Math 2374
Fall 2010
Final Exam
December 16, 2010
Time Limit: 3 hours

Name (Print): ____________
Student ID: ____________
Section Number: ______
Teaching Assistant: ____________
Signature: ____________

This exam contains 14 pages (including this cover page) and 12 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

Solutions.

Please report errors.

1	20 pts	
2	25 pts	
3	25 pts	
4	20 pts	
5	25 pts	
6	30 pts	
7	25 pts	
8	20 pts	
9	25 pts	
10	30 pts	
11	25 pts	
12	30 pts	
TOTAL	300 pts	

TA sections:

Section	TA	Discussion time
011	Chen	T 9:05am
012	Chen	T 11:15am
013	Klein	T 1:25pm
014	Klein	T 3:35pm
015	Bu	T 4:40pm
016	Bu	T 6:45pm
021	Bashkirov	Th 8:00am
022	Bashkirov	Th 10:10am
023	He	Th 12:20pm
024	He	Th 2:30pm
025	Lee	Th 4:40pm
026	Lee	Th 6:45pm

1. (20 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (xe^y, ye^x, e^{x+y^2})$. Suppose $g : \mathbb{R}^3 \to \mathbb{R}^2$ is a function whose matrix of partial derivatives at $(0, 0, 1)$ is given by

$$(\mathbf{D}g)(0,0,1) = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 5 \end{bmatrix}.$$

Find the matrix of partial derivatives of the function $g \circ f$ at $\cancel{(0,0,0)}$. error: $(0,0)$

$f(0,0) = (0,0,1)$

$$Df(x,y) = \begin{bmatrix} e^y & xe^y \\ ye^x & e^x \\ e^{x+y^2} & 2ye^{x+y^2} \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Chain rule says:

$$D(g\circ f)(0,0) = Dg(f(0,0)) \cdot Df(0,0)$$

$$= Dg(0,0,1) \cdot Df(0,0)$$

$$= \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ 6 & 0 \end{bmatrix}.$$

2. (25 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = 5xy - x^3 - y^3$.

a) Write a unit vector **u** that indicates the direction in which f is decreasing the fastest at the point $(-2, 1)$.

$\nabla f(x,y) = [5y - 3x^2, 5x - 3y^2]$

$\nabla f(-2,1) = [-7, -13]$. for decrease: $[7, 13]$

divide by length:

$$\frac{1}{\sqrt{49+169}}[7, 13] = \frac{1}{\sqrt{218}}[7, 13] = \left[\frac{7}{\sqrt{218}}, \frac{13}{\sqrt{218}}\right]$$

b) Find the directional derivative of f at the point $(-2, 1)$ in the direction **u** that you found in part (a).

Directional derivative is

$\vec{u} \cdot \nabla f(2,1)$

$$= \left[\frac{7}{\sqrt{218}}, \frac{13}{\sqrt{218}}\right] \cdot [-7, -13]$$

$$= \frac{-49}{\sqrt{218}} + \frac{-169}{\sqrt{218}} = \frac{-218}{\sqrt{218}} = -\sqrt{218}.$$

3. (25 points) Consider the wire parametrized by $\mathbf{c}(t) = (t\cos t, t\sin t, t)$, $1 \le t \le 2$.

a) Set up, but **do not evaluate**, the integral for the length of the wire.

$$c'(t) = (\cos t - t\sin t,\ \sin t + t\cos t,\ 1)$$

$$\|c'(t)\| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1^2}$$

$$= \sqrt{\cos^2 t \cancel{- 2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t \cancel{+ 2t\sin t\cos t} + t^2\cos^2 t + 1}$$

$$= \sqrt{2 + t^2}$$

$$\int_1^2 \sqrt{2+t^2}\, dt.$$

b) Suppose the density of the wire at the point (x, y, z) is $\delta(x, y, z) = z$. Find the mass of the wire.

