**Math 2374**
**Spring 2008**
**Exam 2 solutions**

1. (30 points) (a) (3 points each)

   $\text{curl}(\nabla f)$ – YES

   $\text{curl}(\text{div}\,\mathbf{F})$ – NO; $\text{div}\,\mathbf{F}$ is a real-valued function, and curl applies to vector fields.

   $\text{div}(\text{curl}\,\mathbf{F})$ – YES

   $\nabla \times (\nabla \times \mathbf{F})$ – YES

   $\nabla \times (\nabla \cdot \mathbf{F})$ – NO; as above.

   (b) (15) Since $||\mathbf{c}(t)||^2 = \mathbf{c}(t) \cdot \mathbf{c}(t) = 1$, we have $\frac{d}{dt}(\mathbf{c}(t) \cdot \mathbf{c}(t)) = 0$, and $0 = \frac{d}{dt}(\mathbf{c}(t) \cdot \mathbf{c}(t)) = \mathbf{c}'(t) \cdot \mathbf{c}(t) + \mathbf{c}(t) \cdot \mathbf{c}'(t) = 2\mathbf{c}'(t) \cdot \mathbf{c}(t)$, which implies that $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ are perpendicular.

2. (20 points) Change the order of integration:

$$\int_0^2 \int_0^{y^2} e^{(y^3)}\, dx\, dy = \int_0^2 y^2 e^{(y^3)}\, dy = \frac{1}{3}\int_0^8 e^u\, du = \frac{e^8 - 1}{3}.$$

3. (20 points) $\mathbf{F} = (yz^2, xy^2, zx^2)$ is not the gradient of a function $f(x, y, z)$, because $\text{curl}\,\mathbf{F} = (0, 2yz - 2xz, y^2 - z^2) \neq \mathbf{0}$.

4. (30 points) (a) (5) $||\mathbf{c}(4\pi) - \mathbf{c}(2\pi)|| = ||(1, 0, 8\pi) - (1, 0, 4\pi)|| = 8\pi - 4\pi = 4\pi$.

   (b) (10) Since $\mathbf{c}'(t) = (-\sin t, \cos t, 2)$, we have

$$L = \int_{2\pi}^{4\pi} ((-\sin t)^2 + (\cos t)^2 + 2^2)^{1/2}\, dt = \int_{2\pi}^{4\pi} \sqrt{5}\, dt = 2\pi\sqrt{5}.$$

   (c) (15) Since $||\mathbf{c}'(t)|| = \sqrt{5}$ as above, we have

$$M = \int_{2\pi}^{4\pi} \frac{1}{z}\sqrt{5}\, dt = \frac{\sqrt{5}}{2}\int_{2\pi}^{4\pi} \frac{1}{t}\, dt = \frac{\sqrt{5}}{2}(\ln(4\pi) - \ln(2\pi)) = \frac{\sqrt{5}}{2}\ln 2.$$

5. (20 points) Since $\mathbf{F} = (yz^2, xz^2, 2xyz)$ is the gradient of the function $f(x, y, z) = xyz^2$, by the fundamental theorem we have

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = f(3, 1, 2) - f(1, -2, -3) = 12 - (-18) = 30.$$

   One may also parametrize the path and calculate the line integral directly.

6. (20 points) For $\mathbf{F} = (P, Q) = (x^2, xy)$, we have $Q_x - P_y = y$. Then by Green's Theorem,

$$\int_{C^+} \mathbf{F} \cdot d\mathbf{s} = \iint_R y\, dA,$$

   where $R$ is the triangular region bounded by $C^+$. That region is bounded by the $x$- and $y$-axes and the line $x = 3 - \frac{3}{2}y$. So we have

$$\int_{C^+} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 \int_0^{3-\frac{3}{2}y} y\, dx\, dy = \int_0^2 (3y - \frac{3}{2}y^2)\, dy = 2.$$

   One may also parametrize the path (in three parts) and calculate the line integral directly.