Math 2374	**Name (Print):**	Solutions, 12/16/10
Fall 2010	**Student ID:**	Please report errors
Midterm 3	**Section Number:**	
December 1, 2010	**Teaching Assistant:**	
Time Limit: 1 hour	**Signature:**	

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	20 pts	
2	25 pts	
3	20 pts	
4	25 pts	
5	25 pts	
6	25 pts	
TOTAL	140 pts	

TA sections:

Section	TA	Discussion time
011	Chen	T 9:05am
012	Chen	T 11:15am
013	Klein	T 1:25pm
014	Klein	T 3:35pm
015	Bu	T 4:40pm
016	Bu	T 6:45pm
021	Bashkirov	Th 8:00am
022	Bashkirov	Th 10:10am
023	He	Th 12:20pm
024	He	Th 2:30pm
025	Lee	Th 4:40pm
026	Lee	Th 6:45pm

1. (20 points) Use spherical coordinates to give a parametrization of the ellipsoid determined by the equation $x^2 + 4y^2 + 9z^2 = 1$.

In spherical coords,

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

so the equation becomes

$$(\rho\cos\theta\sin\phi)^2 + 4(\rho\sin\theta\sin\phi)^2 + 9(\rho\cos\phi)^2 = 1$$

$$\rho^2(\cos^2\theta\sin^2\phi + 4\sin^2\theta\sin^2\phi + 9\cos^2\phi) = 1$$

$$\rho = \frac{1}{\sqrt{\cos^2\theta\sin^2\phi + 4\sin^2\theta\sin^2\phi + 9\cos^2\phi}}$$

So for our parametrization, we can define $\theta = u$, $\phi = v$, and

$$\Phi(u,v) =$$

$$\frac{1}{\sqrt{\cos^2 u\sin^2 v + 4\sin^2 u\sin^2 v + 9\cos^2 v}} \cdot (\cos u\sin v, \sin u\sin v, \cos v)$$

2. (25 points) Suppose we have a fluid in $\mathbb{R}^3$ with flow given by the vector field $F(x, y, z) = (x, y, 1)$. Suppose we have a surface parametrized by $\phi(u, v) = (u + v, v, uv)$ for $0 \le u \le 1$, $2 \le v \le 3$. Find the total rate of flow of fluid through this surface (in the direction given by the normal vector of this parametrization).

First we find the tangent vectors:

$T_u = (1, 0, v)$

$T_v = (1, 1, u)$

$$n = \begin{vmatrix} i & j & k \\ 1 & 0 & v \\ 1 & 1 & u \end{vmatrix} = (-v, v-u, 1)$$

and so the surface integral is

$$\iint F \cdot dS = \int_0^1 \int_2^3 F(u+v, v, uv) \cdot (-v, v-u, 1)\, dv\, du$$

$$= \int_0^1 \int_2^3 (u+v, v, 1) \cdot (-v, v-u, 1)\, dv\, du$$

$$= \int_0^1 \int_2^3 (-uv \cancel{-v^2} \cancel{+v^2} - uv + 1)\, dv\, du$$

$$= \int_0^1 \int_2^3 (1 - 2uv)\, dv\, du$$

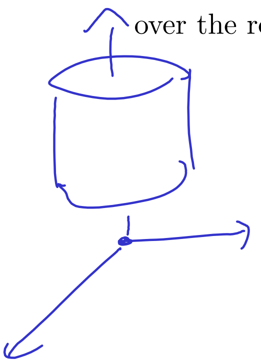
$$= \int_0^1 (v - uv^2)\Big|_2^3 du = \int_0^1 (3 - 9u) - (2 - 4u)\, du = \int_0^1 (1 - 5u)\, du$$

$$= \left(u - \frac{5}{2}u^2\right)\Big|_0^1 = 1 - \frac{5}{2} = \frac{-3}{2}$$

3. (20 points) Evaluate the integral

$$\iiint_D e^{x^2+y^2}\, dx\, dy\, dz$$

over the region described by the equations $x^2 + y^2 \le 2$, $1 \le z \le 5$.



Since the region is a cylinder, let's try cylindrical coords (especially since x^2+y^2 appears!)

$$\iiint_D e^{x^2+y^2}\, dx\, dy\, dz = \int_1^5 \int_0^{2\pi} \int_0^2 e^{r^2} \cdot r\, dr\, d\theta\, dz$$

(Substitute $u = r^2$ $du = 2r\,dr$

$$= \int_1^5 \int_0^{2\pi} \int_0^4 \frac{1}{2} e^u\, du\, d\theta\, dz = \int_1^5 \int_0^{2\pi} \frac{1}{2} e^u \Big|_0^4 d\theta\, dz$$

$$= \int_1^5 \int_0^{2\pi} \frac{1}{2}(e^4 - 1)\, d\theta\, dz = \int_1^5 \frac{1}{2}(e^4-1)\cdot 2\pi\, dz$$

$$= \frac{1}{2}\cdot(e^4-1)\cdot 2\pi \cdot 4 = 4\pi(e^4-1)$$

4. (25 points) Let D be the region in $\mathbb{R}^2$ bounded by the curves $x+y=2$, $x+y=4$, $x-y=0$, $x-y=4$. Use the change of coordinates $x=u+v$, $y=u-v$ to evaluate the integral

$$\iint_D \frac{x-y}{x+y}\,dx\,dy.$$

The Jacobian determinant of this change is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\right| = |(-2)| = 2.$$

We have $x+y=2u$, $x-y=2v$.

