HOMOTOPY FUNCTORIALITY FOR KHOVANOV SPECTRA

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Abstract. We prove that the Khovanov spectra associated to links and tangles are functorial up to homotopy and sign.

Contents

1. Introduction

The goal of this paper is to prove that the Khovanov spectrum [LS14a, LLS20, HKK16], an object in the homotopy category of spectra, is natural with respect to link cobordisms, up to sign. That is:

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Theorem 1. *If L_0 and L_1 are oriented link diagrams in $\mathbb{R}^2$ and $\Sigma\colon L_0 \to L_1$ is an oriented cobordism then there is an induced homotopy class of maps of spectra*

$$\mathscr{X}(\Sigma)\colon \mathscr{X}^j(L_0) \to \mathscr{X}^{j-\chi(\Sigma)}(L_1)$$

from the Khovanov spectrum of L_0 to the Khovanov spectrum of L_1, well-defined up to sign. Given another oriented link cobordism $\Sigma'\colon L_1 \to L_2$,

$$\mathscr{X}(\Sigma') \circ \mathscr{X}(\Sigma) = \pm\mathscr{X}(\Sigma' \circ \Sigma).$$

Further, if Σ consists of a single Reidemeister move then the map $\mathscr{X}(\Sigma)$ is homotopic to the map in the original proof of invariance of $\mathscr{X}(L)$ [LS14a]*, and if Σ consists of a single birth, death, or saddle then $\mathscr{X}(\Sigma)$ is homotopic to the map defined previously in those cases* [LS14b].

(For spectra, "up to sign" means the following. Roughly, reflection across the first coordinate of $\mathbb{R}^{n+1}$ induces an automorphism $(-1)\colon \mathbb{S} \to \mathbb{S}$ of the sphere spectrum; more precisely, to make this automorphism commute with the structure maps of $\mathbb{S}$ one takes a cofibrant-fibrant replacement of the sphere spectrum first. Then, for any spectrum X, there is an induced map $X = \mathbb{S} \wedge X \xrightarrow{(-1)\wedge \mathrm{Id}} \mathbb{S} \wedge X = X$, which plays the role of multiplication by -1.)

Functoriality of Khovanov homology up to sign was first established by Jacobsson [Jac04], by checking directly that the map Khovanov had associated to elementary cobordisms [Kho00, Section 6.3] were invariant under Carter-Saito's movie moves [CS93]. Shortly after, Khovanov and Bar-Natan gave new proofs of this result, using extensions of Khovanov homology to tangles to simplify checking most of the movie moves [Kho06, Bar05]. Soon after, detailed analyses of Jacobsson's proof led to better understanding of the sign ambiguity, and ways to remove it [CMW09, Cap08]. Recently, Blanchet [Bla10] gave another approach to avoiding the sign ambiguity of Khovanov homology, using Lee's deformation [Lee05]. A spectral refinement of part of Blanchet's work was given by Krushkal-Wedrich [KW].

The strategy to prove Theorem 1 is generally similar to Khovanov's proof of naturality. In a previous paper, we gave a spectral refinement of Khovanov's tangle invariants [LLSb]. (By contrast, a spectral refinement of Bar-Natan's tangle invariant is not currently known, nor is a spectral refinement of the Lee deformation.) Much of Khovanov's argument reduces to understanding the automorphisms of the bimodule associated to the identity braid, and a few similar arguments. In the spectral case, this bimodule has too many grading-preserving automorphisms for Khovanov's argument to go through. (See Section 3 for further discussion of this point.) We avoid this problem by localizing further, analogous to Bar-Natan's canopoly. In this more local form, the essence of Khovanov's argument goes through.

This strategy gives somewhat more than Theorem 1. Like Khovanov's and Bar-Natan's proofs, it gives an extension of Theorem 1 to tangle cobordisms (Theorem 4). Additionally, it shows that Khovanov homology and the Khovanov spectrum are also functorial under non-orientable cobordisms, though the grading shifts are harder to track. Along the way, we also prove two structural results about the Khovanov spectrum (as well as their analogues for Khovanov homology): the Khovanov spectral bimodule associated to the mirror of T is the dual to the Khovanov spectral bimodule associated to T, and the Khovanov spectral modules satisfy a planar algebra-like gluing property. (For Khovanov's arc algebras, the analogous properties seem to be well-known—see, for instance, [Rob17, Section 5.3] for the latter—but we do not have a specific citation for them.)

