Math 5378, Differential Geometry
Practice questions for Test 1

The exam itself will be closed book, no notes.

Note: There are more practice questions appearing here than would appear on an actual exam. The actual exam will have 4 questions, and one of them will be off this list.

Solutions will be posted on Monday, March 3.

1. Find a parametrized curve whose trace is the set of points (x, y) in $\mathbb{R}^2$ with $xy = 1$, $x > 0$.

2. Find the arc length of the curve

$$\alpha(t) = \left(t \sin t, t \cos t, \frac{\sqrt{8}}{3} t^{3/2} \right)$$

between $t = 0$ and $t = 1$.

3. Show that the curve

$$\alpha(t) = (\sin t, t, -\cos t)$$

has constant speed. Then find a reparametrization of this curve by arc length.

4. Give the Frenet formulas for the derivatives of the tangent, normal, and binormal of a curve parametrized by arc length.

5. If $\alpha(s)$ is a curve parametrized by arc length, prove that $\alpha'(s)$ is perpendicular to $\alpha''(s)$.

6. Find all real numbers c so that the set

$$\{(x, y, z) \mid x^2 - y^2 + z^3 - z = c\}$$

is a smooth surface in $\mathbb{R}^3$.

7. Consider the coordinate chart

$$\mathbf{x}(u,v) = (u^2, u^3 + v^3, v^2).$$

for $u, v > 0$. Find the coefficients E, F, and G of the first fundamental form in these coordinates.

8. Suppose we have a coordinate chart $\mathbf{x}$ on the open set

$$\{(u,v) \in \mathbb{R}^2 \mid u < 0, -\pi < v < \pi\}$$

such that the coefficients of the first fundamental form are:

- $E = e^u$,
- $F = 0$,
- $G = e^u$.

Find the length of the image of the curve $\alpha(t) = (-1, t)$ between $t = 0$ and $t = 1$.

9. With the same coordinate chart as the previous problem, find the area of the image of the entire region under $\mathbf{x}$.

10. Consider the coordinate chart

$$\mathbf{x}(u,v) = (u, u^2 + v^2, -v).$$

Find a field N of unit normal vectors for this coordinate chart.

11. Given a surface S with unit normal vector field N, give the mathematical definition of the second fundamental form $\mathrm{II}_p(\mathbf{v})$ for a vector v in the tangent space $T_p(S)$.

12. Prove that a point p of a smooth surface is umbilical if and only if the Gaussian curvature K and the mean curvature H satisfy $H^2 = K$.