

Math 2374
Spring 2009
Midterm 2
April 1, 2009
Time Limit: 1 hour

Name (Print): _____
Student ID: _____
Section Number: _____
Teaching Assistant: _____
Signature: _____

This exam contains 4 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch \times 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π , or $\sqrt{2}$. However, you should simplify $\cos \frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals.** To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit.** Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit;** an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	25 pts	
3	20 pts	
4	25 pts	
5	20 pts	
6	25 pts	
TOTAL	140 pts	

SKETCH OF THE SOLUTION

there might be some typos; tell me (Francisco) if you find one

1. (25 points) Consider the region enclosed by the cylinder $x^2 + z^2 = 9$, the plane $x - y + z = 0$ and the plane $x - y + z + 4 = 0$. Compute the volume of this region by setting up the triple integral as $dy dx dz$.

Points of this region can be described as the points inside the cylinder (that is $x^2 + z^2 \leq 9$) that lie between the planes. Knowing the (x, z) -coordinates we can give lower and upper limits for y using the equations for the planes. The region can then be

$$-3 \leq z \leq 3, \quad -\sqrt{9-z^2} \leq x \leq \sqrt{9-z^2}, \quad x+z \leq y \leq x+z+4.$$

For the volume we integrate:

$$\begin{aligned} \iiint_W 1 \, dV &= \int_{-3}^3 \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{x+z}^{x+z+4} dy dx dz = \int_{-3}^3 \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} 4 dx dz \\ &= \int_{-3}^3 8\sqrt{9-z^2} dz = 36\pi. \end{aligned}$$

The last integral can be computed with the substitution $z = 3 \cos \theta$. Alternatively, we can recognize that

$$\int_{-3}^3 \sqrt{9-z^2} dz$$

is the arc of half a circle of radius 3, which is $\frac{1}{2}9\pi$.

2. (25 points) Let \mathbf{c} be the straight line path from $(1, 1, 3)$ to $(2, 1, 5)$ and let $\mathbf{F}(x, y, z) = (xz, e^x \cos(y+z), z-2x-y)$. Compute

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

We can parametrize the segment as

$$\mathbf{c}(t) = (1, 1, 3) + t((2, 1, 5) - (1, 1, 3)) = (1, 1, 3) + t(1, 0, 2) = (t+1, 1, 2t+3),$$

for $0 \leq t \leq 1$. The velocity vector is $\mathbf{c}'(t) = (1, 0, 2)$. The value of the vector field on points of the path is given by

$$\mathbf{F}(\mathbf{c}(t)) = ((t+1)(2t+3), e^{t+1} \cos(1+2t+3), 2t+3-2t-2-1) = (2t^2+5t+3, e^{t+1} \cos(2t+4), 0).$$

The value of the line integral is

$$\begin{aligned} \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_0^1 (2t^2 + 5t + 3, e^{t+1} \cos(2t+4), 0) \cdot (1, 0, 2) dt \\ &= \int_0^1 (2t^2 + 5t + 3) dt = \left. \frac{2}{3}t^3 + \frac{5}{2}t^2 + 3t \right|_{t=0}^1 = \frac{37}{6}. \end{aligned}$$

3. (20 points) Consider the triangle with vertices $(1, 1)$, $(4, 1)$ and $(4, 2)$ and let \mathbf{c} be the *negatively oriented* closed path along the boundary of the triangle. Consider the vector field $\mathbf{F}(x, y) = (\cos x - xy^2, e^y + 2x^2y)$. Using Green's Theorem, compute

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

If we write $\mathbf{F} = (P, Q)$, Green's Theorem says that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = - \int_{\mathbf{c}^+} \mathbf{F} \cdot d\mathbf{s} = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = - \iint_D (4xy - (-2xy)) dA = - \iint_D 6xy dA.$$

