Math 2374
Spring 2009
Final
May 11, 2009
Time Limit: 3 hours

Name (Print): ____________________
Student ID: ____________________
Section Number: ______
Teaching Assistant: ____________________
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This exam contains 8 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	35 pts	
3	35 pts	
4	40 pts	
5	25 pts	
6	25 pts	
7	40 pts	
8	25 pts	
9	25 pts	
10	25 pts	
TOTAL	300 pts	

SKETCH OF THE SOLUTION
there might be some typos; tell me (Francisco) if you find one

1. (25 points) Let $g(x, y, z) = x^2 - y^2 + z^2$.

   (a) Find the tangent plane to $g(x, y, z) = 12$ at the point $(2, 1, -3)$ (write your answer in the form $Ax + By + Cz + D = 0$).

   We first compute the gradient of g: $\nabla g = (2x, -2y, 2z)$. At the point $(2, 1, -3)$ we have

   $$\nabla g(2, 1, -3) = (4, -2, -6).$$

   The tangent plane at this point is then

   $$(4, -2, -6) \cdot (x - 2, y - 1, z + 3) = 0$$

   or equivalently

   $$4x - 2y - 6z - 24 = 0.$$

   (b) Find all the points of the surface $g(x, y, z) = 12$ where the tangent plane is horizontal.

   If the tangent plane is horizontal, then ∇g has to be of the form $(0, 0, C)$. Then $(2x, -2y, 2z) = (0, 0, C)$ which means that $x = y = 0$. We look for points on the surface satisfying these conditions: $z^2 = 12$. We obtain two points

   $$(0, 0, 2\sqrt{3}) \qquad \text{and} \qquad (0, 0, -2\sqrt{3}).$$

2. (35 points) Consider the path $\mathbf{c}(t) = (\cos(5\pi t), 6\sin(5\pi t), (1 + t)^2)$ for $0 \leq t \leq 1$ and the vector field $\mathbf{F}(x, y, z) = (x + yz, y + zx, z + xy)$. Compute

   $$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

   (**Hint.** If you show that $\mathbf{F}$ is conservative the computation can be made simpler).

   We first show that $\mathbf{F}$ is conservative by checking that $\nabla \times \mathbf{F} = \mathbf{0}$

   $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + yz & y + xz & z + `xy \end{vmatrix} = \mathbf{i}(x - x) - \mathbf{j}(y - y) + \mathbf{k}(z - z) = (0, 0, 0).$$

   There are several ways to compute f such that $\nabla f = \mathbf{F}$. In any case, the result is

   $$f = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + xyz$$

   (plus any constant we might want to add). Finally, because $\mathbf{F} = \nabla f$ we can compute the line integral as follows:

   $$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{c}(1)) - f(\mathbf{c}(0)) = f(-1, 0, 4) - f(1, 0, 1) = \frac{15}{2}.$$

3. (35 points) Calculate the intersection of the surfaces (specify the result with equations)

$$y = \sqrt{x^2+z^2}, \qquad y = 4 - \sqrt{x^2+z^2}.$$

Then compute the volume of the solid enclosed by these two surfaces.

The surfaces are two opposed cones along the y axis and with tips in $(0,0,0)$ and $(0,4,0)$. The intersection of the surfaces occurs at

$$\sqrt{x^2+z^2} = 4 - \sqrt{x^2+z^2} \qquad \Leftrightarrow \qquad x^2+z^2 = 4.$$

Then the intersection is a circle in the plane $y = 2$ with center in the origin and radius 2. We can describe the solid W as

$$(x,z) \in D = \{x^2+z^2 \leq 4\}, \qquad \sqrt{x^2+z^2} \leq y \leq 4 - \sqrt{x^2+z^2}$$

and then the volume is

$$\mathrm{Vol}(W) = \iiint_W dV = \iint_D \Big(\int_{\sqrt{x^2+z^2}}^{4-\sqrt{x^2+z^2}} dy \Big) dx\,dz = \iint_D (4 - 2\sqrt{x^2+z^2} = dx\,dz.$$

We now can evaluate this integral using polar coordinates (in the $xz-$plane) to describe D:

$$x = r\cos\theta, \qquad z = r\sin\theta, \qquad 0 \leq r \leq 2, \qquad 0 \leq \theta \leq 2\pi, \qquad dx\,dz = r\,dr\,d\theta$$

and we get

$$\mathrm{Vol}(W) = \int_0^{2\pi} \int_0^2 (4-2r)\,r\,dr\,d\theta = \int_0^{2\pi} \Big(2r^2 - \frac{2}{3}r^3\Big)\Big|_{r=0}^2 d\theta = \int_0^{2\pi} \frac{8}{3} d\theta = \frac{16\pi}{3}.$$

4. (40 points) Consider the function

$$f(x,y) = e^{2x+2y+2}\,(5y - y^2 - 3).$$

(a) Give a linear and a quadratic approximation of f near $(-2,1)$.

