Math 8307, Algebraic Topology II
Homework 9
Due in-class on **Wednesday, April 8**

1. Show that the only possible nontrivial natural transformations $H^n(X;\mathbb{Z}) \to H^m(X;\mathbb{Q})$ occur when $m = nd$, and are of the form $\alpha \mapsto a\alpha^d$ for some $a \in \mathbb{Q}$.

2. Use rational cohomology to compute the rational homotopy groups $\pi_k(S^3 \vee S^3) \otimes \mathbb{Q}$ in dimensions $k = 1 \ldots 7$.

3. (This question is worth double.) A *local coefficient system* $\mathcal{A}$ on a space X consists of the following data:

   - A set of abelian groups $\{A_x\}$ for $x \in X$.
   - For every path γ starting at x and ending at y, an isomorphism of abelian groups $\gamma_* : A_x \to A_y$ that only depends on the homotopy class of the path.

   Given a local coefficient system $\mathcal{A}$ on X, define the singular chain complex with values in $\mathcal{A}$ by

   $$C_n(X;\mathcal{A}) = \bigoplus_{\sigma:\Delta[n]\to X} A_{\sigma(1,\ldots,1)}.$$

   Recall that the standard n-simplex $\Delta[n]$ is

   $$\{(t_1,\ldots,t_n) | 0 \le t_1 \le \cdots \le t_n \le 1\}.$$

   Use the structure of a local coefficient system to define a boundary map $\partial : C_n(X,\mathcal{A}) \to C_{n-1}(X,\mathcal{A})$, and show that it satisfies $\partial \circ \partial = 0$.

   (The homology of the resulting chain complex is the *homology of X with coefficients in the coefficient system* $\mathcal{A}$. In particular, if $E \to B$ is a fibration, the homology groups of the fibers form local coefficient systems on B, and there is a version of the Serre spectral sequence that works with no assumptions on $\pi_1(B)$ acting trivially on $H_*(F)$.)