# HOPF ALGEBROIDS AND STACKS

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Abstract. These are notes I prepared for a talk on producing stacks from Hopf algebroids. In no way am I an expert on this material, so there are lots of gaps and mistakes, all due to me of course. The main reference can be found on the web, see (1).

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## 1. Background

Recall that a <u>*stack*</u> is a sheaf of groupoids that satisfies the descent condition. In more detail; let

$$F : \mathcal{C}^{\mathrm{op}} \to Groupoids$$

be a sheaf where $\mathcal{C}$ is a Grothendieck category. The descent condition means that for any cover

$$\mathcal{U} = \{U_i \to U\}$$

in $\mathcal{C}$ there is an equivalence of categories

$$F(U) \cong Desc(\mathcal{U}, F)$$

Now we need to define the descent data category.

Let $\mathcal{U} = \{U_i \to U\}$ be a covering of an object $U$ in $\mathcal{C}$. We'll define a new category, $Desc(\mathcal{U}, F)$:

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(1) objects will be pairs $(\{x_i\}, \{\phi_{i,j}\})$ where $x_i$ is an object of $F(U_i)$ and

$$\phi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ji}}$$

are isomorphisms in $F(U_{ij})$ that satisfy the cocycle condition

$$\phi_{jk} \circ \phi_{ij} \cong \phi_{ik}$$

(2) The morphisms are a compatible collection of maps from $x_i \to x_i'$.

Some comments are in order. Each object is a collection of data, namely class of objects together with restriction maps. The idea is supposed to be similar to having a vector bundle and looking at its local trivilazations.

## 2. Hopf Algebroids

First a Hopf algebroid is a cogroupoid object in the category of rings. Let's right this out in much more detail.

A Hopf algebroid is a pair $(A, \Gamma)$ or rings together with some structure maps such that for any commutative ring $B$, the sets $\mathrm{Hom}(A, B)$ and $\mathrm{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid. Call this groupoid $G_\Gamma(B)$.

The structure maps are

$\eta_l : A \to \Gamma$ (co-source)
$\eta_r : A \to \Gamma$ (co-target)
$\nabla : \Gamma \to \Gamma \otimes_A \Gamma$ (coproduct)
$\epsilon : \Gamma \to A$ (co-unit)
$c : \Gamma \to \Gamma$ (co-inverse)

There are some diagrams containing all these maps that need to commute, but we'll leave that out for now. The source map $\eta_l$ will be assumed to be a flat map of left $A$-modules. In this case, there are enough injective comodules to do homological algebra and is part of the hypothesis for the change of rings theorem.

**Definition 2.1.** *Let $(A, \Gamma)$ be a Hopf algebroid. A left $A$-module $M$ is a $\Gamma$ co-module if there is a left $A$-linear map*

$$\psi : M \to \Gamma \otimes_A M$$

*that is co-associative and counitary.*

## 3. Schemes

An affine scheme is a representable functor from the category of rings to sets. Let $R$ be a commutative ring with unit, then $\mathrm{Spec}(R)$ is the functor $\mathrm{Hom}(R, -)$. It's classical that the opposite category of rings is equivalent to the category of affine schemes. The inverse map is given by taking global sections.

In order to get a stack from a Hopf algebroid we will put a topology on $Rings^{op}$. There are a few topologies to choose from, but I will follow what Goerss does in (1) and start with the flat topology.

**Definition 3.1.** *A map of affine schemes* $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is flat if the map of rings* $f : A \to B$ *is flat*

A map of affine schemes $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is of *locally of finite type* if there exists an affine open cover of locally ringed spaces

$$\mathrm{Spec}(A) = \bigcup \mathrm{Spec}(A_i) = U_i$$

such that if we have an open cover

$$f^{-1}(U_i) = \bigcup \mathrm{Spec}(B_{ij})$$

of the pre-image, then each $B_{ij}$ is a finitely generated $A_i$-algebra.

With this information, a covering of a scheme $X$ is a collection

$$\left\{ X_i \xrightarrow{f_i} X \right\}$$

such that each $f_i$ is a flat and locally of finite type map of affine schemes and

$$X = \bigcup f_i(U_i)$$

as sets. This definition comes from Milne's E'tale cohomology book.

**Lemma 3.2.** *A map* $A \to B$ *is faithfully flat iff* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is flat and surjective.*

If $U_i = \mathrm{Spec}(A_i)$ and $X = \mathrm{Spec}(R)$, this is equivalent to $S \to \coprod A_j$ being faithfully flat for some finite subcollection of the original cover. I think this follows from the fact that $\mathrm{Spec}(A)$ is quasi-compact; every open cover has a finite subcover. This puts the flat topology on $Rings^{op}$. There are other topologies, for example the Zariski and E'tale.

