Study guide for third exam

Math 2374, Fall 2006

1. Cylindrical and spherical coordinates (Section 1.4)
   (a) We'll use them for changing variables.
   (b) Sample book problems: 1.4 #4, #12
2. Change of variables (Chapter 6)
   (a) In double integrals
      i. Key idea: Evaluate integral in new region over new coordinates with new area measure $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$.
      ii. Formula: $\iint_D f(x,y) dx\, dy = \iint_{D^*} f(\mathbf{T}(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$.
      iii. Important special case: polar coordinates, where $dx\, dy = r\, dr\, d\theta$.
   (b) In triple integrals
      i. Key idea: Evaluate integral in new region over new coordinates with new volume measure $dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$.
      ii. Formula: $\iiint_W f(x,y,z) dx\, dy\, dz = \iiint_{W^*} f(\mathbf{T}(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$.
      iii. Important special cases: cylindrical coordinates ($dx\, dy\, dz = r\, dr\, d\theta\, dz$), spherical coordinates ($dx\, dy\, dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$)
   (c) Sample book problems: 6.1 #2, 6.2 #2, #14, #23, #26, #30
3. Parametrized surfaces (Section 7.3)
   (a) Parametrize key surfaces: spheres, cylinders, cones, planes, surface of form $z = h(x,y)$.
   (b) Tangent vectors: $\mathbf{T}_u = \frac{\partial \mathbf{\Phi}}{\partial u}$ and $\mathbf{T}_v = \frac{\partial \mathbf{\Phi}}{\partial v}$
   (c) Normal vector: $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$
   (d) Use normal vector to find equation for tangent plane
   (e) Unit normal vector $\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}$ specifies orientation. Positive side of surface is side with normal.
   (f) Sample book problems: 7.3 #3, #7, #14
4. Surface area of a parametrized surface (Section 7.4)
   (a) Key idea: surface area element of $\mathbf{x} = \mathbf{\Phi}(u,v)$ is $dS = \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$
   (b) Formula: Surface area $= \iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, du\, dv$.
   (c) Sample book problems: 7.4 #5, #8, #13

5. Surface integrals of scalar-valued function (Section 7.5)
   (a) Key idea: Integrate $f(\mathbf{x})$ across surface (i.e., $f(\mathbf{\Phi}(u,v))$) using above dS.
   (b) Formula: $\iint_S f\,dS = \iint_D f(\mathbf{\Phi}(u,v))\,\|\mathbf{T}_u \times \mathbf{T}_v\|\,du\,dv$.
   (c) Formula for graphs (special case of above formula):
   $\iint_S f\,dS = \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}dx\,dy$
   (d) Sample book problems: 7.5 #3, #8, #14

6. Surface integrals of vector-valued functions (Section 7.6)
   (a) Key idea: Integrate normal component of $\mathbf{F}(\mathbf{x})$ across surface (i.e., $\mathbf{F}(\mathbf{\Phi}(u,v))\cdot\mathbf{n}$) using above dS.
   (b) Formula: $\iint_S \mathbf{F}\cdot d\mathbf{S} = \pm\iint_D \mathbf{F}(\mathbf{\Phi}(u,v))\cdot(\mathbf{T}_u\times\mathbf{T}_v)\,du\,dv$. (minus sign if $\mathbf{T}_u\times\mathbf{T}_v$ points in the opposite direction as $\mathbf{n}$)
   (c) Formula for graphs (special case of above formula, upward normal):
   $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(x,y,g(x,y))\cdot\left(-\frac{\partial g}{\partial x},-\frac{\partial g}{\partial y},1\right)dx\,dy$
   (d) Sample book problems: 7.6 #1, #6, #16

7. Stokes' Theorem (Section 8.2)
   (a) Key idea 1: to calculate circulation of $\mathbf{F}$ around closed curve C, you can choose any surface with boundary C and calculate flux integral of $\nabla\times\mathbf{F}$ over surface.
   (b) Key idea 2: if calculating the surface integral of a vector field $\mathbf{G}=\nabla\times\mathbf{F}$ over a surface S, then you can either
      i. convert it to the integral of $\mathbf{F}$ over the boundary ∂S, or
      ii. change the surface S to any other surface S' with the same boundary $\partial S'=\partial S$ and compute the integral over S' rather than over S.
   (c) Need positively oriented boundary: orient using the right hand rule (alternatively, walk on positive side of surface near boundary and surface is on left).
   (d) Formula: $\int_{\partial S}\mathbf{F}\cdot d\mathbf{s} = \iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}$
   (e) Sample book problems: 8.2 #1, #9, #21, #25

8. Conservative vector fields (Section 8.3)
   (a) Key idea: test if a vector field is conservative. If it is, your life got a lot easier (that is, if you're trying to compute a line integral of the vector field).
   (b) Fact: if a vector field $\mathbf{F}$ is conservative, then
      i. its line integral depends only on the endpoints (so is zero over closed curves)
      ii. $\mathbf{F}=\nabla f$ for some potential function f.
      iii. $\int_C \mathbf{F}\cdot d\mathbf{s} = f(\mathbf{q})-f(\mathbf{p})$, where $\mathbf{p}$ and $\mathbf{q}$ are the endpoints of the path.
   (c) Test for conservative vector fields:

i. In 2D where $\mathbf{F}$ is defined in all $\mathbf{R}^2$: $\mathbf{F}$ is path-independent if and only if $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$.

ii. In 3D where $\mathbf{F}$ is defined in all $\mathbf{R}^3$ except for possibly a finite number of points: $\mathbf{F}$ is path-independent if and only if $\nabla \times \mathbf{F} = \mathbf{0}$.

(d) Don't forget the consequence of having a hole through the domain in $\mathbf{R}^2$.

(e) If $\mathbf{F}$ is conservative, find potential function f so that $\nabla f = \mathbf{F}$.

(f) Sample book Problems: 8.3 #2, #3, #9, #13(b)