Math 8306, Algebraic Topology
Homework 2
Due in-class on **Monday, October 6**

1. In class, we defined subdivision maps $s_n^i : \Delta[n+1] \to \Delta[n] \times [0,1]$ for $0 \leq i \leq n$ by

   $$s_n^i(t_1, \cdots, t_{n+1}) = ((t_1, \ldots, \widehat{t_{i+1}}, \ldots, t_{n+1}), t_{i+1}).$$

   . Show that these satisfy the relations

   - $s_n^i d_{n+1}^j = \begin{cases} (d_n^{j-1}, id) \circ s_{n-1}^i & \text{if } i < j-1 \\ (d_n^j, id) \circ s_{n-1}^{i-1} & \text{if } i > j \end{cases}$
   - $s_n^0 d_{n+1}^0 = i_0$
   - $s_n^n d_{n+1}^0 = i_1$
   - $s_n^{i-1} d_{n+1}^i = s_n^i d_{n+1}^i$ for $i \geq 1$.

   Use this to show that the operator $h : C_n(X) \to C_{n+1}(X \times [0,1])$ given by

   $$h(\sum a_\sigma \sigma) = \sum a_\sigma \sum_{i=0}^{n} (-1)^i (\sigma, id) \circ s_n^i$$

   satisfies $\partial h(x) + h\partial\sigma(x) = i_0(x) - i_1(x)$.

2. Let C_* be the chain complex with

   $$C_n = \begin{cases} \mathbb{Z} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Let D_* be the chain complex with

   $$D_n = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

   such that the boundary map $\partial : D_1 \to D_0$ sends m to $2m$.

   Show that the natural projection $\pi : D_* \to C_*$ is a map of chain complexes and it induces the zero map $H_*(D_*) \to H_*(C_*)$. Show that there is no chain homotopy h with $\partial h + h\partial = \pi$ (from π to zero).

3. For $Z \subset Y \subset X$ spaces, show that there is a short exact sequence of singular chain complexes

$$0 \to C_*(Y,Z) \to C_*(X,Z) \to C_*(X,Y) \to 0.$$

What does the resulting long exact sequence of homology groups look like?

4. (Formal Mayer-Vietoris sequence) Suppose that there is a map of long exact sequences as follows:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_{n+1} & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & F_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr & & \downarrow & & \\ \cdots & \longrightarrow & G_{n+1} & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & G_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \end{array}$$

Here all the maps $F_n \to G_n$ are isomorphisms. Show that there is a long exact sequence:

$$\cdots \to D_{n+1} \to A_n \to B_n \oplus C_n \to D_n \to A_{n-1} \to \cdots$$

(Define the maps first.)