Math 8301, Manifolds and Topology
Homework 10
Due in-class on **Monday, November 26**

1. For maps $f : A \to B$ and $g : B \to C$ of abelian groups, show that there is an exact sequence

   $$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \mathrm{coker}(f) \to \mathrm{coker}(gf) \to \mathrm{coker}(g) \to 0.$$

2. If M is the Möbius strip and ∂M is its boundary, calculate the relative homology groups $H_*(M, \partial M)$.

3. Show that there exists a natural transformation $\epsilon : C_0(X) \to \mathbb{Z}$, from the zero'th singular chain group of X to $\mathbb{Z}$, satisfying $\epsilon \circ \partial = 0$, sending any point of X to 1.

4. For a space X, we define the *reduced* singular chain complex of X to be the chain complex

   $$\cdots \to C_2(X) \to C_1(X) \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

   (i.e. defining $\widetilde{C}_{-1}(X) = \mathbb{Z}$), and the *reduced homology groups* $\widetilde{H}_n(X)$ to be the homology groups of this complex. For *most* spaces, but not all, there is an isomorphism

   $$H_0(X) \cong \mathbb{Z} \oplus \widetilde{H}_0(X).$$

   Figure out what the exceptional case is, and determine what the reduced homology groups are in this case.

5. A simplicial complex $(\mathcal{V}, \mathcal{F})$ is *locally finite* if, for all vertices $v \in \mathcal{V}$, there are only finitely many faces $\sigma \in \mathcal{F}$ containing v. In this circumstance, show that you can define groups C_n^{BM} whose elements are *arbitrary* sums of n-simplices, together with boundary operators $\partial : C_n^{BM} \to C_{n-1}^{BM}$ satisfying $\partial \circ \partial = 0$. Calculate the associated homology groups H_n^{BM} for a triangulation of $\mathbb{R}$. (These groups are called the *Borel-Moore* homology groups.)