Math 8301, Manifolds and Topology
Homework 3
Due in-class on **Friday, Sep 26**

1. Recall that the standard two-simplex Δ^2 is the subspace
$$\{(t_0, t_1, t_2) \in \mathbb{R}^3 \mid t_i \geq 0, \Sigma t_i = 1\}.$$
If X is a space with points p, q, and r, α is a path from p to q, β is a path from q to r, and γ is a path from p to r, show that the path composite $\alpha * \beta$ is homotopic to γ if and only if there is a continuous map $\sigma : \Delta^2 \to X$ such that $\sigma(1-t, t, 0) = \alpha(t), \sigma(0, 1-t, t) = \beta(t)$, and $\sigma(1-t, 0, t) = \gamma(t)$.

2. Using the previous exercise, show that if $K : [0,1] \times [0,1] \to X$ is a continuous map, and we define
$$\alpha(t) = K(t, 0), \qquad \beta(t) = K(1, t),$$
$$\gamma(t) = K(0, t), \qquad \delta(t) = K(t, 1)$$
then $\alpha * \beta$ is homotopic to $\gamma * \delta$.

3. Suppose that a topological space X has a function $m : X \times X \to X$. Show that if α and β are *any* paths in X, the definition
$$(\alpha \cdot \beta)(t) = m(\alpha(t), \beta(t))$$
is homotopy invariant, in the sense that $[\alpha] * [\beta] = [\alpha * \beta]$ is well-defined on homotopy classes of paths.

4. Show that the product of the previous problem satisfies an interchange law
$$(\alpha \cdot \beta) * (\gamma \cdot \delta) = (\alpha * \gamma) \cdot (\beta * \delta)$$
whenever the left-hand side is defined.

5. ("Don't worry about space-filling curves") Suppose M is a n-dimensional manifold and that $\gamma : [0,1] \to M$ is a path in M. Show that there is a homotopic path $\gamma' \sim \gamma$ and an integer N satisfying the following: for all $0 \leq k < N$, there is an open set $U_k \subset M$ and a homeomorphism ϕ_k from U_k to an open disc in $\mathbb{R}^n$ such that
   - $\gamma'([k/N, (k+1)/N]) \subset U_k$ and
   - the composite function $\phi_k \circ \gamma' : [k/N, (k+1)/N] \to \mathbb{R}^n$ is linear.