1. (20 points) **Evaluate**

$$\int_0^1 \int_{x^2}^{x} (2xy + x) dy\, dx.$$

Solution:

$$\begin{aligned}
\int_0^1 \int_{x^2}^{x} (2xy + x) dy\, dx &= \int_0^1 \left(xy^2 + xy \Big|_{x^2}^{x} \right) dx \\
&= \int_0^1 (x^3 + x^2) - (x^5 + x^3) dx \\
&= \int_0^1 x^2 - x^5 dx \\
&= \frac{1}{3}x^3 - \frac{1}{6}x^g \Big|_0^1 \\
&= \frac{1}{3} - \frac{1}{6} \\
&= \frac{1}{6}.
\end{aligned}$$

There are two integrals that need to be evaluated in this problem, each one was worth 10 points. Correctly taking the anti-derivative was 5 points, and correctly evaluating at the limits was 5 points.

2. (30 points) **Consider the wire parametrized by $\mathbf{r}(t) = (t\cos t, t\sin t, t)$ for $\sqrt{2} \le t \le \sqrt{7}$.**

   (i) (15 points) **Set up, but do not evaluate, the integral for the length of the wire.**

   $r'(t) = (-t\sin t + \cos t, t\cos t + \sin t, 1)$ **5 points**

   $l(t) = \int_r ds = \int_{\sqrt{2}}^{\sqrt{7}} ||r'(t)||dt$ **5 points form; 1 point bounds**

   $l(t) = \int_{\sqrt{2}}^{\sqrt{7}} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}dt = \int_{\sqrt{2}}^{\sqrt{7}} \sqrt{2 + t^2}dt$ **4 points**

   (ii) (15 points) **Suppose the density of the wire at the point (x, y, z) is $\delta(x, y, z) = z$. Find the mass of the wire. (This should be an easy integral to calculate.)**

$\int_{\sqrt{2}}^{\sqrt{7}} \delta(r(t))||r'(t)||dt$ **5 points**

$\int_{\sqrt{2}}^{\sqrt{7}} t\sqrt{2 + t^2}dt$ **(2 points for t, 3 points for $\sqrt{2 + t^2}$)**

$= 1/2 \int_{\sqrt{2}}^{\sqrt{7}} 2t\sqrt{2 + t^2}dt = (1/2)(2/3)(2 + t^2)^{3/2}|_{\sqrt{2}}^{\sqrt{7}} = 1/3(9^{3/2} - 4^{3/2}) = 19/3$ **5 points**

Notes: -2 for algebra mistakes

3. (20 points) **Reverse the order of integration for this integral:**

$$\int_0^1 \int_0^{3-3y} (3x - y + 1) dx\, dy$$

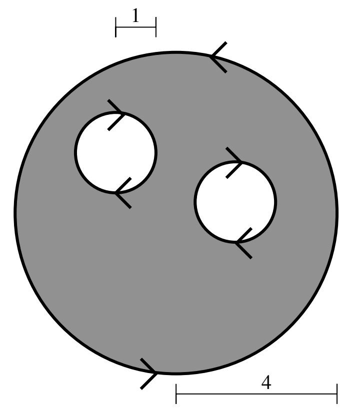
Solution:
The region of integration is a right triangle bounded by the x-axis, the y-axis and the line

$x = 3 - 3y$, which can be re-written $y = 1 - \frac{x}{3}$.
The x-intercept of the line $y = 1 - \frac{1}{3}$ is $x = 3$. Thus changing the order of integration gives the integral

$$\int_0^3 \int_0^{1-\frac{x}{3}} (3x - y + 1) dy\, dx.$$

There are 4 bounds that need to be correctly specified (upper and lower bounds in x and y respectively). 5 points were given for each correct bound.