We start by computing all the partial derivatives we need for the approximations

$$\begin{aligned}
f &= e^{2x+2y+2}\,(5y-y^2-3) & f(-2,1) &= 1\\
\frac{\partial f}{\partial x} &= 2e^{2x+2y+2}\,(5y-y^2-3) & \frac{\partial f}{\partial x}(-2,1) &= 2\\
\frac{\partial f}{\partial y} &= e^{2x+2y+2}\,(8y-2y^2-1) & \frac{\partial f}{\partial y}(-2,1) &= 5\\
\frac{\partial^2 f}{\partial x^2} &= 4e^{2x+2y+2}\,(5y-y^2-3) & \frac{\partial^2 f}{\partial x^2}(-2,1) &= 4\\
\frac{\partial^2 f}{\partial x \partial y} &= 2e^{2x+2y+2}\,(8y-y^2-1) & \frac{\partial^2 f}{\partial x \partial y}(-2,1) &= 10\\
\frac{\partial^2 f}{\partial y^2} &= e^{2x+2y+2}\,(12y-4y^2+6) & \frac{\partial^2 f}{\partial y^2}(-2,1) &= 14
\end{aligned}$$

Now the linear approximation is

$$z = f(-2,1) + \frac{\partial f}{\partial x}(-2,1)(x+2) + \frac{\partial f}{\partial y}(-2,1)(y-1) = 1 + 2(x+2) + 5(y-1)$$

and the quadratic approximation is

$$z = 1 + 2(x+2) + 5(y-1) + \frac{1}{2}\Big(4(x+2)^2 + 20(x+2)(y-1) + 14(y-1)^2\Big).$$

(b) Compare the value of the linear and quadratic approximations to $f(-1.9,\,1) = 1.2214...$
We simply have to evaluate the two expressions of (a) at the point $x = -1.9$, $y = 1$. Note that $x + 2 = 0.1$ and $y - 1 = 0$, so the expressions are very simple

$$z = 1 + 2\,\cdot(0.1) + 5\,\cdot 0 = 1.2 \qquad \text{for the linear approximation}$$

$$z = 1 + 2\cdot(0.1) + 5\,\cdot 0 + \frac{1}{2}\Big(4\cdot(0.1)^2 + 20\cdot(0.1)\cdot 0 + 14\cdot 0^2\Big) = 1.22 \qquad \text{for the quadratic appr.}$$

5. (25 points) Consider the path

$$\mathbf{c}(t) = \big(2t\sin t, \tfrac{1}{3}t^3, 2t\cos t\big), \qquad 0 \le t \le 2.$$

(a) Show that the speed along the path is increasing with time.
The velocity vector is

$$\mathbf{c}'(t) = \big(2\sin t + 2t\cos t,\ t^2,\ 2\cos t - 2t\sin t\big).$$

The speed is the norm of the velocity vector

$$\begin{aligned} \|\mathbf{c}'(t)\|^2 &= (4\sin^2 t + 4t^2\cos^2 + 8t\sin t\cos t) + t^4 + (4\cos^2 t + 4t^2\sin^2 t - 8t\sin t\cos t) \\ &= 4 + 4t^2 + t^4 = (2+t^2)^2 \end{aligned}$$

Therefore, the speed is $\|\mathbf{c}'(t)\| = 2 + t^2$ which is an increasing function of t for $t > 0$.

(b) Compute the length of the path.
We just have to calculate the following integral

$$\int_{\mathbf{c}} ds = \int_0^2 (2+t^2)dt = \Big(2t + \frac{t^3}{3}\Big)\Big|_{t=0}^{2} = \frac{20}{3}.$$

6. (25 points) Change the order of integration in

$$\int_0^2 \int_0^{2x-x^2} \sqrt{1-y}\,dy\,dx$$

and compute the resulting integral. (**Hint.** The expression $2x - x^2 = 1 - (x-1)^2$ might be helpful in your calculations.)