This paper is organized as follows. Section 2 recalls Khovanov's arc algebras and aspects of their spectral refinements. Section 3 discusses why Khovanov's proof of invariance does not immediately translate to the spectral case. The failure motivates the rest of this paper. Section 4 gives the planar algebra-like gluing property of the Khovanov modules and their spectral refinements, using the language of multicategories. Section 5 proves the duality between the Khovanov bimodules of a tangle and its mirror, and the spectral refinement of this duality. Section 6 combines these to prove functoriality of the Khovanov spectra, Theorems 1 and 4. We also give the analogous proof of functoriality of Khovanov homology, Theorem 3. Section 7 gives an example of how to extract an explicit invariant of cobordisms from the functor, in the spirit of the Hopf invariant.

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2. Background and grading conventions

Wherein we summarize expeditiously Khovanov's ARC ALGEBRAS AND BIMODULES including their key GLUING and INVARIANCE properties. We then recall the SPECTRAL REFINEMENTS of these algebraic objects, and corresponding properties of these spectral refinements. We conclude with a helpful summary of the paper's GRADING CONVENTIONS.

2.1. **Khovanov's arc algebras and modules.** Let $V = \mathbb{Z}[X]/(X^2)$ denote Khovanov's Frobenius algebra. The comultiplication on V is given by $\Delta(1) = 1 \otimes X + X \otimes 1$ and $\Delta(X) = X \otimes X$, and the counit is $\epsilon(1) = 0$, $\epsilon(X) = 1$. Equivalently, we can view V as a $(1+1)$-dimensional TQFT. So, given a closed 1-manifold Z, we have an abelian group $V(Z)$, which is generated by all ways of labeling the components of Z by 1 or X, and a cobordism from Z to Z' induces a homomorphism from $V(Z)$ to $V(Z')$ (which is the multiplication in V if the cobordism is a single merge and the comultiplication Δ if the cobordism is a single split.)

Let $\underline{2}$ be the category with two objects, 0 and 1, and a single morphism from 0 to 1,

$$\underline{2} = \big(0 \longrightarrow 1\big).$$

Given a link diagram L with N crossings $\mathfrak{C}$, applying the TQFT V to the cube of resolution of L gives a commutative cube $\underline{2}^{\mathfrak{C}} \to \mathsf{Ab}$, the category of abelian groups. Traditionally, the Khovanov complex is defined as the total complex or iterated mapping cone of this cube. To avoid choosing a sign assignment or ordering of the crossings, we will take another version of the iterated mapping cone. Let $\underline{2}_+^{\mathfrak{C}}$ be the category obtained by adding one more object $*$ to $\underline{2}^{\mathfrak{C}}$ and a morphism from each object except the terminal one in $\underline{2}^{\mathfrak{C}}$ to $*$. Extend V to a functor $\underline{2}_+^{\mathfrak{C}} \to \mathsf{Ab}$ by sending $*$ to the trivial group. Let $\mathcal{C}(L) = \bigoplus_{i,j} \mathcal{C}_{i,j}(L)$ be the homotopy colimit of this diagram (see, e.g., [LLS20, Section 4.2] or [HLS16, Definition 3.11]), with an internal (quantum) grading and a homological shift that use the orientation of L or other auxiliary data (see Section 2.4). That is,

$$\mathcal{C}(L) = \operatorname{hocolim}_{w \in \underline{2}_+^{\mathfrak{C}(L)}} V(L_w). \tag{2.1}$$

This complex is homotopy equivalent to the usual Khovanov complex, though the signs in the homotopy equivalence seem to depend on some choices. The Khovanov homology $Kh(L)$ is the homology of $\mathcal{C}(L)$.

