1. (25 points) **Let $g(x,y) = e^{x+y}$ and $\mathbf{r} : \mathbf{R} \to \mathbf{R}^2$ be a function where $\mathbf{r}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{r}'(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $F'(0)$ where $F(t) = g(\mathbf{r}(t))$.**

   We use the special case of the chain rule:

$$F'(0) = \nabla g(\mathbf{r}(0)) \cdot \mathbf{r}'(0) = \nabla g(1,-1) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

   For this, we compute (check!)

$$\nabla g(x,y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x+y} \\ e^{x+y} \end{bmatrix}.$$

   Thus, plugging in $x = 1, y = -1$, $e^{x+y} = e0 = 1$ and thus

$$F'(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1)(1) + (1)(2) = 3.$$

2. (20 points) **Find an equation for the plane containing the line parametrized by $(x,y,z) = \mathbf{c}_1(t) = (2+t, 2t, 1-t)$ and the line parametrized by $(x,y,z) = \mathbf{c}_2(t) = (3-t, -2t-1, t-6)$. Write your answer in the form $Ax + By + Cz + D = 0$.**

   In order to find the equation of the plane, we will select a point on it, say $c_1(0) = (2,0,1)$, find a *non-zero* normal vector $\mathbf{n}$, and write

$$\mathbf{n} \cdot (x-2, y-0, z-1) = 0.$$

   The first attempt is to take the direction vectors

$$\mathbf{v}_1 = (1,2,-1), \quad \text{and} \quad \mathbf{v}_2 = (-1,-2,1)$$

   of the lines $\mathbf{c}_1(t)$ and $\mathbf{c}_2(t)$ respectively, and take theire cross-products. However, in the present case we see right away that $\mathbf{v}_2 = -\mathbf{v}_1$, which means that the lines are *parallel.* Therefore, the cross-product gives

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}.$$

   Thus, we do not get in this way a *non-zero* normal vector.

   Note also that the lines are *not* the same, they are just parallel.[1]

   Instead, we will keep, say, $\mathbf{v}_1$ and replace $\mathbf{v}_2$ by a vector connecting line 1 to line 2. In this way, we are sure that these two vectors are not parallel. Let's take:

$$\mathbf{v}_2' = c_2(0) - c_1(0) = (3,-1,-6) - (2,0,1) = (1,-1,-7).$$

   Then,

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2' = (1,2,-1) \times (1,-1,-7) = -15\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}.$$

[1] To see this, choose $t = -2$ so as to make the first component in $\mathbf{c}_1(t)$ equal to 0: $\mathbf{c}_1(-2) = (0,-4,3)$. On the other hand, in order to make the first component of $\mathbf{c}_2(t)$ equal to 0, we need $t = 3$ (so far, no trouble; the fact that the value for t is different for $\mathbf{c}_1(t)$ and for $\mathbf{c}_2(t)$ is *not* the issue). In turn, $\mathbf{c}_2(3) = (0,-7,-3)$. We conclude that $(0,-4,3) \neq (0,-7,-3)$ and so the lines are not identical.

Thus,

$$\begin{aligned} \mathbf{n} \cdot (x-2, y-0, z-1) &= 0 \\ -15(x-2) + 6(y-0) - 3(z-1) &= 0 \\ -15x + 6y - 3z + 33 &= 0 \\ -5x + 2y - z + 11 &= 0 \end{aligned}$$

3. (12 points) Match the equation with its graph. Give reasons for your choice.

   (a) $y = \sqrt{x^2 + z^2}$ (b) $y = x^2 + z^2$ (c) $1 = x^2 + z^2$ (d) $y = x^2 - z^2$

   a) Graph I. The xz-cross sections are circles while the xy- and yz-cross sections are lines. Because y cannot be negative, this is a half-cone.

   b) Graph IV. The xz-cross sections are circles while the xy- and yz-cross sections are parabolas. This is a paraboroid.

   c) Graph III. The xz-cross sections are circles while the xy- and yz-cross sections are lines with x, z constant. This is a cylinder.

   d) Graph II. The xy- and yz-cross sections are parabolas while the xz-cross sections are hyperbolas. This is a hyperboroid.

