Math 2374
Spring 2009
Midterm 3
April 29, 2009
Time Limit: 1 hour

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exam contains 5 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	30 pts	
3	30 pts	
4	30 pts	
5	25 pts	
TOTAL	140 pts	

SKETCH OF THE SOLUTION
there might be some typos; tell me (Francisco) if you find one

1. (25 points) Construct a change of variables $T : D^* \to D$ that maps the unit square $D^* = [0,1] \times [0,1]$ to the parallelogram D with vertices $(1,1)$, $(3,3)$, $(2,8)$ and $(0,6)$. Use T to compute the area of D.

   We choose the vertex $(1,1)$ as the image point of the vertex $(0,0)$ of D^*. Now we have to find the two adjacent vertices, that is, $(3,3)$ and $(0,6)$. The change of variables is then

$$(x,y) = T(u,v) = (1,1) + u\big((3,3)-(1,1)\big) + v\big((0,6)-(1,1)\big) = (1+2u-v, 1+2u+5v).$$

   We now compute the derivative of T and the absolute value of its determinant

$$\mathbf{D}T(x,y) = \begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix}, \qquad |\det \mathbf{D}T(x,y)| = 12.$$

   Finally we apply the change of variables in the integral over D that we set up to calculate the area of D

$$\text{Area}(D) = \iint_D dx\,dy = \iint_{D^*} 12\,du\,dv = \int_0^1\int_0^1 12\,du\,dv = 12.$$

2. (30 points) Compute the volume of the part of the sphere $x^2+y^2+z^2=4$ that lies outside the surface $x^2+z^2=3$.

   The volume can be described as follows: the coordinates (x,z) satisfy

$$3 \le x^2+z^2 \le 4$$

   and given (x,z), we know that

$$-\sqrt{4-(x^2+z^2)} \le y \le \sqrt{4-(x^2+z^2)}.$$

   The first option is to start here, set up the integral in (x,y,z):

$$\begin{aligned}\text{Volume}(W) &= \iiint_W dV = \iint_{3\le x^2+z^2\le 4} \int_{-\sqrt{4-(x^2+z^2)}}^{\sqrt{4-(x^2+z^2)}} dy\,dx\,dz \\ &= \iint_{3\le x^2+z^2\le 4} 2\sqrt{4-(x^2+z^2)}\,dx\,dz\end{aligned}$$

   and then make a change of variables to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ (with $\sqrt{3} \le r \le 2$ and $0 \le \theta \le 2\pi$) to obtain

$$\begin{aligned}\iint_{3\le x^2+z^2\le 4} 2\sqrt{4-(x^2+z^2)}\,dx\,dz &= \int_0^{2\pi}\int_{\sqrt{3}}^2 2\sqrt{4-r^2}\; r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{-2}{3}(4-r^2)^{3/2}\Big|_{r=\sqrt{3}}^2 d\theta = \int_0^{2\pi}\frac{2}{3}\,d\theta = \frac{4\pi}{3}.\end{aligned}$$

   As second option consists of using cylindrical coordinates (with y as the vertical axis). In this way

$$\begin{array}{lcc} x = r\cos\theta & \sqrt{3} \le r \le 2 & \\ y = y & 0 \le \theta \le 2\pi & dx\,dy\,dz = r\,dr\,d\theta\,dy \\ z = r\sin\theta & -\sqrt{4-r^2} \le y \le \sqrt{4-r^2} & \end{array}$$

and

$$\text{Volume}(W) \;=\; \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dy\,dr\,d\theta = \ldots = \frac{4\pi}{3}$$

(the integrals are exactly the same as before).

3. (30 points) Let S be the part of the plane $x+y+z=1$ that lies in the first octant ($x \geq 0$, $y \geq 0$ and $z \geq 0$), with the normal oriented 'upwards' (the $z-$component is positive). Let

$$\mathbf{F}(x,y,z) = (y,\; xyz\cos(x+y+z)e^{2x+3y},\; 0).$$

Using Stokes' Theorem, compute

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

Stokes' Theorem says that

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s},$$

where $\mathbf{c}$ is the closed path that goes along the boundary of S, 'leaving S on the left'. In this case. $\mathbf{c}$ is composed of three straight line segments: $\mathbf{c}_1$ goes from $(1,0,0)$ to $(0,1,0)$; $\mathbf{c}_2$ goes from $(0,1,0)$ to $(0,0,1)$ and $\mathbf{c}_3$ goes from $(0,0,1)$ to $(1,0,0)$.

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{c}_3} \mathbf{F}\cdot d\mathbf{s}$$

We now parametrize the three segments (in all cases $0 \leq t \leq 1$):

$$\begin{aligned}
\mathbf{c}_1(t) &= (1,0,0)+t\big((0,1,0)-(1,0,0)\big) = (1-t,t,0)\\
\mathbf{c}_2(t) &= (0,1,0)+t\big((0,0,1)-(0,1,0)\big) = (0,1-t,t)\\
\mathbf{c}_3(t) &= (0,0,1)+t\big((1,0,0)-(0,0,1)\big) = (t,0,1-t).
\end{aligned}$$