## 4. Stacks from Hopf Algebroids

Suppose we have a Hopf algebroid $(A, \Gamma)$, the assignment

$$G_\Gamma : \mathrm{Spec}(R) \mapsto G_\Gamma(R)$$

is a presheaf of groupoids (with source category $Rings^{op}$. The presheaf $G_\Gamma$ is typically not a stack, write $M(A, \Gamma)$ for the associated stack. This all depends on the chosen topology for $Rings^{op}$.

**Example 4.1.** *Another example is the degenerate case of the above; we take the Hopf algebroid* $(A, A)$ *for some commutative ring A. It's a Hopf algebroid where all the structure maps are the identity. Thus we can look at the sheaf of groupoids it produces. It's a sheaf since it is representable and in the flat topology, representable presheaves are sheaves.*

**Claim 4.2.** *: The only morphisms in the groupoid given by* $(\mathrm{Hom}(A, R), \mathrm{Hom}(A, R))$ *are the identity isomorphisms.*

*Proof.* Recall the groupoid structure, given two objects $f \in \mathrm{Hom}(A, R)$ and $g \in \mathrm{Hom}(A, R)$ in our groupoid, we get a morphism between them if

$$\begin{array}{ccccc} & & f & & \\ & s & & \phi & \\ A & & A & & R \\ & t & & & \\ & & g & & \end{array}$$

can be filled in so that things commute. In the case that $s = t = id$ we see that $\phi = f = g$. There is a unique map that does make the diagram commute, which is the identity arrow in the groupoid. □

*Now given a cover*

$$\{\mathrm{Spec}(S_i) \to \mathrm{Spec}(B)\}$$

*the associated descent data is a collection of objects $s_i$ that* **agree** *on intersections, thus by the sheaf condition it lifts to a global object. This produces the equivalence of categories needed to be a stack.*

The whole point here is given a ring $R$ we can construct a stack, called $\mathrm{Spec}(R) = G_R$ . There is actually a moduli problem floating around here. Suppose $(A, \Gamma)$ is a Hopf algebroid. A morphism of stacks

$$\mathrm{Spec}(R) \to M(A, \Gamma).$$

corresponds to an equivalence class of pairs

$$(f : A \to S, \phi : \Gamma \to S \otimes_r S)$$

where $R \to S$ is a faithfully flat extension.

I don't really understand this observation, see the Goerss notes for more stuff.

**Example 4.3.** *Let $L$ be the ungraded Larzard ring, so that* $\mathrm{Hom}(L, R)$ *is naturally isomorphic to the set of formal group laws over $R$. If we let $W$ be the ring that represents isomorphisms, then $(L, W)$ is a Hopf algebroid. The associated stack $M(L, W) = \mathcal{M}_{FGL}$ has the cool property that morphisms*

$$\mathrm{Spec}(R) \to \mathcal{M}_{FGL}$$

*classify equivalence classes of formal group laws over faithfully flat extensions of $R$.*

I think the reason for this is tied up with what Haynes Miller was trying to tell me at the end of the talk. In particular, we need to understand the associated stack functor in this case to get the result.

**Remark 4.4.** *There is also an example for elliptic curves using the Weirestrass forms of the curves.*

## 5. Representable Morphisms

**Definition 5.1.** *A stack $\mathcal{M}$ is representable if it is isomorphic to* $\mathrm{Spec}(R)$ *for some ring $R$.*

A representable morphism of stacks is one whose fibers are schemes. The first step is to define the fibre product of a groupoids. Given a diagram

$$\begin{array}{ccc} & & G_2 \\ & & \downarrow{\scriptstyle g} \\ G_1 & \xrightarrow{f} & H \end{array}$$

of groupoids, the "homotopy pull-back" $G_1 \times_G^h G_2$ is the groupoid with objects consisting of triples

$$(x, y, \phi)$$

with $x$ an object in $G_1$ and $y$ an object of $G_2$ and

$$\phi : f(x) \to g(y)$$

is an isomorphism in $H$. A morphism from $(x, y, \phi)$ to $(x', y', \phi')$ is a pair of morphisms $x \to x'$ and $y \to y'$ so that

$$\begin{array}{ccc} f(x) & \longrightarrow & f(x') \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi'} \\ g(y) & \longrightarrow & g(y') \end{array}$$

commutes in $H$. Comments from the audience suggested that to lift this construction to presheaves of groupoids, just compute it point wise. He also mentioned that there are some subtleties in producing a stack in this way. The consequence of course, is a notion of pullback for stacks.