4. (20 points) **On a hill which is shaped like the surface $z = \sin(xy)$, a drop of rain falls and lands at the point $(1, 0, 0)$. In which direction along the xy plane would that drop of rain head from that point to get down the hill as quickly as possible? Justify your answer.**

   The direction along the xy plane that the rain drop will head downwards the fastest is opposite to the direction of the gradient of z with respect to x, y. This is because the directional derivative along a vector u is minimum when the dot product $u \cdot \nabla z = |u||\nabla z| cos\theta$ is the smallest, i.e. when $\theta = 180$ degree.

$$\nabla z = (\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = (y\cos(xy), x\cos(xy))$$

   At $(1, 0, 0), \nabla z = (0, 1)$. Hence the direction we want is $(0, -1)$ or along the negative y-axis.

5. (20 points) **Consider the surface defined by $z = f(x, y) = 2x^2 + xy + y^2 - 3$. Find the equation for the tangent plane at the point $(x, y, z) = (1, 2, 5)$.**

   $\frac{\partial f}{\partial x}(x, y) = 4x + y$, $\frac{\partial f}{\partial x}(1, 2) = 6$, $\frac{\partial f}{\partial y}(x, y) = x + 2y$, $\frac{\partial f}{\partial y}(1, 2) = 5$, $f(1, 2) = 5$

   Therefore, the equation for the tangent plane was

$$z = 5 + 6(x-1) + 5(y-2)$$
$$z = -11 + 6x + 5y$$

6. (11 points) **Consider the function of question 5: $f(x, y) = 2x^2 + xy + y^2 - 3$. Find a linear approximation to $f(x, y)$ at the point $(x, y) = (1, 2)$. Use this linear approximation to estimate $f(1.1, 1.9)$.**

The linear approximation is the tangent plane above $L(x, y) = -11+6x+5y$, and $L(1.1, 1.9) = 5 + 6(.1) + 5(-.1) = 5.1$.

7. (12 points) **Match the equation with its level curve plot. In each plot, the level curves $c = 0, 1, 2, 3, 4, 5, 6$ are shown. Give reasons for your choice.**

   (a) $f(x, y) = x^2 + y^2$ (b) $f(x, y) = x^2 - y^2 + 3$
   (c) $f(x, y) = \sqrt{x^2 + y^2}$ (d) $f(x, y) = 3x + y + 3$

   I is the level curve of (d), for (a) is the only function with linear level curve.

   II is the level curve of (b). Let h be the height. Then level curve of (b) is $x^2 - y^2 + 3 = h$ which is hyperbola.

   III is the level curve of (c). We have $\sqrt{x^2 + y^2} = h$, that is $x^2 + y^2 = h^2$. h= radius of the circle. So when heiht is raised by 1, radius of the circle is also increased by 1.

   IV is the level curve of (a). We have $x^2 + y^2 = h$. $\sqrt{h}$= radius . So when height is raised by 1, the increasing of the radius is less than 1.

   1. 3 points off if your answer is I(d), II(b), III(a), IV(c), but, you indicate the level curves for (a), (c) should be circles.
   2. 4 points off if your answer is I(d), II(c), III(b), IV(a).
   3. 9 points off if the only correct match I(d). But if you have put down words of correct characters of other functions, you receive 3 more points.

8. (20 points) **For the $g(x, y, z) = z - x^2 + y^2$, the level surfaces will be hyperbolic paraboloids. Find the tangent plane to the level surface $g(x, y, z) = 42$ at the point $(2, 7, -3)$.**

   $$\nabla g(x, y, z) = (-2x, 2y, 1)$$
   $$\nabla g(2, 7, -3) = (-4, 14, 1)$$

   So the tangent plane is given by

   $$-4(x - 2) + 14(y - 7) + (z + 3) = 0$$

   An alternative way.
   The level surface defined by

   $$z = x^2 - y^2 + 42.$$

   So we can view the problem as finding out tangent plane to $z = x^2 - y^2 + 42$ at the point $(2, 7, -3)$.

   1. 5 points off if 42 appears in your equation for the plane .
   2. 3 points off if your evaluation of the gradient if (-4, 14, -3) but not (-4, 14, 1).
   3. 5 points off if you didn't evaluate (-2x, 2y, 1) at the point (2, 7, -3), and gave a quadratic equation.
   4. 1 or 2 points for slightly computation mistakes.
   5. A small number of you don't know how to do the differentiate. But if all the remaining parts are more or less correct, you receive 10 points. If there are essential mistakes in remaining parts, you reveive 5 or less.