Khovanov extended this construction to tangles as follows [Kho02]. Given an even integer n, let $\mathsf{B}(n)$ denote the set of crossingless matchings of n points. View an element $a \in \mathsf{B}(n)$ as a $(0,n)$-tangle, and let $\widehat{a}$ denote its mirror, an $(n,0)$-tangle. Let $\mathcal{C}(n)$ denote the linear category with:

- Objects $\mathsf{B}(n)$,
- $\mathcal{C}(n)(a,b) \coloneqq \mathrm{Hom}_{\mathcal{C}(n)}(a,b) = V(a\widehat{b})$, and
- Composition $\mathrm{Hom}_{\mathcal{C}(n)}(b,c) \times \mathrm{Hom}_{\mathcal{C}(n)}(a,b) \to \mathrm{Hom}_{\mathcal{C}(n)}(a,c)$ induced by the TQFT V and the *canonical saddle cobordism* $\widehat{a} \amalg a \to \mathrm{Id}$, the identity braid on n points.

Equivalently, we can view $\mathcal{C}(n)$ as an algebra, by taking

$$\bigoplus_{a,b \in \mathrm{Ob}(\mathcal{C}(n))} \mathcal{C}(a,b)$$

with multiplication $(x \cdot y) = y \circ x$ when defined and 0 otherwise. (Some elementary concepts related to linear categories are recalled in Section 2.2.)

Given an (m,n)-tangle diagram T with N crossings, there is a differential module $\mathcal{C}(T)$ over $\mathcal{C}(m)$ and $\mathcal{C}(n)$ defined by

$$\mathcal{C}(T)(a,b) = \mathcal{C}(aT\widehat{b}) = \mathrm{hocolim}_{v \in \underline{2}_+^{\mathfrak{c}(T)}} V(aT_v b), \tag{2.2}$$

where T_v is the resolution of T associated to v, and for $v = *$ we define $V(aT_v b) = 0$. The module structure is induced by the canonical saddle cobordisms, and the differential comes from the *crossing change cobordisms*. Far-commutativity of these cobordisms implies that the module structure is associative and respects the differential.

Khovanov proves:

Lemma 2.1. [Kho02] *The module $\mathcal{C}(T_v)$ associated to each resolution T_v of T is left-projective and right-projective. In fact, for each $a \in \mathsf{B}(m)$ there is a crossingless matching $a' \in \mathsf{B}(n)$ and an integer j so that $\mathcal{C}(T_v)(a,\cdot) \cong V^{\otimes j} \otimes \mathcal{C}(a')$, and similarly in the other factor.*

Theorem 2.2. [Kho02] *Up to quasi-isomorphism, the differential graded bimodule $\mathcal{C}(T)$ is invariant under Reidemeister moves.*

In fact, Theorem 2.2 holds up to homotopy equivalence of differential graded bimodules, which could be used to simplify some of the discussion below in the algebraic, but not the spectral, case; see Remark 4.19. In the proof, Khovanov associates specific homomorphisms to the Reidemeister moves.

The other key property is that gluing of tangles corresponds to tensor product of bimodules:

Theorem 2.3. [Kho02] *Given an (m,n)-tangle T_1 and an (n,p)-tangle T_2, there is a quasi-isomorphism*

$$\mathcal{C}(T_1) \otimes_{\mathcal{C}(n)} \mathcal{C}(T_2) \simeq \mathcal{C}(T_1 T_2).$$

2.2. **Terminology for linear and spectral categories.** Since we are working mainly in the language of linear or spectral categories, we recall how some constructions and terminology for rings extends to this setting. In the linear case, verifying that these extensions have the expected properties is elementary; for the spectral case, see for instance Blumberg-Mandell [BM12, Section 2].

We call a linear category *finite* if it has finitely many objects and each morphism space is a finitely-generated free abelian group. A spectral category is finite if it has finitely many objects and each morphism space is weakly equivalent to a finite CW spectrum.

