Math 8301, Manifolds and Topology
Homework 5
Due in-class on **Friday, Oct 10**

1. Show that the fundamental group of the n-sphere S^n is trivial for $n > 1$ by directly showing that any loop γ is homotopic to the trivial loop.

2. Now give a proof of the same using the Seifert-Van Kampen theorem.

3. Suppose $f(z)$ is a monic polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ whose coefficients are complex numbers. Recall $S^1 = \{w \in \mathbb{C} \mid |w| = 1\}$. Show that there is a sufficiently large real number $R > 0$ such that

   (a) $f(z) \neq 0$ when $|z| = R$, and

   (b) the resulting function $S^1 \to \mathbb{C} \setminus \{0\}$, given by $w \mapsto f(Rw)$, is homotopic to the map $w \mapsto (Rw)^n$.

4. Suppose you are given a simplicial complex with a finite set $\mathcal{V}$ of vertices and set $\mathcal{F}$ of faces. Let X be the space you get by realizing this simplicial complex. For definiteness, we'll let $\mathbb{V}$ be the vector space with basis $\mathcal{V}$, and define

$$X = \bigcup_{U \in \mathcal{F}} \left\{ \sum_{v \in U} t_v \cdot v \,\middle|\, t_v \geq 0, \sum t_v = 1 \right\} \subset \mathbb{V}.$$

   Show that none of the faces of dimension 3 or greater have any effect on the fundamental group: you can put them in or take them of $\mathcal{F}$ without changing π_1. (This is also true if the set is infinite.)

5. A *graph* is a simplicial complex with only vertices and edges, i.e. where no faces have dimension higher than one. A *tree* is a graph, with at least one vertex, such that for any vertices $p \neq q$, there exists a *unique* sequence $e_1, e_2, \cdots, e_n$ of edges such that

   (a) $e_i \neq e_j$ for $i \neq j$,

   (b) e_i and e_{i+1} always share a common vertex,

   (c) p is a vertex of e_1, and

   (d) q is a vertex of e_n.

Show that any tree gives rise to a space with trivial fundamental group. (If you want, you can instead show the stronger statement that this space is contractible.)