Mass =

$$\int_1^2 \underbrace{f(t)}_{\text{density}} \cdot \|c'(t)\|\, dt$$

$$= \int_1^2 t \cdot \sqrt{2+t^2}\, dt$$

$u = 2 + t^2$

$du = 2t\,dt$

$$= \int_3^6 \frac{1}{2}\sqrt{u}\, du = \frac{1}{3} u^{\frac{3}{2}} \Big|_3^6 = \frac{6^{3/2}}{3} - \frac{3^{3/2}}{3}$$

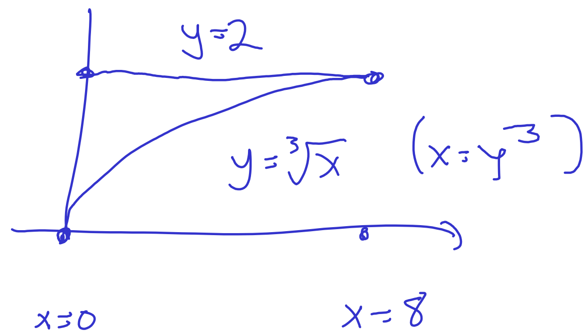
$$= \sqrt{24} - \sqrt{3}$$

4. (20 points) Compute

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \cos(y^4)\, dy\, dx.$$

Switch order of integration.

This becomes



$$\int_0^2 \int_0^{y^3} \cos(y^4)\, dx\, dy$$

$$= \int_0^2 x \cos(y^4) \Big|_0^{y^3} dy$$

$$= \int_0^2 y^3 \cos(y^4)\, dy$$

$u = y^4$

$du = 4y^3\, dy$

$$= \int_0^{16} \frac{1}{4} \cos(u)\, du$$

$$= \frac{1}{4} \sin(u) \Big|_0^{16} = \frac{\sin(16)}{4} - \cancel{\frac{\sin(0)}{4}} = \frac{\sin(16)}{4}$$

5. (25 points) Find the volume of the region bounded by the cylinder $x^2 + y^2 = 9$ and the planes $x + y + z = 1$ and $x + y + z = 4$.

Volume: (using cylindrical coordinates, but can be done in ordinary coordinates too):

this becomes $0 \le r \le 3$

$0 \le \theta \le 2\pi$

$1 - r\cos\theta - r\sin\theta \le z \le 4 - r\cos\theta - r\sin\theta$

$$\int_0^3 \int_0^{2\pi} \int_{1-r\cos\theta - r\sin\theta}^{4 - r\cos\theta - r\sin\theta} 1 \cdot r \, dz \, d\theta \, dr$$

$$= \int_0^3 \int_0^{2\pi} rz \Big|_{1-r\cos\theta-r\sin\theta}^{4-r\cos\theta-r\sin\theta} d\theta \, dr$$

$$= \int_0^3 \int_0^{2\pi} r\cdot(4 - r\cos\theta - r\sin\theta) - r(1 - r\cos\theta - r\sin\theta) \, d\theta \, dr$$

$$= \int_0^3 \int_0^{2\pi} 3r \, d\theta \, dr$$

$$= \int_0^3 3r\theta \Big|_0^{2\pi} dr$$

$$= \int_0^3 6\pi r \, dr = 3\pi r^2 \Big|_0^3 = 27\pi.$$

6. (30 points) Is $F = (y^2 + 2xz, 2xy, x^2 + 2z)$ the gradient of a function $f(x, y, z)$? If so, find such a function $f(x, y, z)$, and if not, show why not.

(If you take the curl, you'll get zero.)

Antiderivatives:

$y^2 + 2xz \longrightarrow xy^2 + x^2 z + c$

$2xy \longrightarrow xy^2 + d$

$x^2 + 2z \longrightarrow x^2 z + z^2 + e$

So, collecting terms together, we get

$f(x,y,z) = xy^2 + x^2 z + z^2.$

Check:

$\nabla f = (y^2 + 2xz, 2xy, x^2 + 2z)$

as desired.

7. (25 points) Evaluate

$$\iint \frac{x}{\sqrt{x^2+y^2}}\,dx\,dy$$

over the region, between angles 0 and $\frac{5\pi}{4}$, enclosed by the circles of radius 2 and 3 around the origin.