Let $\mathscr{C}$ and $\mathscr{D}$ be linear categories. The *tensor product* $\mathscr{C} \otimes \mathscr{D}$ has objects $\mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$ and $\mathrm{Hom}_{\mathscr{C}\otimes\mathscr{D}}((c_1, d_1), (c_2, d_2)) = \mathrm{Hom}_{\mathscr{C}}(c_1, c_2) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}}(d_1, d_2)$. A *dg* $(\mathscr{C}, \mathscr{D})$*-bimodule* is a *dg* functor $\mathscr{C}^{\mathrm{op}} \otimes \mathscr{D} \to \mathsf{Kom}$, where Kom denotes the category of chain complexes of free abelian groups (and the morphism spaces in $\mathscr{C}$ and $\mathscr{D}$ have trivial differential). We will often drop the term *dg* even though we are considering *dg* bimodules. If $\mathscr{C}$ and $\mathscr{D}$ are spectral categories, their tensor product and bimodules are defined similarly, with smash product in place of tensor product and spectra in place of chain complexes.

Given a $(\mathscr{C}, \mathscr{D})$-bimodule M and a $(\mathscr{D}, \mathscr{E})$-bimodule N, the tensor product of M and N is the $(\mathscr{C}, \mathscr{E})$-bimodule $M \otimes_{\mathscr{D}} N$ with

$$(M \otimes_{\mathscr{D}} N)(c, e) = \Big(\bigoplus_{d\in\mathrm{Ob}(\mathscr{D})} M(c, d) \otimes_{\mathbb{Z}} N(d, e)\Big)/(f_*(m) \otimes n \sim m \otimes f^*(n))$$

for $f \in \mathscr{D}(d, d')$, with the obvious structure maps. There is an analogous tensor product in the spectral case.

Given $(\mathscr{C}, \mathscr{D})$-bimodules M, N, a *chain map* from M to N is a natural transformation. Explicitly, a chain map consists of chain maps $F_{c,d}\colon M(c, d) \to N(c, d)$ for each $c \in \mathrm{Ob}(\mathscr{C})$ and $d \in \mathrm{Ob}(\mathscr{D})$ so that for any objects $(c_1, d_1), (c_2, d_2) \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}} \times \mathscr{D})$, $m \in M(c_1, d_1)$, $f \in \mathscr{C}(c_2, c_1)$, and $g \in \mathscr{D}(d_1, d_2)$,

$$F_{c_2,d_2}(M(f^{\mathrm{op}}, g)(m)) = N(f, g)(F_{c_1,d_1}(m_1)); \tag{2.3}$$

if we write $M(f^{\mathrm{op}}, g)(m)$ in the perhaps more suggestive notation $f \cdot m_1 \cdot g$, and similarly for N, this equation becomes

$$F_{c_2,d_2}(f \cdot m \cdot g) = f \cdot F_{c_1,d_1}(m) \cdot g.$$

Similarly, a chain homotopy from a chain map F to a chain map G consists of chain homotopies $H_{c,d}$ from $F_{c,d}$ to $G_{c,d}$ for each $(c, d) \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}} \times \mathscr{D})$ satisfying the same compatibility condition (2.3). One can also define the homology of a $(\mathscr{C}, \mathscr{D})$-bimodule, and hence a *quasi-isomorphism* of $(\mathscr{C}, \mathscr{D})$-bimodules. The set of chain homotopy classes of chain maps is not invariant under quasi-isomorphism, but is if one takes a *projective resolution* of M first. Similarly, in the spectral case, we define $\mathrm{Hom}(M, N)$ to be the path components of the space of natural transformations from a cofibrant resolution of M to a fibrant resolution of N. This notion is invariant under weak equivalence of M and N. Similarly, tensor products are not invariant under quasi-isomorphism or weak equivalence, but if one takes projective (cofibrant) replacements first then they become so.