4. (25 points) **Let $\mathbf{F}(x, y) = (2y, -x)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{s}$ where C is the boundary of the shaded region below, oriented as pictured. Note that the outer circle has radius 4 and the two smaller circles have radius 1.**



The only way to get full credit on this problem was to use Green's Theorem

Note that C is positively oriented as the boundary of D, where D represents the shaded region. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C 2y dx - x dy = \int\int_D \frac{\partial(-x)}{\partial x} - \frac{\partial(2y)}{\partial y} dx dy = \int\int_D -1 - 2 dx dy$$

This last integral is $-3*\text{Area}(D) = -3 * 14\pi = -42\pi$.

Problem 4 grading rubric:

- +5 if solution was coordinate independent.
- If the student used Green, the other 20 points were distributed roughly as follows:
  - +15 for proper use of Green's theorem, including calculating the partials, getting the domain correct and getting the orientations (signs) correct.
  - +5 for correctly computing the area.
- If the student parametrized the curves, he/she automatically lost 5 points for using coordinates. The remaining 20 points were distributed roughly as follows:
  - +5 for parametrizations.
  - +10 for correctly setting up the line integrals, including using the correct formula, getting the orientations correct, and getting the proper integrand.
  - +5 computation points

5. (20 points) **Find the value of the line integral $\int_C y\,dx + x^2\,dy$ along the parabola C defined by $y = x^2$ from the point $(0,0)$ to the point $(1,1)$.**

C follows the parabola $y = x^2$, so one possible parametrization is $\mathbf{c}(t) = (t, t^2)$, where $0 \leq t \leq 1$. Let $\mathbf{F}(x,y) = (y, x^2)$. Then,

$$\int_C y dx + x^2 dy = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_0^1 (t^2, t^2) \cdot (1, 2t) dt =$$

$$\int_0^1 t^2 + 2t^3 dt = \frac{t^3}{3} + \frac{t^4}{2}\Big|_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Problem 5 grading rubric:

- +5 for getting a correct parametrization.
- +5 for using the formula for evaluating line integrals.
- +5 for getting the correct integrand.
- +5 computation points.
- -20 if a student used Green's Theorem.

6. (25 points) **Let W be the region in $\mathbf{R}^3$ which is inside the cylinder $y^2 + z^2 = 1$ and bounded by the yz-plane and the plane $z + x = 1$.**

   (i) (15 points) **Set up the integral $\iiint_W f(x,y,z)\, dV$ as an iterated integral with order $dx\,dz\,dy$.**

   The surface described by the equation is a cylinder whose center axis is the x-axis; here we are asked to set it up as an iterated integral with order $dx\ dz\ dy$.

   Notice that the projection of the cylinder into the yz-plane is just a unit circle (the whole circle, not just the top half!), so the dz and dy limits are just over that circle. So we have

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{?}^{?} f(x,y,z) dx dz dy.$$

   To find the dx limits, observe that we want to integrate over the distance from the yz-plane (i.e. $x = 0$) to the given plane $z + x = 1$ for the entire circle described above. So the lower limit is $x = 0$, and the upper limit is described by the plane, which we solve for x to get $x = 1 - z$. The final answer is

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{1-z} f(x,y,z) dx dz dy.$$

   (ii) (10 points) **Set up the integral in (i) as an iterated integral with order $dy\,dz\,dx$.**

Now we are asked to find the integral with the bounds $dy\ dz\ dx$. For the outer two limits, we are again integrating over the distance from the yz-plane to the plane $z+x = 1$. The outermost limit, in the x-direction, cannot involve y or z, so we need to see the minimum and maximum values that x takes on for the volume in question. The minimum is clearly 0, but the maximum is where the plane leaves the sphere; this happens when $z = -1$, or at $x = 2$.

The dz-bounds, on the other hand, go from the lowest value that z takes on over the relevant volume, which here is -1, to the plane. Here we need the plane's equation solved for z: $z = 1 - x$. What we have so far is

$$\int_0^2 \int_{-1}^{1-x} \int_?^? f(x, y, z) dydzdx.$$

Finally, the bounds for y need to ensure that we are only integrating over the circle that is the cross-section of the cylinder, which means we use the standard integrating-over-a-circle bounds here. The final answer is

$$\int_0^2 \int_{-1}^{1-x} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} f(x, y, z) dydzdx.$$