The curve $y = 2x - x^2$ is a parabola with vertex at $x = 1$ $(y = 1)$ and pointing down. We can write x in terms of y

$$(x-1)^2 = 1 - y \qquad \iff \qquad x - 1 = \pm\sqrt{1-y} \qquad \iff \qquad x = 1 \pm \sqrt{1-y}.$$

The limits are then $0 \le y \le 1$ and $1 - \sqrt{1-y} \le x \le 1 + \sqrt{1-y}$. Therefore the integral becomes

$$\begin{aligned} \int_0^2 \int_0^{2x-x^2} \sqrt{1-y}\, dy\, dx &= \int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} \sqrt{1-y}\, dx\, dy \\ &= \int_0^1 \sqrt{1-y}\,(1 + \sqrt{1-y} - 1 + \sqrt{1-y})dy \\ &= \int_0^1 2(1-y)dy = -(1-y)^2\Big|_{y=0}^{1} = 1. \end{aligned}$$

7. (40 points) The points

$$\mathbf{p}_1 = (0,0,1), \qquad \mathbf{p}_2 = (1,0,1), \qquad \mathbf{p}_3 = (0,2,1), \qquad \mathbf{p}_4 = (1,2,1)$$

$$\mathbf{p}_5 = (0,0,2), \qquad \mathbf{p}_6 = (1,0,4), \qquad \mathbf{p}_7 = (0,2,8), \qquad \mathbf{p}_8 = (1,2,10)$$

are the vertices of a truncated prism with rectangular base. The surface of the top face (the one with $\mathbf{p}_5$, $\mathbf{p}_6$, $\mathbf{p}_7$ and $\mathbf{p}_8$ as vertices) will be called S_1.

(a) Find the equation of the plane that contains S_1 (write your answer in the form $z = Ax + By + C$).

We pick the point $\mathbf{p}_5 = (0,0,2)$ and define the vectors

$$\mathbf{t}_1 = \mathbf{p}_6 - \mathbf{p}_5 = (1,0,2), \qquad \mathbf{t}_2 = \mathbf{p}_7 - \mathbf{p}_5 = (0,2,6).$$

These vectors are tangent to the plane and

$$\mathbf{t}_1 \times \mathbf{t}_2 = (-4,-6,2)$$

is normal to the plane. The equation is therefore

$$(-4,-6,2)\cdot(x-0, y-0, z-2) = 0 \qquad \iff \qquad -4x - 6y + 2z - 4 = 0,$$

which can be simplified to $z = 2x + 3y + 2$.

(b) Compute the flux integral

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S},$$

where the normal has been oriented upwards and $\mathbf{F}(x,y,z) = (y,x,z)$.

The surface can be described as the part of the graph of $z = 2x + 3y + 2$ corresponding to $0 \le x \le 1$ and $0 \le y \le 2$. The parametrization is the $\mathbf{\Phi} = (x, y, 2x + 3y + 2)$. The tangent vectors are then

$$\mathbf{T}_x = (1,0,2), \qquad \mathbf{T}_y = (0,1,3)$$

and the normal vector is $\mathbf{T}_x \times \mathbf{T}_y = (-2, -3, 1)$. Then, the flux integral is

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^2 (y, x, 2x+3y+2) \cdot (-2,-3,1) dy\, dx \\
&= \int_0^1 \int_0^2 (-2y - 3x + 2x + 3y + 2)\, dy\, dx = \int_0^1 \int_0^2 (y - x + 2)\, dy\, dx \\
&= \int_0^1 \left(\frac{y^2}{2} - xy + 2y \right)\Big|_{y=0}^2 = \int_0^1 (6 - 2x) dx = \left(6x - x^2\right)\Big|_{x=0}^1 = 5.
\end{aligned}$$

(c) Let now S_2 be the surface formed by the remaining five faces of the truncated prism with normal pointing outwards. Use Gauss' Theorem to evaluate, for the same vector field $\mathbf{F}$, the flux integral

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

Using Gauss' Theorem we can write our problem as

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \mathrm{div}\mathbf{F}\, dV - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iiint_W 1\, dV - 5,$$

where we have called W to the solid truncated prism and have calculated $\operatorname{div} \mathbf{F} = 1$. The solid volume can be described as the volume between the plane $z = 1$ and the plane $z = 2x+3y+2$ for $0 \le x \le 1$ and $0 \le y \le 2$. Therefore

$$\begin{aligned}
\iiint_W dV &= \int_0^1 \int_0^2 \int_1^{2x+3y+2} dx\, dy\, dx = \int_0^1 \int_0^2 (2x + 3y + 1)\, dy\, dx \\
&= \int_0^1 \left(2xy + \frac{3}{2}y^2 + y\right)\Big|_{y=0}^2 dx = \int_0^1 (4x+8) dx = \left(2x^2 + 8x\right)\Big|_{x=0}^1 = 10.
\end{aligned}$$

Finally, we plug this result into our initial formula to obtain

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 10 - 5 = 5.$$

8. (25 points) Show that $(-1, 1)$ is a critical point of

$$f(x, y) = x^2 + 5x + 5 + 2y^2 - 7y - 3xy$$

and classify it as maximum, minimum or saddle–point.