2.3. **Spectral arc algebras and modules.** The construction of the spectral Khovanov algebras and modules uses Elmendorf-Mandell's K-theory of permutative categories [EM06], and the first step is translating the notion of algebras and modules into that language. For each even integer n, there is a *arc algebra shape multicategory* $\mathcal{S}_n$ with an object for each pair of crossingless matchings $(a_1, a_2) \in \mathsf{B}(n) \times \mathsf{B}(n)$—these remember the Hom-spaces in the arc algebra $\mathcal{C}(n)$—and morphisms encoding when Hom's can be composed [LLSb, Section 2.3]. That is, there is a unique multimorphism

$$(a_1, a_2), (a_2, a_3), \dots, (a_{\alpha-1}, a_\alpha) \to (a_1, a_\alpha)$$

for each n-tuple of crossingless matchings $a_1, \dots, a_\alpha \in \mathsf{B}(n)$. Khovanov's arc algebra $\mathcal{C}(n)$ is a multifunctor from $\mathcal{S}_n$ to abelian groups (and multilinear maps). To define the spectral arc algebra $\mathscr{X}(n)$, it suffices to lift the arc algebra multifunctor to a functor $\mathcal{S}_n \to \mathscr{S}$, the multicategory of spectra. Similarly, there is a *tangle shape multicategory* $\mathcal{T}_{m;n}$ so that multifunctors from the tangle shape multicategory to chain complexes or spectra encode the notion of a pair of linear categories with the object sets $\mathsf{B}(m)$ and $\mathsf{B}(n)$ and a differential bimodule or spectral bimodule over them. Khovanov's bimodules $\mathcal{C}(T)$ define a functor from $\mathcal{T}_{m;n}$ to chain complexes, and to construct the spectral tangle invariants it suffices to lift these bimodules to $\mathscr{S}$. (See also the discussion in [LLSa, Section 3.3].) There are also groupoid-enriched versions $\widetilde{\mathcal{S}_n}$, $\widetilde{\mathcal{T}_{m;n}}$ of these shape multicategories. It is easier to define functors from the groupoid-enriched versions (because this encodes a kind of lax multifunctor), and Elmendorf-Mandell's rectification theorem implies that the space of functors from the groupoid-enriched versions is equivalent to the space of functors from the honest versions [LLSb, Sections 2.4 and 2.9].

The construction of the functors from $\mathcal{S}_n$ and $\mathcal{T}_{m;n}$ to spectra proceeds in several steps. Elmendorf and Mandell's K-theory is a multifunctor from the category of permutative categories to spectra. The Burnside category $\mathscr{B}$ (of the trivial group) is the multicategory enriched in groupoids with objects finite sets, morphisms $\mathrm{Hom}(X_1, \dots, X_k; Y)$ the finite correspondences from $X_1 \times \cdots \times X_k$ to Y, and 2-morphisms bijections of correspondences. There is a functor from the Burnside category to permutative categories sending a set X to the category of sets over X. So, to construct functors $\mathcal{S}_n \to \mathscr{S}$ and $\mathcal{T}_{m;n} \to \mathscr{S}$ it suffices to give functors to the Burnside category.

The *embedded cobordism category* has objects closed 1-manifolds embedded in $\mathbb{R}^2$ or $(0,1)^2$, 1-morphisms cobordisms embedded in $[0,1] \times \mathbb{R}^2$, and 2-morphisms isotopies of embedded cobordisms. In a previous paper [LLS20, Section 2.11], we constructed the *Khovanov-Burnside functor*, from the embedded cobordism category to the Burnside category. (See also [LLSb, Section 2.11] and [HKK16].)

To avoid needing to check that no loops of cobordisms where the Khovanov-Burnside functor has nontrivial monodromy appear in the construction of the tangle invariants, we introduce another auxiliary category, the *divided cobordism category* [LLSb, Section 3.1]. The following is a trivial generalization of that definition:

Definition 2.4. Let U be a subset of $\mathbb{R}^2$. The *divided cobordism category of* U, denoted $\mathsf{Cob}_d(U)$, is the category enriched in groupoids defined as follows:

(1) An object of $\mathsf{Cob}_d(U)$ is an equivalence class of the following data:
- A smooth, closed 1-manifold Z embedded in the interior of U.
- A compact 1-dimensional submanifold-with-boundary $A \subset Z$, the *active arcs*, satisfying the following: If I denotes the closure of $Z \setminus A$, then each component of A and I is an interval. The components of I are the *inactive arcs*.