Under this change of coords the region becomes the region D^* defined by $1\le u\le 2$, $0\le v\le 2$

This integral then becomes

$$\iint_D \frac{x-y}{x+y}\,dx\,dy = \int_1^2\int_0^2 \frac{v}{u}\cdot(2)\,dv\,du$$

$$= \int_1^2 \frac{v^2}{u}\Big|_0^2\,du = \int_1^2 \frac{4}{u}\,du = 4\ln|u|\Big|_1^2$$

$$= 4\ln(2) - \underbrace{4\ln(1)}_{0}$$

$$= 4\ln(2)$$

5. (25 points) Given a surface S described by the equation $z = g(x, y)$ as (x, y) varies over a region $D \subset \mathbb{R}^2$, **give a parametrization of** S and **use it to derive the formula**

$$\text{area}(S) = \iint_D \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}\, dx\, dy$$

The parametrization of the surface is

$$\Phi(x, y) = (x, y, g(x, y)) \quad \text{for} \quad (x, y) \in D.$$

In this parametrization,

$$T = \left(1, 0, \frac{\partial g}{\partial x}(x, y)\right)$$

$$T = \left(0, 1, \frac{\partial g}{\partial y}(x, y)\right)$$

$$n = T \times T = \begin{vmatrix} i & j & k \\ 1 & 0 & \partial g/\partial x \\ 0 & 1 & \partial g/\partial y \end{vmatrix}$$

$$\left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

$$\|n\| = \sqrt{\left(-\frac{\partial g}{\partial x}\right)^2 + \left(-\frac{\partial g}{\partial y}\right)^2 + 1}$$

and so the surface area is

$$\iint_D dS = \iint_D \|n\|\, dx\, dy = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy$$

as desired.

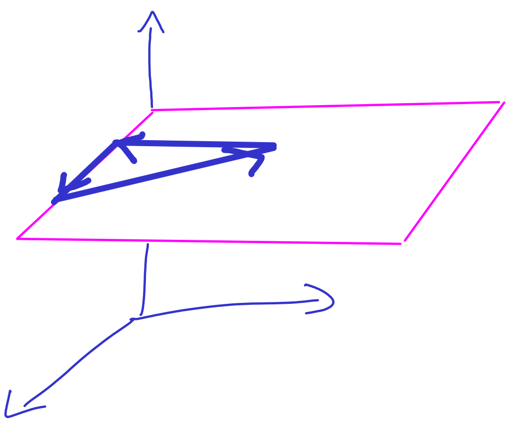
6. (25 points) Let C be the perimeter of the triangle in $\mathbb{R}^3$ with vertices $(1,0,3)$, $(2,0,3)$, and $(1,1,3)$ in order. If $F(x,y,z) = (xe^x, y^3 + x^2, xz)$, use Stokes' theorem to evaluate the line integral

$$\int_C F \cdot ds.$$

To use Stokes theorem, first we find curl(F):

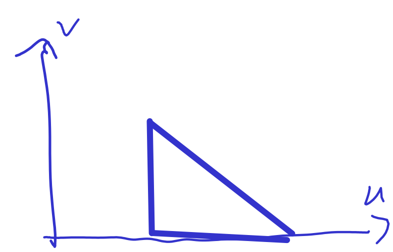
$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^x & y^3+x^2 & xz \end{vmatrix} = (0, -z, 2x)$$

Next, we parametrize this triangle. All the points lie in the plane $z=3$, and so we can use the parametrization $\Phi(u,v) = (u,v,3)$.



The x-y coordinates cover the triangle with vertices $(1,0)$, $(2,0)$, and $(1,1)$, which have limits of integration

$$1 \le u \le 2 \qquad 0 \le v \le 2-u$$



The normal vector for this param. is calculated by:

$$T_u = (1,0,0) \qquad T_v = (0,1,0) \qquad n = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$$

which is in the correct direction for Stokes' theorem.

The integral, by Stokes theorem, is

$$\iint \operatorname{curl}(F)\cdot dS = \int_1^2 \int_0^{2-u} (0,3,2u)\cdot(0,0,1)\,dv\,du = \int_1^2 \int_0^{2-u} 2u\,dv\,du$$

$$= \int_1^2 2u(2-u)\,du = \int_1^2 4u - 2u^2\,du$$

$$= \left(2u^2 - \frac{2}{3}u^3\right)\Big|_1^2 = \left(8 - \frac{16}{3}\right) - \left(2 - \frac{2}{3}\right)$$

$$= \frac{4}{3}$$