A morphism

$$\mathcal{M} \to \mathcal{N}$$

of stacks is representable (with respect to affines) if whenever we form the homotopy pull-back

$$\begin{array}{ccc} \mathrm{Spec}(R) \times_{\mathcal{N}}^h \mathcal{M} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & \mathcal{N} \end{array}$$

there is an equivalence of groupoid sheaves

$$\mathrm{Spec}(R) \times_{\mathcal{N}}^h \mathcal{M} \cong Spec(S)$$

for some commutative ring $S$, in other words, pull-backs along affines are affine.

**Example 5.2.** *If $(A, \Gamma)$ is a Hopf algebroid then*

$$\alpha : Spec(R) \to M(A, \Gamma)$$

*is representable.*

*Proof.* First suppose that $\alpha$ is actually represented by a map $\alpha : A \to R$. Likewise, suppose that our test map $\beta : \mathrm{Spec}(S) \to \mathcal{M}$ is represented by $\beta : A \to R$. Then

$$\mathrm{Spec}(R) \times^h_{\mathcal{M}} \mathrm{Spec}(S) \simeq \mathrm{Spec}(R \otimes \Gamma \otimes S)$$

and so the pull-back is affine. We can reduce to this case using the following trick. Suppose we have started with some general map $x : \mathrm{Spec}(R) \to \mathcal{M}$. Then there is a faithfully flat extension $i : R \to R'$ so that

$$i^*x : \mathrm{Spec}(R') \to \mathcal{M}$$

is represented by a map $A \to R'$. Again, this has to do with the associated stack functor and depends on the topology chosen. I don't understand this yet. Do the same for $\alpha$ and use faithfully flat descent to get the pull-back to be affine. □

**Definition 5.3.** *A map $f : \mathrm{Spec}(S) \to \mathcal{M}$ of stacks is flat if $f$ is representable, and when ever we pull back along a morphism of the form $g : \mathrm{Spec}(R) \to \mathcal{M}$ the induced maps out of the pull-back are induced by flat maps of rings.*

## 6. Structure Sheaf

**Definition 6.1.** *A stack $\mathcal{M}$ is an algebraic stack if we can chose a flat (and thus representable) surjective morphism*

$$\mathrm{Spec}(S) \to \mathcal{M}$$

*for some commutative ring $S$.*

If $\mathcal{M}$ is an algebraic stack define the category

$$\mathrm{FLAT}/\mathcal{M}$$

to have objects the flat maps

$$\mathrm{Spec}(R) \to \mathcal{M}.$$

A morphism

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{x} & \mathcal{M} \\ {\scriptstyle f} \downarrow & \nearrow_{y} & \\ \mathrm{Spec}(S) & & \end{array}$$

is a pair $(f, \phi)$ where $f : S \to R$ is a ring homomorphism and $\phi$ is an isomorphism $\phi : x \to f^*y$ (remember that *Stacks* is a 2-category. This means that the diagram only commutes up to isomorphism and this isomorphism is part of the data.

As an example, consider the case when $\mathcal{M} = {}_{FGL}$. Then $x$ and $y$ are represented by formal group laws $G$ and $H$ over faithfully flat extensions of $R$ and $S$. We are then specifying a isomorphism $f^*H \cong G$ over a faithfully flat extension of $R$.

Next, define a cover of an object $\mathrm{Spec}(R) \to \mathcal{M}$ of FLAT/$\mathcal{M}$ to be a faithfully flat map

$$\mathrm{Spec}(S) \to \mathrm{Spec}(R).$$

Now that we have a topology on FLAT/$\mathcal{M}$, a sheaf is a presheaf of sets that satisfies the the standard equalizer diagram for all covers.

**Definition 6.2.** *The* <u>*structure*</u> *sheaf on $\mathcal{M}$ is the functor*

$$\mathrm{Spec}(R) \to \mathcal{M} \mapsto R.$$

*It's actually a sheaf of rings.*

*Proof.* I guess it goes like this. Which I now think is wrong. Let $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be a covering in FLAT/$\mathcal{M}$. The pull-back in the diagram

$$\begin{array}{ccc} & & \mathrm{Spec}(S) \\ & & \\ & & \mathrm{Spec}(R) \\ & & \\ \mathrm{Spec}(S) & \mathrm{Spec}(R) & \mathcal{M} \end{array}$$