The triangle D is the area delimited by the lines $y = 1$, $x = 4$ and $y = (x + 2)/3$. A possible description of this region is given by the inequalities

$$1 \leq x \leq 4, \quad 1 \leq y \leq \frac{x+2}{3}.$$

Then we compute the resulting integral as

$$\begin{aligned} - \int_1^4 \int_1^{(x+2)/3} 6xy \, dy \, dx &= - \int_1^4 3xy^2 \Big|_{y=1}^{(x+2)/3} dx = \int_1^4 \left(-\frac{1}{3}x^3 - \frac{4}{3}x^2 + \frac{5}{3}x \right) dx \\ &= \left(-\frac{1}{12}x^4 - \frac{4}{9}x^3 + \frac{5}{6}x^2 \right) \Big|_{x=1}^4 = -\frac{147}{4}. \end{aligned}$$

4. (25 points) Consider the path $\mathbf{c}(t) = (3t \cos t, 3t \sin t, \sqrt{8}t^{3/2})$ for $0 \leq t \leq 2$.

- (a) Show that the speed is increasing with time.

The velocity vector is:

$$\mathbf{c}'(t) = (3 \cos t - 3t \sin t, 3 \sin t + 3t \cos t, 3\sqrt{2}t^{1/2}).$$

The speed is its norm

$$\begin{aligned} \|\mathbf{c}'(t)\| &= \sqrt{(3 \cos t - 3t \sin t)^2 + (3 \sin t + 3t \cos t)^2 + (3\sqrt{2}t^{1/2})^2} \\ &= 3\sqrt{\cos^2 t + t^2 \sin^2 t - 2t \cos t \sin t + \sin^2 t + t^2 \cos^2 t + 2t \sin t \cos t + 2t} \\ &= 3\sqrt{1 + t^2 + 2t} = 3\sqrt{(1+t)^2} = 3(1+t). \end{aligned}$$

It is clear that this function is increasing linearly with time.

- (b) Compute the length of the path.

$$\text{length} = \int_0^2 \|\mathbf{c}'(t)\| dt = \int_0^2 3(1+t) dt = \frac{3}{2}(1+t)^2 \Big|_{t=0}^2 = 12.$$

5. (20 points) Show that the path $\mathbf{c}(t) = (\cos t, \sin t, -\cos 2t)$ is a flow line of the vector field $\mathbf{F}(x, y, z) = (-y, x, 4xy)$.

We just have to verify that $\mathbf{F}(\mathbf{c}(t)) = \mathbf{c}'(t)$ for all values of t . On the one hand

$$\mathbf{c}'(t) = (-\sin t, \cos t, 2\sin 2t)$$

and on the other hand

$$\mathbf{F}(\mathbf{c}(t)) = (-\sin t, \cos t, 4\sin t \cos t).$$

These functions coincide because $\sin 2t = 2\sin t \cos t$.

6. (25 points) Change the order of integration in the integral

$$\int_0^{1/\sqrt{2}} \int_{\arccos x}^{\pi/2} dy dx$$

and then compute the resulting integral.

As given, the limits are

$$0 \leq x \leq \frac{1}{\sqrt{2}}, \quad \arccos x \leq y \leq \frac{\pi}{2}.$$

The region is therefore delimited by the lines $y = \pi/2$, $x = 1/\sqrt{2}$ and the curve $y = \arccos x$. Note that $\arccos(1/\sqrt{2}) = \pi/4$. The region can then be described as

$$\frac{\pi}{4} \leq y \leq \frac{\pi}{2}, \quad \cos y \leq x \leq \frac{1}{\sqrt{2}}.$$

The integral is then

$$\begin{aligned} \int_0^{1/\sqrt{2}} \int_{\arccos x}^{\pi/2} dy dx &= \int_{\pi/4}^{\pi/2} \int_{\cos y}^{1/\sqrt{2}} dx dy = \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sqrt{2}} - \cos y \right) dy \\ &= \frac{\pi}{4\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}} \right). \end{aligned}$$