Polar coordinates!

$$\int_0^{5\pi/4}\int_2^3 \frac{r\cos\theta}{r}\, r\,dr\,d\theta$$

$$= \int_0^{5\pi/4}\int_2^3 r\cos\theta\,dr\,d\theta$$

$$= \int_0^{5\pi/4} \frac{r^2}{2}\cos\theta\Big|_2^3\,d\theta$$

$$= \int_0^{5\pi/4}\left(\frac{9}{2}-2\right)\cos\theta\,d\theta$$

$$= \frac{5}{2}\sin\theta\Big|_0^{5\pi/4}$$

$$= \frac{5}{2}\sin\left(\frac{5\pi}{4}\right) - \frac{5}{2}\sin(0)$$

$$= \frac{5}{2}\cdot\left(\frac{-1}{\sqrt{2}}\right) = \frac{-5}{2\sqrt{2}}$$

8. (20 points) Let $F(x, y, z)$ be the vector field

$$(\sin(x)e^y - 3xz^2, \cos(y) - \cos(x)e^y, z^3 + z\sin(y)).$$

Calculate the total flow of this vector field out of cylinder bounded by the surfaces $x = 0$, $x = 9$, and $y^2 + z^2 = 4$.

Divergence theorem.

$$\nabla \cdot F = (\cancel{\cos(x)e^y} - \cancel{3z^2}) + (\cancel{-\sin(y)} - \cancel{\cos(x)e^y}) + (\cancel{3z^2} + \cancel{\sin(y)})$$

$$= 0.$$

so the total flow out is

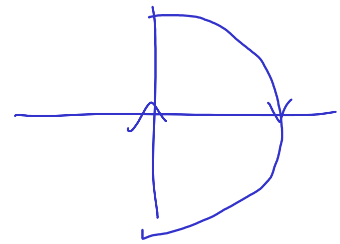
$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{0}^{9} 0 \, dx\, dy\, dz = 0.$$

9. (25 points) Calculate the total work done by the vector field

$$F(x,y) = (xe^x + yx, e^y + x)$$

on an object that travels *clockwise* around the boundary of the half-circle $x^2 + y^2 \leq 4$, $x \geq 0$.

Green's theorem: this is the negative of the double integral of



$$\frac{\partial}{\partial x}(e^y + x) - \frac{\partial}{\partial y}(xe^x + yx)$$

$$= 1 - x.$$

Integrating in polar coords:

$$-\int_{-\pi/2}^{\pi/2}\int_0^2 (1 - r\cos\theta)\, r\,dr\,d\theta$$

$$= -\int_{-\pi/2}^{\pi/2} \left(\frac{r^2}{2} - \frac{r^3}{3}\cos\theta\right)\Big|_0^2 \, d\theta$$

$$= -\int_{-\pi/2}^{\pi/2} \left(2 - \frac{8}{3}\cos\theta\right) d\theta$$

$$= -\left(2\theta - \frac{8}{3}\sin\theta\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= -\left[\left(2\frac{\pi}{2} - \frac{8}{3}\right) - \left(\frac{-2\pi}{2} - \frac{8}{3}(-1)\right)\right] = -\pi + \frac{8}{3} - \pi + \frac{8}{3}$$

$$= -2\pi + \frac{16}{3}$$

10. (30 points) Suppose we have a fluid with flow given by the vector field

$$F(x, y, z) = (e^z + \sin(x), 2\sin(y/2), e^{x+y^2})$$

through the rectangle with vertices $(0, 0, 0)$, $(0, 0, 3)$, $(1, 2, 3)$, and $(1, 2, 0)$. Give the total flow through this surface, in the form of both the magnitude of the total flow and a vector pointing in the same direction as the total flow.

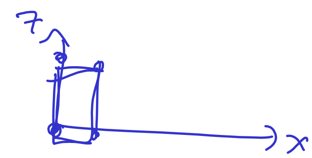
Formula for plane:

$\left.\begin{array}{l}(0,0,3)-(0,0,0)\\(1,2,0)-(0,0,0)\end{array}\right\}$ directions $\longrightarrow$ normal $\begin{vmatrix} i & j & k \\ 0 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} = (-6, 3, 0)$

plane is $0 = (-6,3,0)\cdot((x,y,z)-(0,0,0)) = -6x+3y.$

$$\boxed{y = 2x.}$$

parametrization:
rectangle, ignoring y-coordinate, goes from $(0,0)$ to $(0,3)$ to $(1,3)$ to $(1,0)$.



so $\boxed{\begin{array}{c}\Phi(u,v) = (u, 2u, v)\\ 0 \le u \le 1\\ 0 \le v \le 3\end{array}}$ parametrizes it.