(2) A morphism from (Z,A) to (Z',A') is an equivalence class of pairs (Σ,Γ) where
   - Σ is a smoothly embedded cobordism in $[0,1]\times\mathring{U}$ from $\{0\}\times Z$ to $\{1\}\times Z'$, vertical near $\{0,1\}\times\mathring{U}$.
   - $\Gamma\subset\Sigma$ is a collection of properly embedded arcs in Σ, vertical near $\partial\Sigma$, with $(\partial A\cup\partial A')=\partial\Gamma$, and so that every component of $\Sigma\setminus\Gamma$ has one of the following forms:
     (I) A rectangle, with two sides components of Γ and two sides components of $A\cup A'$.
     (II) A $(2k+2)$-gon, $k\geq 0$, with $(k+1)$ sides in Γ, one side in I', and the other k sides in I.

   The pairs (Σ,Γ) and (Σ',Γ') are equivalent if there is an orientation-preserving diffeomorphism $\phi\colon[0,1]\to[0,1]$ so that $(\phi\times\mathrm{Id}_U)(\Sigma)=\Sigma'$ and $(\phi\times\mathrm{Id}_U)(\Gamma)=\Gamma'$.

(3) There is a unique 2-morphism from (Σ,Γ) to (Σ',Γ') whenever (Σ,Γ) is isotopic to (Σ',Γ') rel boundary.

(4) Composition of divided cobordisms is defined in the obvious way.

In the case that U is a square $(0,1)^2$, the diffeomorphism group of the first $(0,1)$-factor acts on the divided cobordism category, and we quotient by this action. More precisely, we quotient the object set by the action of the orientation-preserving diffeomorphisms of $(0,1)$ which are the identity near $\{0,1\}$ and the morphism sets by the group of orientation-preserving diffeomorphisms of $[0,1]\times(0,1)$ which are the identity near $[0,1]\times\{0,1\}$ and which are independent of the first coordinate near $\{0,1\}\times(0,1)$. (This last condition ensures that the diffeomorphisms preserve the property of the cobordisms being vertical near the boundary.) Then concatenation in the first $(0,1)$-factor gives a strictly associative multiplication or *horizontal composition* $\amalg$ on $\mathsf{Cob}_d((0,1)^2)$. This horizontal composition allows us to view $\mathsf{Cob}_d((0,1)^2)$ as a multicategory, with multimorphisms from $(Z_1,A_1),\dots,(Z_n,A_n)$ to (Z,A) given by the morphisms in $\mathsf{Cob}_d((0,1)^2)$ from $(Z_1,A_1)\amalg\cdots\amalg(Z_n,A_n)$ to (Z,A). (In the language of Hu-Kriz-Kriz, this is an example of a $\star$-category [HKK16].)

Another case of composition is if $U=D^2\setminus(D_1\cup\cdots\cup D_k)$ and $V=D^2\setminus(D'_1\cup\cdots\cup D'_\ell)$ are complements of disjoint round disks inside D^2 then we can form the composition $U\circ_i V$ by rescaling and translating V to identify the outer D^2 of V with D_i in U; see Definition 4.22 below for more details. We will sometimes also call this composition *horizontal*, to distinguish it from composition of cobordisms.

The category Cob_d has a *canonical groupoid enrichment* $\widetilde{\mathsf{Cob}_d}$ [LLSb, Section 2.4].

Lemma 2.5. *For any U, the Khovanov-Burnside functor induces a functor $\widetilde{\mathsf{Cob}_d}(U)\to\mathscr{B}$. In the case $U=(0,1)^2$, this functor respects the action of the diffeomorphism group of $(0,1)$ on the first factor, and for U, V equal to the complements of disks in D^2 there is a natural isomorphism between the horizontal composition of the Khovanov-Burnside functors for U and V and the Khovanov-Burnside functor for $U\circ_i V$.*

Proof. The first statement is a trivial generalization of [LLSb, Proposition 3.2]. The statement about invariance under the diffeomorphism action is immediate from the construction of the Khovanov-Burnside functor (see [LLSb, Section 2.11]), as are the statements about gluing. □