is the intersection. The fact that $\mathcal{M}$ is algebraic implies that the pull-back is affine, say of the form $\mathrm{Spec}(T)$. More over, the two maps $\mathrm{Spec}(T) \to \mathrm{Spec}(S)$ are the two restriction maps and we see that

$$\mathrm{Spec}(T) \qquad \mathrm{Spec}(S) \qquad \mathrm{Spec}(R)$$

is a coequalizer, and so

$$R \qquad S \qquad T$$

is an equilzer. □

## 7. Quasi-coherent Sheaves

**Definition 7.1.** *A quasi-coherent sheaf $F$ on an algebraic stack $\mathcal{M}$ is a sheaf on $\mathcal{M}$ such that $F(R,x)$ is a $R$ module for every $x : \mathrm{Spec}(R) \to \mathcal{M}$ and for every diagram of the form*

$$\begin{array}{ccc} \mathrm{Spec}(R) & x & \mathcal{M} \\ f & y & \\ \mathrm{Spec}(S) & & \end{array}$$

*in* **FLAT**/$\mathcal{M}$ *the induced map of $S$ modules $f^*F(R,x) \to F(S,y)$ is an isomorphism.*

I think we need a cocycle condition in this definition. But it is not mentioned in (1). Explicitly, the cocycle condition says that if we have a chain of covers

$$\mathrm{Spec}(R) \xrightarrow{f} \mathrm{Spec}(S) \xrightarrow{g} \mathrm{Spec}(T)$$

over $\mathcal{M}$ then

$$\phi_{g\circ f} = \phi_f \circ f^*\phi_g : F_R \xrightarrow{\simeq} (g\circ f)^*F_T = f^*(g^*F_T).$$

Again, suppose $(A, \Gamma)$ is a Hopf algebroid and let $C$ be a $\Gamma$ comodule. We would like to get a quasi-coherent sheaf from such a gadget.

The idea is something like this. If

$$\mathrm{Spec}(R) \xrightarrow{x} \mathcal{M} = M(A, \Gamma)$$

is actually represented by a map $x : A \to R$ define

$$\mathcal{F}_C(R, x) = R \otimes_A C.$$

The problem is, all objects in FLAT/$\mathcal{M}$ need not be of this form. There is an argument to reduce to this case though.

We are allowed to restrict to such maps above because for any general morphism $\mathrm{Spec}(R) \to \mathcal{M}$ there is a cover $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ so that the composition

$$\mathrm{Spec}(S) \to \mathrm{Spec}(R) \to \mathcal{M}$$

can be represented by a map of rings $A \to S$. So, if you are keeping count, this is the third time such a claim has reared its head. We need to figure this argument out at some point. Having defined the presheaf on all the covers, the sheaf condition ensures a definition on $\mathrm{Spec}(R) \to \mathcal{M}$.

I don't understand the transition functions, but it goes like this. Again see (1) A morphism in **FLAT**/$\mathcal{M}$ is a diagram of the form

$$\begin{array}{ccc} \mathrm{Spec}(R) & {}^{x} & \mathcal{M} = M(A,\Gamma) \\ {}_{f} & {}^{y} & \\ \mathrm{Spec}(S) & & \end{array}$$

where $f : S \to R$ is a map of rings. This corresponds to a diagram

$$\begin{array}{cccccc} & & A & {}^{y} & S \\ & & {}_{\eta_L} & {}^{x} & {}_{f} \\ A & {}^{\eta_R} & \Gamma & & R \end{array}$$

which i guess is saying that there is a morphism from $x$ to $f^*y$ in the groupoid given by the Hopf algebroid. Either way, taking a push-out produces a map

$$\alpha : S \otimes_R \Gamma \to R$$

and the transition function

$$F_C(S, y) \to F_C(R, x)$$

is given by

$$S \otimes_A C \xrightarrow{id\otimes\psi} S \otimes_A \Gamma \otimes_A C \xrightarrow{\alpha\otimes id} R \otimes_A C$$

I guess one question to ask is, why is the map induced by applying $-\otimes_A C$ to the ring map $S \to R$ not good enough? It must have to do with getting an isomorphism at the end of the day.

**Theorem 7.2.** *There is an equivalence of categories*

$$\{\textit{Quasi-coherent sheaves on } M(A,\Gamma)\} \cong (A,\Gamma) - \textit{(comodules)}$$

## 8. Cohomology

## References

1. P. Goerss, *(pre-)sheaves of ring spectra over the moduli stack of formal group laws*, `http://www.math.northwestern.edu/~pgoerss/papers/newton.pdf`.