### 18.905 Problem Set 2

Due Wednesday, September 20 in class

1. A simplicial complex consists of a pair $(V, F)$, where $V$ is a set of vertices and $F$ (the set of faces) is a collection of finite subsets of $V$ satisfying the following properties.

- We have $\{v\} \in F$ for all $v \in V$.
- If $S \subset T$ and $T \in F$, then $S \in F$.
(Course 6 people might know this as a "hereditary hypergraph".) From a simplicial complex $(V, F)$, we form a space $X$ by starting with the vertices $V$ and, for every $S \in F$ of size $n+1$, we glue in a unique $n$-simplex whose vertices are the elements of $S$. The faces of this simplex correspond to the subsets of $S$.

More precisely, let $W$ be the vector space with basis $\left\{e_{v} \mid v \in V\right\}$, and let

$$
X=\bigcup_{S \in F}\left\{\sum_{v \in S} t_{v} v \mid 0 \leq t_{v} \leq 1, \sum t_{v}=1\right\}
$$

Suppose that we have chosen a partial order on $V$ such that for any $S \in$ $F$, the elements of $S$ are totally ordered. Use this to give a $\Delta$-complex structure on $X$. (You may assume that for any $v_{0}, \ldots, v_{n}$ in a vector space $W$, there is a unique affine transformation $f: \Delta^{n} \rightarrow W$ such that $f$ takes the $i$ 'th vertex of $\Delta^{n}$ to $v_{i}$.)
Update. It has been pointed out to me that I need to be explicit about what the topology on the vector space $W$ is; it's not the metric topology or the product topology. The topology on $W$ is a limit topology: A subspace $A \subset W$ is closed if and only if $A \cap U$ is closed for any finite-dimensional subspace $U$ of $W$.
2. Hatcher, exercise 8 on page 131 .
3. In class, we defined face maps $d_{n}^{i}: \Delta^{n-1} \rightarrow \Delta^{n}$ for $0 \leq i \leq n$, and subdivision maps $s_{n}^{j}: \Delta^{n+1} \rightarrow \Delta^{n} \times[0,1]$ for $1 \leq j \leq n+1$. These satisfy the following relations.

$$
\begin{aligned}
s_{n}^{j} \circ d_{n+1}^{i} & = \begin{cases}\left(d_{n}^{i-1}, i d\right) \circ s_{n-1}^{j} & \text { if } j<i \\
\left(d_{n}^{i}, i d\right) \circ s_{n-1}^{j-1} & \text { if } j>i+1\end{cases} \\
s_{n}^{1} \circ d_{n+1}^{0} & =(i d, 1) \\
s_{n}^{n+1} \circ d_{n+1}^{n+1} & =(i d, 0) \\
s_{n}^{i} \circ d_{n+1}^{i} & =s_{n}^{i+1} \circ d_{n+1}^{i}
\end{aligned}
$$

If $H: X \times[0,1] \rightarrow Y$ is a homotopy between the maps $f$ and $g$, we then defined a homotopy operator $h: C_{n}(X) \rightarrow C_{n+1}(Y)$ by

$$
h\left(\sum n_{\sigma}[\sigma]\right)=\sum n_{\sigma} \sum_{j=1}^{n+1}(-1)^{j}\left[H \circ(\sigma, i d) \circ s_{n}^{j}\right]
$$

where the map $\left[H \circ(\sigma, i d) \circ s_{n}^{j}\right]$ is the composite map $\Delta^{n+1} \rightarrow Y$. Use the given relations to show that for any $\sigma: \Delta^{n} \rightarrow X$, we have

$$
\partial h(\sigma)=f_{*}(\sigma)-g_{*}(\sigma)-h(\partial \sigma)
$$

in $C_{n+1}(Y)$.
4. Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$ are homomorphisms of abelian groups. Show that there is an exact sequence
$0 \rightarrow \operatorname{ker}(f) \rightarrow \operatorname{ker}(g f) \rightarrow \operatorname{ker}(g) \rightarrow \operatorname{coker}(f) \rightarrow \operatorname{coker}(g f) \rightarrow \operatorname{coker}(g) \rightarrow 0$.