Given crossingless matchings $a_1,a_2\in\mathsf{B}(m)$, there is an associated object of $\mathsf{Cob}_d((0,1)^2)$ with underlying 1-manifold $a_1\widehat{a_2}$ and inactive arcs a small neighborhood of $\partial a_1=\partial\widehat{a_2}$. The canonical saddle cobordisms have natural choices of divides, giving divided cobordisms $a_1\widehat{a_2}\amalg a_2\widehat{a_3}\to a_1\widehat{a_3}$ [LLSb, Section 3.2], where $\amalg$ means concatenation and then rescaling in the first $(0,1)$-direction. This gives a functor $\widetilde{\mathcal{S}_m} \to \widetilde{\mathsf{Cob}_d}((0,1)^2)$. Composing with the Khovanov-Burnside functor then gives the spectral arc algebra.

For the spectral tangle invariants, given an (m,n)-tangle T with crossings $\mathfrak{C}$ (and m, n even), there is a multicategory $\underline{2}^{\mathfrak{C}} \tilde{\times} \widetilde{\mathcal{T}}_{m;n}$ enriched in groupoids, a kind of thickened product of the cube category and the tangle shape multicategory [LLSb, Section 3.2.4]. There is a multifunctor $\underline{2}^{\mathfrak{C}} \tilde{\times} \widetilde{\mathcal{T}}_{m;n} \to \widetilde{\mathsf{Cob}_d}((0,1)^2)$ which sends an object (v,a,T,b) to the 1-manifold $aT_v\widehat{b}$ with active arcs at the boundary of T_v, a region around each crossing of T which was given the 0-resolution, and a small neighborhood of at least one point in the interior of each segment of T. (The last active arcs come from giving T *pox*; to reduce clutter, we will suppress the pox in this paper.) Composing with the Khovanov-Burnside functor and K-theory gives a functor $\underline{2}^{\mathfrak{C}} \tilde{\times} \widetilde{\mathcal{T}}_{m;n} \to \mathscr{S}$. Applying Elmendorf-Mandell's rectification procedure gives a functor $\underline{2}^{\mathfrak{C}} \times \mathcal{T}_{m;n} \to \mathscr{S}$ from an ordinary, non-enriched multicategory. On the full subcategories $\mathcal{S}_m$ and $\mathcal{S}_n$, this functor agrees with the arc algebra multifunctor. (This uses the fact that those categories are *blockaded* [LLSb, Proposition 2.39].) For pair of crossingless matchings a, b we can restrict the functor to the subcategory spanned by objects (v,a,T,b), i.e., to the different resolutions of T capped-off by a and b, to get a map $\underline{2}^{\mathfrak{C}} \to \mathscr{S}$. Take the iterated mapping cone of this functor by extending it to $\underline{2}^{\mathfrak{C}}_+$ by sending $*$ to a one-point space and then taking the homotopy colimit. Doing this for all pairs (a,b) gives a functor $\mathcal{T}_{m;n} \to \mathscr{S}$, which corresponds to the spectral Khovanov tangle bimodule $\mathscr{X}(T)$.

A key property is that applying singular chains to these spectral invariants gives the ordinary Khovanov algebras and chain complexes of bimodules up to chain homotopy equivalence [LLSb, Proposition 4.2]. (In fact, the chain homotopy equivalences are canonical up to homotopy.) So, by Whitehead's theorem, to verify invariance of the bimodules, it suffices to construct maps associated to Reidemeister moves which induce Khovanov's homotopy equivalences at the level of singular chains. Doing so is straightforward [LLSb, Sections 3.5 and 4.2].

