Math 2374
Spring 2009
Final
May 11, 2009
Time Limit: 3 hours

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exam contains 15 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	35 pts	
3	35 pts	
4	40 pts	
5	25 pts	
6	25 pts	
7	40 pts	
8	25 pts	
9	25 pts	
10	25 pts	
TOTAL	300 pts	

1. (25 points) Let $g(x, y, z) = x^2 - y^2 + z^2$.

   (a) Find the tangent plane to $g(x, y, z) = 12$ at the point $(2, 1, -3)$ (write your answer in the form $Ax + By + Cz + D = 0$).

   (b) Find all the points of the surface $g(x, y, z) = 12$ where the tangent plane is horizontal.

2. (35 points) Consider the path $\mathbf{c}(t) = (\cos(5\pi\, t),\, 6\sin(5\pi\, t),\, (1+t)^2)$ for $0 \le t \le 1$ and the vector field $\mathbf{F}(x,y,z) = (x + yz, y + zx, z + xy)$. Compute

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

(**Hint.** If you show that $\mathbf{F}$ is conservative the computation can be made simpler).

3. (35 points) Calculate the intersection of the surfaces (specify the result with equations)

$$y = \sqrt{x^2 + z^2}, \qquad y = 4 - \sqrt{x^2 + z^2}.$$

Then compute the volume of the solid enclosed by these two surfaces.

4. (40 points) Consider the function

$$f(x,y) = e^{2x+2y+2}\,(5y - y^2 - 3).$$

(a) Give a linear and a quadratic approximation of f near $(-2,1)$.

(b) Compare the value of the linear and quadratic approximations to $f(-1.9, 1) = 1.2214...$

5. (25 points) Consider the path

$$\mathbf{c}(t) = \left(2t\sin t, \tfrac{1}{3}t^3, 2t\cos t\right), \qquad 0 \le t \le 2.$$

(a) Show that the speed along the path is increasing with time.

(b) Compute the length of the path.

6. (25 points) Change the order of integration in

$$\int_0^2 \int_0^{2x-x^2} \sqrt{1-y}\, dy\, dx$$

and compute the resulting integral. (**Hint.** The expression $2x - x^2 = 1 - (x-1)^2$ might be helpful in your calculations.)

7. (40 points) The points

$$\mathbf{p}_1 = (0,0,1), \qquad \mathbf{p}_2 = (1,0,1), \qquad \mathbf{p}_3 = (0,2,1), \qquad \mathbf{p}_4 = (1,2,1)$$

$$\mathbf{p}_5 = (0,0,2), \qquad \mathbf{p}_6 = (1,0,4), \qquad \mathbf{p}_7 = (0,2,8), \qquad \mathbf{p}_8 = (1,2,10)$$

are the vertices of a truncated prism with rectangular base. The surface of the top face (the one with $\mathbf{p}_5$, $\mathbf{p}_6$, $\mathbf{p}_7$ and $\mathbf{p}_8$ as vertices) will be called S_1.

(a) Find the equation of the plane that contains S_1 (write your answer in the form $z = Ax + By + C$).

(b) Compute the flux integral

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S},$$

where the normal has been oriented upwards and $\mathbf{F}(x, y, z) = (y, x, z)$.

(c) Let now S_2 be the surface formed by the remaining five faces of the truncated prism with normal pointing outwards. Use Gauss' Theorem to evaluate, for the same vector field $\mathbf{F}$, the flux integral

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

8. (25 points) Show that $(-1, 1)$ is a critical point of

$$f(x, y) = x^2 + 5x + 5 + 2y^2 - 7y - 3xy$$

and classify it as maximum, minimum or saddle–point.

9. (25 points) Consider the function

$$\mathbf{F}(x,y,z) = (2x - y^2 + z,\ x - y - z^2)$$

and a path $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ for which we know that

$$\mathbf{g}(0) = (1,1,2) \qquad \mathbf{g}'(0) = (0,-1,3).$$

Compute $\mathbf{D}[\mathbf{F} \circ \mathbf{g}](0)$. (**Hint.** To properly understand the derivative of $\mathbf{g}$, it might be convenient to think of how many variables and components $\mathbf{g}$ has).

10. (25 points) The height of a mountain as a function of the horizontal coordinates is given by the formula
$$h(x, y) = \frac{15}{x^2 + 9y^2 + 2}.$$

(a) If a mountaineer is in the point with horizontal coordinates $(2, 1)$ and wants to descend as fast as possible, what direction should she head to? (Give your answer as a unit vector).

(b) The summit is located at the point with coordinates $(0, 0)$. Give the equation of the tangent plane at the summit.