Due Wednesday, September 20 in class

- 1. A simplicial complex consists of a pair (V, F), where V is a set of vertices and F (the set of faces) is a collection of finite subsets of V satisfying the following properties.
  - We have  $\{v\} \in F$  for all  $v \in V$ .
  - If  $S \subset T$  and  $T \in F$ , then  $S \in F$ .

(Course 6 people might know this as a "hereditary hypergraph".) From a simplicial complex (V, F), we form a space X by starting with the vertices V and, for every  $S \in F$  of size n + 1, we glue in a unique n-simplex whose vertices are the elements of S. The faces of this simplex correspond to the subsets of S.

More precisely, let W be the vector space with basis  $\{e_v | v \in V\}$ , and let

$$X = \bigcup_{S \in F} \left\{ \sum_{v \in S} t_v v \Big| 0 \le t_v \le 1, \sum t_v = 1 \right\}$$

Suppose that we have chosen a partial order on V such that for any  $S \in F$ , the elements of S are totally ordered. Use this to give a  $\Delta$ -complex structure on X. (You may assume that for any  $v_0, \ldots, v_n$  in a vector space W, there is a unique affine transformation  $f : \Delta^n \to W$  such that f takes the *i*'th vertex of  $\Delta^n$  to  $v_i$ .)

**Update.** It has been pointed out to me that I need to be explicit about what the topology on the vector space W is; it's not the metric topology or the product topology. The topology on W is a limit topology: A subspace  $A \subset W$  is closed if and only if  $A \cap U$  is closed for any finite-dimensional subspace U of W.

- 2. Hatcher, exercise 8 on page 131.
- 3. In class, we defined face maps  $d_n^i : \Delta^{n-1} \to \Delta^n$  for  $0 \le i \le n$ , and subdivision maps  $s_n^j : \Delta^{n+1} \to \Delta^n \times [0,1]$  for  $1 \le j \le n+1$ . These satisfy the following relations.

$$\begin{split} s_n^j \circ d_{n+1}^i &= \begin{cases} (d_n^{i-1}, id) \circ s_{n-1}^j & \text{if } j < i \\ (d_n^i, id) \circ s_{n-1}^{j-1} & \text{if } j > i+1 \end{cases} \\ s_n^1 \circ d_{n+1}^0 &= (id, 1) \\ s_n^{n+1} \circ d_{n+1}^{n+1} &= (id, 0) \\ s_n^i \circ d_{n+1}^i &= s_n^{i+1} \circ d_{n+1}^i \end{cases} \end{split}$$

If  $H: X \times [0,1] \to Y$  is a homotopy between the maps f and g, we then defined a homotopy operator  $h: C_n(X) \to C_{n+1}(Y)$  by

$$h(\sum n_{\sigma}[\sigma]) = \sum n_{\sigma} \sum_{j=1}^{n+1} (-1)^{j} [H \circ (\sigma, id) \circ s_{n}^{j}].$$

where the map  $[H \circ (\sigma, id) \circ s_n^j]$  is the composite map  $\Delta^{n+1} \to Y$ . Use the given relations to show that for any  $\sigma : \Delta^n \to X$ , we have

$$\partial h(\sigma) = f_*(\sigma) - g_*(\sigma) - h(\partial \sigma)$$

in  $C_{n+1}(Y)$ .

4. Suppose  $f:A\to B$  and  $g:B\to C$  are homomorphisms of abelian groups. Show that there is an exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(gf) \to \operatorname{coker}(g) \to 0.$$