Math 8301, Manifolds and Topology
Homework 6
Due in-class on **Friday, Oct 17**

1. Suppose $f : H \to G$ is a group homomorphism. Show that the amalgamated product $G *_H \{e\}$ is always isomorphic to the quotient G/N, where N is the normal subgroup generated by the image of f.

2. Suppose X is path-connected and p, q are points in X. Construct a new space X' by taking a disjoint union of X and $[0, 1]$, then gluing 0 to p and 1 to q. Show that $\pi_1(X', p)$ is the free product $\pi_1(X, p) * \mathbb{Z}$. (Hint: Seifert-van Kampen is hard to use directly here. Start by finding a loop $S^1 \to X'$ and show X' is a deformation retract of one obtained by gluing in $S^1 \times [0, 1]$ along $S^1 \times \{0\}$.)

3. Suppose X is path-connected and p, q are points in X. We know that $\pi_1(X, p)$ and $\pi_1(X, q)$ are isomorphic, but that this isomorphism depends on a choice of path from p to q. Show that there is a *canonical* isomorphism between $\pi_1(X, p)_{ab}$ and $\pi_1(X, q)_{ab}$ (in the sense that it does not depend on any choices).

4. Express the abelianization of the group

$$\langle a, b, c \mid abc^4 = a^4c^2a^4 = a^2b^8c^8 = e\rangle$$

as a product of cyclic abelian groups. (You do not need to give explicit generators.)

(If you are using a double-sided printer, note that this is not the last problem on the assignment.)

5. ("Sometimes homotopies don't preserve basepoints") Suppose we have spaces X and Y, together with two continuous maps $f, g : X \to Y$ and a basepoint $x \in X$. Suppose that there is a homotopy $H : X \times [0,1] \to Y$ starting at f and ending at g, but that H does not necessarily preserve the basepoint. Show that if we define

$$\alpha(t) = H(x,t)$$

then there is an identity

$$g_*([\gamma]) = [\alpha^{-1}] * f_*([\gamma]) * [\alpha]$$

and so an identification of maps $\pi_1(X,x) \to \pi_1(Y, g(x))$.

Use this to show that if $i : X \to Y$ and $j : Y \to X$ are maps such that ij and ji are each homotopic to the identity, then $i_* : \pi_1(X,x) \to \pi_1(Y, i(x))$ is an isomorphism.