Math 8302, Smooth Manifolds and Smooth Topology II
Smooth Homework 5
Due smoothly in-class on **Monday, Smooth March 11**

Smooth Problem 1. Suppose M and N are a smooth manifolds with a smooth map $f : M \to N$, X is a smooth vector field on M, Y is another smooth vector field on M, X' is a smooth vector field on N, Y' is another smooth vector field on N, and $f : M \to N$ is a smooth map such that for all $p \in M$ the map on smooth tangent spaces df_p satisfies $df_p(X(p)) = X'(f(p))$ and $df_p(Y(p)) = Y'(f(p))$. In this situation, we say that the smooth map f *carries* the smooth vector field X to the smooth vector field X' (and similarly, the smooth map f carries the smooth vector field Y to the smooth vector field Y'). Show that the smooth map f carries the smooth vector field $[X, Y]$ to the smooth vector field $[X', Y']$.

Smooth Problem 2. Suppose that M and N are smooth manifolds, $f : M \to N$ is a smooth function, X is a smooth vector field on M, X' is a smooth vector field on N, and the smooth function f carries the smooth vector field X to the smooth vector field X'. Show that, for any smooth map $c : (a, b) \to M$ which defines a smooth flow line for the smooth vector field X, the function $f \circ c : (a, b) \to N$ is a smooth flow line for the smooth vector field X'.

Smooth Problem 3. Fix a vector $\vec{v}$ in $\mathbb{R}^3$. We view $\mathbb{R}^3$ as a smooth manifold using the standard smooth structure. First, show that the function which sends a point p of the smooth manifold $\mathbb{R}^3$ to the vector $\vec{v} \times p$ based at p defines a smooth vector field $X_{\vec{v}}$ on $\mathbb{R}^3$. Second, if $\vec{w}$ is another vector, determine the smooth Lie bracket $[X_{\vec{v}}, X_{\vec{w}}]$ of the smooth vector field $X_{\vec{v}}$ and the smooth vector field $X_{\vec{w}}$.

Problem of Smoothness 4. In this problem, we view $\mathbb{R}^4$ as a smooth manifold using the standard smooth structure; we write points of the smooth manifold $\mathbb{R}^3$ in the form (x, y, x', y'). Find the smooth flow lines of the smooth vector field

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - y'\frac{\partial}{\partial x'} + x'\frac{\partial}{\partial y'},$$

and show that these smooth flows are defined for all times t in the smooth manifold $\mathbb{R}$. (Smooth hint: Complex numbers!)

Smoove Problem 5. Suppose M is a smooth manifold of dimension n with a chosen point p, and $X_1, \ldots, X_n$ are smooth vector fields on M so that $\{X_i(p)\}$ is a basis of the smooth tangent space $T_p(M)$.

For each i, let $\theta_i : U_i \to M$ be a smooth flow for the smooth vector field X_i (defined on some smooth open submanifold $M \times \{0\} \subset U_i \subset M \times \mathbb{R}$).

Inductively define smooth functions f_j on an open neighborhood of $\vec{0} \in \mathbb{R}^j$ as follows. The smooth function $f_0 : \mathbb{R}^0 \to M$ sends 0 to p. Then

$$f_j(t_1, \ldots, t_j) = \theta_j(f_{j-1}(t_1, \ldots, t_{j-1}), t_j).$$

From these smooth functions, we get smooth differentials $(df_j)_{\vec{0}} : \mathbb{R}^j \to T_p(M)$. Show that the expression of this in the basis $\{X_i(p)\}$ is a matrix with ones on the diagonal and zeroes elsewhere.

⋆ **Challenge.** Read carefully through the assignment to see if I still managed to miss the adjective "smooth" anywhere. (Note: I don't want to know the answer to this problem.)