Math 8301, Manifolds and Topology
Homework 11
Due in-class on **Wednesday, December 5**

1. For a space X, use the Mayer-Vietoris sequence to compute the homology groups of $X \times S^1$.

2. For a space X with subspaces $A \subset B \subset X$, show that there is a short exact sequence of chain complexes
$$0 \to C_*(B, A) \to C_*(X, A) \to C_*(X, B) \to 0.$$
Explain how this relates the three associated types of relative homology groups.

3. Use the previous exercise to show that if X is a space, $X = U \cup V$ where U, V are open subsets, and $A \subset U \cap V$, there is a Mayer-Vietoris sequence relating $H_*(X, A)$, $H_*(U, A)$, $H_*(V, A)$, and $H_*(U \cap V, A)$.

4. Suppose X has a sequence of subspaces $A_0 \subset A_1 \subset \cdots$ such that $X = \cup A_i$, and so that a subset $U \subset X$ is closed if and only if $U \cap A_i$ is closed for all i. (In this case, we say that X has the *direct limit* topology determined by these subspaces.) Show that every element in $H_k(X)$ is the image of an element in $H_k(A_i)$ for some i, and that two elements in $H_*(A_i)$ become the same in $H_k X$ if and only if there is some $j \geq i$ such that their images in $H_k(A_j)$ coincide. In this case, we say $H_k(X)$ is the *direct limit* of the sequence of groups $H_k(A_i)$. (Hint: Show that a map $\Delta^n \to X$ always factors through some map $\Delta^n \to A_i$.)

5. Suppose M is a manifold and $p \in M$. Compute the relative homology groups $H_*(M, M \setminus \{p\})$, and use it to show that "dimension" is a well-defined invariant of a connected manifold.