Math 5285H

Midterm 1

No collaboration is allowed. This test is open-book and open-library but no electronic sources may be consulted.

This test is due in-class on **Friday, October 21**.

1. How many elements in the symmetric group Σ_6 have order 6?

2. Let D_{2n} be the dihedral group with $2n$ elements:

$$D_{2n} = \langle a, b \mid a^n = e, b^2 = e, bab = a^{-1} \rangle$$

   Find necessary and sufficient conditions on an integer k so that the two reflections b and ba^k generate the whole group: every element in D_{2n} can be obtained by multiplying together copies of b and ba^k in some order.

3. Find all possible values of x which are solutions to the following equations in modular arithmetic:

   (a) $x^2 = \overline{1}$ in $\mathbb{Z}/5$.

   (b) $x^2 = \overline{-1}$ in $\mathbb{Z}/5$.

   (c) $x^2 + x + \overline{1} = \overline{0}$ in $\mathbb{Z}/7$.

   (d) $x^3 + x^2 - \overline{2}x - \overline{1} = \overline{0}$ in $\mathbb{Z}/13$.

4. For which prime numbers p is the matrix

$$\begin{bmatrix} \overline{1} & \overline{1} & \overline{2} \\ \overline{1} & \overline{2} & \overline{3} \\ \overline{2} & \overline{3} & \overline{47} \end{bmatrix}$$

   an element of $GL_3(\mathbb{Z}/p)$?

5. Let F be a field and V a finite-dimensional vector space over F. The *dual space* of V, called V^*, is the set of linear transformations $T : V \to F$. We define addition and scalar multiplication on V^* as follows:

   (a) $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

   (b) $(a \cdot T)(v) = a(T(v))$

   Show that these rules make V^* into a vector space over F, of the same dimension as V.