Math 8301, Manifolds and Topology
Homework 9
Due in-class on **Monday, Nov 24** (I will be away on the 21st)

1. Write out a proof of the following half of the *four-lemma*. Suppose we have a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \xrightarrow[h']{} & D' \end{array}$$

   such that α and γ are surjective, δ is injective, and the rows are exact. Show that β is surjective.

2. Suppose $(\mathcal{V}, \mathcal{F})$ and $(\mathcal{V}', \mathcal{F}')$ are simplicial complexes. A *map* of simplicial complexes is a function $f : \mathcal{V} \to \mathcal{V}'$ such that, for all $S \in \mathcal{F}$, the set
$$f(S) = \{f(s) \mid s \in S\} \subset \mathcal{V}'$$
   is in $\mathcal{F}'$. Show that this definition makes simplicial complexes into a category, and that geometric realization is a functor from simplicial complexes to spaces.

3. Show that homology is a functor on simplicial complexes: a map of simplicial complexes $X \to Y$ gives well-defined maps $H_n(X) \to H_n(Y)$ on simplicial homology groups. (Warning: You need to somehow account for the fact that the map may not be one-to-one on vertices.)

4. A Δ-complex (or, sometimes, a *semisimplicial complex*) consists of a sequence of sets $(X_n)_{n\in\mathbb{N}}$ (X_n is the set of n-*simplices*), together with functions
$$d_n^i : X_n \to X_{n-1}$$
   for $0 \le i \le n$ (which takes an n-simplex to an $(n-1)$-simplex, obtained by deleting the i'th vertex). These are required to satisfy the relations
$$d_{n-1}^i d_n^j = d_{n-1}^{j-1} d_n^i$$
   whenever $i < j$. (This expresses that the two possible orders for deleting vertices i and j coincide.)

(a) Prove that a simplicial complex $(\mathcal{V}, \mathcal{F})$, together with a chosen *total order* on $\mathcal{V}$, determines a Δ-complex.

(b) Give, explicitly, a Δ-complex corresponding to a Klein bottle with $|X_0| = 1$, $|X_1| = 3$, $|X_2| = 2$, and $|X_n| = 0$ for $n > 2$.

5. A simplicial complex $(\mathcal{V}, \mathcal{F})$ is *locally finite* if, for all vertices $v \in \mathcal{V}$, there are only finitely many faces $\sigma \in \mathcal{F}$ containing v. In this circumstance, show that you can define groups C_n^{BM} whose elements are *arbitrary* sums of n-simplices, together with boundary operators $\partial : C_n^{BM} \to C_{n-1}^{BM}$ satisfying $\partial \circ \partial = 0$. Calculate the associated homology groups H_n^{BM} for a triangulation of $\mathbb{R}$. (These groups are called the *Borel-Moore* homology groups.)