Sample Exam

Exam rules:
Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply to all exams:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.

- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.

- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1. Give an equation for the tangent line to the curve $y^2 = x^3 - x$ at the point $(2, \sqrt{6})$. Write your answer in the form $Ax + By + C = 0$.

2. Suppose that a gas concentration at position (x, y, z) is given by the function

$$C(x, y, z) = x^2 + x^3 y + yz$$

If you were located at position $(1, 1, 1)$, find the direction that you would need to move in order to increase the concentration as quickly as possible. Write your answer in the form of a unit vector.

3. Let $g(s, t) = (x(s, t), y(s, t)) = (\cos(s) + \cos(t), \sin(s) + \sin(t))$, where $s < t < s + \pi$. There exists a vector field $F(x, y)$ such that $F(g(s, t)) = (-\sin(t), \cos(t))$.

   Using the chain rule on $F \circ g$, solve for the Jacobian $DF(g(s, t))$.

4. Suppose that $f(x, y, z)$ satisfies $f(3, 5, 1) = 1$, $\frac{\partial f}{\partial x}(3, 5, 1) = 2$, $\frac{\partial f}{\partial y}(3, 5, 1) = -20$, and $\frac{\partial f}{\partial z}(3, 5, 1) = 3$. Estimate $f(3.01, 4.99, 1.02)$.

5. What is the curvature of the curve $c(t) = (3t, 7t^2)$ at $t = 0$?

6. Find the length of the curve

$$c(t) = (7\cos(t) + 5, t/2, 7\sin(t) - 3)$$

   between $t = 0$ and $t = 2\pi$.

7. Find the maximum and minimum of the function $x + 2y - z$ on the ellipsoid $x^2 + y^2 + 2z^2 = 1$.

8. Show that $(0, 0)$ is the only critical point of the function $f(x, y) = x^2 + xy - y^2$. Is this point a local maximum, local minimum, or a saddle point?

9. Find the integral

$$\iint y \, dA$$

   over the region enclosed by the curves $y = 0$ and $y = x - x^2$.

10. Use the substitution $x = u + 5v$, $y = u - v$ to evaluate the double integral

$$\iint 1 \, dx \, dy$$

   over the region $0 \leq x - y \leq 6$, $-2 \leq x + 5y \leq 2$.

11. Find the integral

$$\iiint e^{x^2+y^2+z^2} \, dV$$

   over the sphere of radius R centered at the origin.

12. Suppose the vector field $F(x, y) = (x^2, y^2)$ expresses the rate of flow of a fluid. Calculate the total rate of flow out of the boundary of the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

13. Set up a double integral to evaluate the area of the skewed cylinder $1 = (x - z)^2 + y^2$ over the region $0 \leq z \leq 2$. (You do not need to evaluate the integral.)

14. If the above skewed cylinder has density at point (x, y, z) given by $(x - z) \cdot y$, find the total mass of the cylinder.

15. Use Stokes' theorem to evaluate the line integral

$$\int y\,dx + x\,dy$$

around the path which moves, in straight lines, from $(2, 0, 0)$ to $(0, 3, 0)$ to $(0, 0, 4)$ to $(2, 0, 0)$.

16. Calculate the total flux of flow of the vector field $F(x, y, z) = (x, y, z)$ through the upper half-sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$. (Measure the total flow *away* from the origin.)

17. If the vector field $F(x, y, z) = (x, y^2, z^2)$ represents a fluid flow, calculate the total rate of flow *out* of the box $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, $-1 \leq z \leq 1$.

18. Suppose we are in $\mathbf{R}^6$, with coordinates (u, v, w, x, y, z). Given the 2-form $\omega = (uv)\,dx \wedge dy + (wz)dy \wedge dz$, find $d\omega$.

19. Suppose we are in $\mathbf{R}^4$, with coordinates (w, x, y, z), and we want to find a 2-form ω so that $d\omega$ is the 3-form

$$x\,dx \wedge dy \wedge dz + w\,dy \wedge dz \wedge dx.$$

Does such a 2-form exist? Explain why or why not.

20. In $\mathbf{R}^4$, with coordinates (x, y, z, t), suppose we have a path than moves around the boundary of a triangle, from the vertex $(0, 0, 1, 1)$ to $(1, 1, 0, 0)$ to $(2, 2, 2, 2)$ and back. Use the general form of Stokes' theorem to evaluate the line integral

$$\int x\,dt + y\,dz$$

over this path. (Make sure to get the sign right.)

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