Flux integral:

$T_u = (1,2,0)$ $\quad T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2,-1,0)$

$T_v = (0,0,1)$

flow: $\displaystyle\int_0^3\int_0^1 (e^v + \sin(u), 2\sin(u), e^{u+4u^2})\cdot(2,-1,0)\,du\,dv$

$\displaystyle = \int_0^3\int_0^1 2e^v\,du\,dv = \int_0^3 2e^v\,dv = 2e^v\Big|_0^3 = 2e^3 - 2.$

This is positive, so it goes in the same direction as the normal vector $T_u \times T_v = (2,-1,0).$

11. (25 points) Find the maximum and minimum values of the function $f(x, y) = x(x^2 + y^2 - 1)$ on the disk $x^2 + y^2 \leq 1$.

$$\nabla f(x,y) = [3x^2 + y^2 - 1, 2xy].$$

Critical points: $0 = \nabla f$ $\quad 0 = 3x^2 + y^2 - 1 \quad \underline{0 = 2xy}$

either $x = 0$ and $y^2 - 1 = 0$ $\quad y = \pm 1 \quad (0,1) \ (0,-1)$

or $y = 0$ and $3x^2 - 1 = 0$ $\quad x = \pm \frac{1}{\sqrt{3}} \quad \left(\frac{1}{\sqrt{3}}, 0\right) \ \left(\frac{-1}{\sqrt{3}}, 0\right)$

values: $f(0,1) = 0 \cdot (1-1) = 0$

$f(0,-1) = 0 \cdot (1-1) = 0$

$f\left(\frac{1}{\sqrt{3}}, 0\right) = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{-2}{3\sqrt{3}}$

$f\left(\frac{-1}{\sqrt{3}}, 0\right) = \frac{-1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{2}{3\sqrt{3}}.$

On boundary $x^2 + y^2 = 1$, so

$$f(x,y) = x \underbrace{(x^2 + y^2 - 1)}_{0} = 0.$$

among all of these, the max is $\frac{2}{3\sqrt{3}}$

the min is $\frac{-2}{3\sqrt{3}}$

12. (30 points) Find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ on the ellipsoid with equation $x^2 + 4y^2 + 9z^2 = 49$.

Constraint $g(x,y,z) = x^2+4y^2+9z^2 = 49$

Lagrange multipliers $\nabla g = \lambda \nabla f$

$\nabla f = [1, 1, -1]$ $\nabla g = [2x, 8y, 18z]$

Solve: $x^2+4y^2+9z^2=49$

$2x = \lambda \cdot 1 \longrightarrow \lambda = 2x$

$8y = \lambda \cdot 1 \longrightarrow 8y = 2x \quad y = x/4$

$18z = \lambda(-1) \longrightarrow 18z = -2x \quad z = -x/9$

$x^2 + 4\left(\frac{x}{4}\right)^2 + 9\left(\frac{-x}{9}\right)^2 = 49$

$\left(1 + \frac{1}{4} + \frac{1}{9}\right)x^2 = 49$

$\left(\frac{36+9+21}{36}\right)x^2 = 49$

$\frac{49x^2}{36} = 49$

$x^2 = 36$

$x = \pm 6$

solutions are at

$\left(6, \frac{6}{4}, \frac{-6}{9}\right) = \left(6, \frac{3}{2}, \frac{-2}{3}\right)$

and $\left(-6, \frac{-3}{2}, \frac{2}{3}\right)$.

Values:

$f\left(6, \frac{3}{2}, \frac{-2}{3}\right) = 6 + \frac{3}{2} + \frac{2}{3}$

$= \frac{36+9+4}{6} = \frac{49}{6}$.

$f\left(-6, \frac{-3}{2}, \frac{2}{3}\right) = -6 - \frac{3}{2} - \frac{2}{3}$

$= \frac{-49}{6}$.

max: $\frac{49}{6}$ min: $\frac{-49}{6}$.