Confirmation of critical point, check to see if the first derivatives are equal to zero at $(-1, 1)$. We first compute the partial derivatives of f:

$$\frac{\partial f}{\partial x} = 2x + 5 - 3y, \qquad \frac{\partial f}{\partial y} = 4y - 7 - 3x.$$

We evaluate them at $(-1, 1)$ to obtain

$$\frac{\partial f}{\partial x}(-1, 1) = 2(-1) + 5 - 3(1) = 0, \qquad \frac{\partial f}{\partial y}(-1, 1) = 4(1) - 7 - 3(-1) = 0.$$

Hence we have a critical point.

To classify the point, we need all second derivatives:

$$f_{xx}(x,y) = 2$$
$$f_{yy}(x,y) = 4$$
$$f_{xy}(x,y) = -3 = f_{yx}(x,y)$$

Now take the determinant of the Hessian matrix, which is

$$f_{xx}\,f_{yy} - \left(f_{xy}\right)^2 = 2 \times 4 - (-3)^2 = -1 < 0.$$

Since this is less than 0, the critical point is a saddle-point.

9. (25 points) Consider the function

$$\mathbf{F}(x,y,z) = (2x - y^2 + z,\; x - y - z^2)$$

and a path $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ for which we know that

$$\mathbf{g}(0) = (1,1,2) \qquad \mathbf{g}'(0) = (0,-1,3).$$

Compute $\mathbf{D}[\mathbf{F}\circ\mathbf{g}](0)$. (**Hint.** To properly understand the derivative of $\mathbf{g}$, it might be convenient to think of how many variables and components $\mathbf{g}$ has).

We have to apply the chain rule:

$$\mathbf{D}[\mathbf{F}\circ\mathbf{g}](0) = \mathbf{DF}(\mathbf{g}(0))\mathbf{D}g(0) = \mathbf{DF}(1,1,2)\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}.$$

The derivative of $\mathbf{F}$ at

$$\mathbf{DF} = \begin{bmatrix} 2 & -2y & 1 \\ 1 & -1 & -2z \end{bmatrix} \qquad \mathbf{DF}(1,1,2) = \begin{bmatrix} 2 & -2 & 1 \\ 1 & -1 & -4 \end{bmatrix}.$$

Finally

$$\mathbf{D}[\mathbf{F}\circ\mathbf{g}](0) = \begin{bmatrix} 2 & -2 & 1 \\ 1 & -1 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}.$$

10. (25 points) The height of a mountain as a function of the horizontal coordinates is given by the formula

$$h(x,y) = \frac{15}{x^2 + 9y^2 + 2}.$$

(a) If a mountaineer is in the point with horizontal coordinates $(2,1)$ and wants to descend as fast as possible, what direction should she head to? (Give your answer as a unit vector).

Compute

$$\mathbf{u} = \frac{-\nabla f}{||\nabla f||} \text{ at } (x,y) = (2,1)$$

because we need the unit vector in the direction opposite to the gradient (direction of fastest increase).

$$\nabla f|_{(2,1)} = -\frac{15}{(x^2+9y^2+2)^2}(2x, 18y)\bigg|_{(2,1)} = -\frac{2}{15}(2,9)$$

The unit vector in the direction of $-(-\frac{2}{15}(2,9)) = \frac{1}{15}(2,9)$ is the same as the unit vector in the direction of $(2,9)$, which is

$$\mathbf{u} = \frac{1}{||(2,9)||}(2,9) = \frac{1}{\sqrt{85}}(2,9).$$

(b) The summit is located at the point with coordinates $(0,0)$. Give the equation of the tangent plane at the summit.

By evaluating ∇f, computed in part (a), at the origin, we see that the partial derivatives of f exist and are zero at the origin. Therefore, the graph of f at the origin does have a tangent plane. Since the summit is a maximum of the function f, the tangent plane is the horizontal plane passing through the point $(x,y,z) = (0,0,f(0,0)) = (0,0,\frac{15}{2})$. So the equation is

$$z = \frac{15}{2}$$