know that $\mathbf{F} = \operatorname{curl} \mathbf{G}$ for some vector field $\mathbf{G}$, and hence we can apply Stokes' Theorem to integrate over the disk D (with upward-pointing normal) having common boundary with the hemisphere:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{G} \cdot d\mathbf{s} = \iint_D \mathbf{F} \cdot d\mathbf{S}.$$

The disk D can be parametrized by $\Phi(r, \theta) = (r\cos\theta, r\sin\theta, 0)$, has normal vector $T_r \times T_\theta = r\mathbf{k}$, and so $\mathbf{F}(\Phi(r, \theta)) \cdot (T_r \times T_\theta) = 2r$. Thus,

$$\iint_D \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 2r \, dr \, d\theta = 2\pi.$$