The respective velocity vectors are

$$\mathbf{c}_1'(t) = (-1,1,0), \qquad \mathbf{c}_2(t) = (0,-1,1), \qquad \mathbf{c}_3'(t) = (1,0,-1)$$

and

$$\begin{aligned}
\mathbf{F}(\mathbf{c}_1(t)) &= \mathbf{F}(1-t,t,0) = (t,0,0),\\
\mathbf{F}(\mathbf{c}_2(t)) &= \mathbf{F}(0,1-t,t) = (1-t,0,0),\\
\mathbf{F}(\mathbf{c}_3(t)) &= \mathbf{F}(t,0,1-t) = (0,0,0).
\end{aligned}$$

Therefore the integral we have to compute is

$$\begin{aligned}
\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} &= \int_0^1 (t,0,0)\cdot(-1,1,0)\,dt + \int_0^1 (1-t,0,0)\cdot(0,-1,1)dt + \int_0^1 (0,0,0)\cdot(1,0,-1)dt\\
&= \int_0^1 (-t)\,dt = \frac{-t^2}{2}\Big|_{t=0}^{1} = -\frac{1}{2}.
\end{aligned}$$

4. (30 points) The surfaces $x^2+y^2+z^2=1$ and $x^2+y^2-z^2=\frac{1}{2}$ intersect in two circles. Let S be the part of the surface of the sphere that lies between these two circles.

(a) Parametrize this surface using spherical coordinates. Include the bounds for the parameters in your answer.

Adding both equations we obtain that

$$2(x^2+y^2)=1+\frac{1}{2} \qquad \Longrightarrow \qquad x^2+y^2=\frac{3}{4},$$

and using now any of both we obtain

$$z^2=\frac{1}{4}, \qquad z=\pm\frac{1}{2}.$$

This means that the intersection is two circles of radii $\sqrt{3}/2$ one placed at $z=1/2$ and the other one at $z=-1/2$. We are then dealing with the part of the spherical surface that lies between $z=-1/2$ and $z=1/2$. We only need to find the latitudes (that is, ϕ) for these two parallels. On the sphere $z=\cos\phi$, so $z=\pm 1/2$ correspond respectively to

$$\phi=\frac{\pi}{3}, \qquad \text{and} \qquad \phi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

and we have that the parametrization of the surface using spherical coordinates is

$$(\cos\theta\,\sin\phi,\sin\theta\,\sin\phi,\,\cos\phi), \qquad 0\le\theta\le 2\pi, \qquad \frac{\pi}{3}\le\phi\le\frac{2\pi}{3}.$$

(b) For the vector field $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, and considering the outwards orientation for the normal vector, compute

$$\iint_S \mathbf{r}\cdot d\mathbf{S}.$$

With the parametrization of the sphere in spherical coordinates, the normal vector we obtain is

$$\mathbf{T}_\theta\times\mathbf{T}_\phi=(-\cos\theta\,\sin^2\phi,-\sin\theta\,\sin^2\phi,-\sin\phi\,\cos\phi)$$

(we have done this computation several times in class). This normal clearly points inwards (for points of the upper hemisphere it is pointing down). Therefore, for what we have been asked to do we need to change its sign. Finally we compute the integral

$$\begin{aligned}\iint_S \mathbf{r}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)\cdot(\cos\theta\sin^2\phi,\sin\theta\sin^2\phi,\cos\phi\sin\phi)d\phi\;d\theta\\ &= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\left(\cos^2\theta\sin^3\phi+\sin^2\theta\sin^3\phi+\cos^2\phi\sin\phi\right)d\phi d\theta\\ &= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\sin\phi\,d\phi d\theta=\int_0^{2\pi}(-\cos\phi)\Big|_{\phi=\frac{\pi}{3}}^{\frac{2\pi}{3}}d\theta=\int_0^{2\pi}1\,d\theta=2\pi.\end{aligned}$$

5. (25 points) Show that the vector field $\mathbf{F}(x, y, z) = (2xyz, x^2z - z\sin y, x^2y + \cos y + z)$ is conservative. Find f such that $\mathbf{F} = \nabla f$.

First we check that $\nabla \times \mathbf{F} = \mathbf{0}$ (which proves that $\mathbf{F}$ is conservative):

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z - z\sin y & x^2y + \cos y + z \end{vmatrix} \\ &= \mathbf{i}\big(x^2 - \sin y - (x^2 - \sin y)\big) - \mathbf{j}\big(2xy - 2xy\big) + \mathbf{k}\big(2xz - 2xz\big) = (0,0,0) \end{aligned}$$

To calculate the potential we can start by using that

$$\frac{\partial f}{\partial x} = 2xyz \qquad \Longrightarrow \qquad f = x^2yz + g(y,z).$$

Then we compare the partial derivatives with respect to y

$$x^2z - z\sin y = \frac{\partial f}{\partial y} = x^2z + \frac{\partial g}{\partial y} \qquad \Longrightarrow \qquad \frac{\partial g}{\partial y} = -z\sin y \qquad \Longrightarrow \qquad g = z\cos y + h(z).$$

So far we know that

$$f = x^2yz + z\cos y + h(z).$$

We finally compare the $z-$partial derivatives:

$$x^2y + \cos y + z = \frac{\partial f}{\partial z} = x^2y + \cos y + h'(z) \quad \Longrightarrow \quad h'(z) = z \quad \Longrightarrow \quad h(z) = \frac{z^2}{2} + C.$$

A potential is then given by (we can take the constant $C = 0$):

$$f = x^2yz + z\cos y + \frac{z^2}{2}.$$