The final basic property of the tangle invariants is that gluing tangles corresponds to tensor product of bimodule spectra. To prove this, we use yet another multicategory: the *gluing shape multicategory* $\mathcal{U}_{m;n;p}$, which encodes the notion of three bimodules X, Y, and Z and a map from the derived tensor product of X and Y to Z [LLSb, Section 5], and its groupoid enrichment $\widetilde{\mathcal{U}}_{m;n;p}$. (See also Section 4.3 for a generalization of this construction.) Given an (m,n)-tangle S and a (n,p)-tangle T, the same scheme as above gives a multifunctor $\underline{2}^{\mathfrak{C}} \tilde{\times} \widetilde{\mathcal{U}}_{m;n;p} \to \mathsf{Cob}_d((0,1)^2)$. Composing with the Khovanov-Burnside functor and K-theory, then rectifying, gives a functor $\widetilde{\mathcal{U}}_{m;n;p} \to \mathscr{S}$, which encodes a map from $\mathscr{X}(S) \otimes^L_{\mathscr{X}(n)} \mathscr{X}(T) \to \mathscr{X}(T \circ S)$. At the level of singular chains, this agrees with Khovanov's gluing map, hence is a weak equivalence [LLSb, Theorem 5].

2.4. **Gradings.** To avoid keeping track of orientations of tangles, we will assign Khovanov complexes to pairs (T,P) where T is a tangle and P is an integer. (This is similar to Khovanov's category $\mathbb{ETL}$ [Kho02].) Given an oriented tangle, we recover the usual Khovanov invariants by letting P be the number of positive crossings of T. Other than this, we follow the grading conventions from our previous paper [LLSb, Section 2.10.1].

Grade the Khovanov Frobenius algebra V by $\mathrm{gr}_q(1) = -1$ and $\mathrm{gr}_q(X) = 1$.

On the arc algebras:

- For the quantum grading on $\mathcal{C}(n)$, we shift $\mathcal{C}(a\widehat{b})$ up by $n/2$, so the lowest-graded elements are idempotents in $\mathcal{C}(a\widehat{a})$ in grading 0.
- For the homological grading, $\mathcal{C}(n)$ lies in grading 0.

Next, fix an (m,n)-tangle T with N crossings and an integer P. Recall that $\mathcal{C}(T)(a,b)$ is the iterated mapping cone (via a homotopy colimit) of a diagram $\underline{2}^{\mathfrak{C}(T)} \to \mathsf{Ab}$ (see Equation (2.2)).

- For the quantum grading, we shift the grading on $\mathcal{C}(aT_v\widehat{b})$ up by $n/2-|v|+2N-3P$. (Here, $|v|$ is the height of v, i.e., the sum of the entries of v.)
- For the homological grading, we let $\mathcal{C}(aT_v\widehat{b})$ lie in homological grading $-P$. (Note that we have not taken the mapping cone yet; after taking the mapping cone, the grading of the term corresponding to $\mathcal{C}(aT_v\widehat{b})$ will be shifted up by $N-|v|$, so it will lie in homological grading $N-|v|-P$.)

In formulas, if we let $\{h,q\}$ denote shifting the quantum grading up by q and the homological grading up by h, then

$$\mathcal{C}(n) = \bigoplus_{a\in \mathsf{B}(n)} V(a\widehat{a})\{0,n/2\}$$

$$\mathcal{C}(T,P)(v,a,b) = V(aT_v\widehat{b})\{-P, n/2-|v|+2N-3P\}$$

$$\mathcal{C}(T,P)(a,b) = \operatorname{hocolim}_{v\in \underline{2}_+^{\mathfrak{C}(T)}} \mathcal{C}(T,P)(v,a,b).$$

The homotopy equivalence for gluing tangles (Theorem 2.3) consists of grading-preserving maps

$$\mathcal{C}(T_1,P_1)\otimes_{\mathcal{C}(n)} \mathcal{C}(T_2,P_2) \xrightarrow{\simeq} \mathcal{C}(T_2\circ T_1, P_1+P_2).$$

Given graded modules M,N, we define a homogeneous morphism $f\colon M\to N$ to have grading k if f increases the grading by k. (This is the opposite of the typical grading convention for cohomology, and would result in the cohomology of a topological space being supported in negative gradings.)

Remark 2.6. With our grading conventions, the graded Euler characteristic of the Khovanov homology of L is the (unnormalized) Jones polynomial of $m(L)$, the mirror of L, and positive knots have Khovanov homology supported in negative gradings. The differential on the Khovanov complex decreases the homological grading.

3. Khovanov's argument and why it does not translate immediately

Wherein we recall key points of Khovanov's proof of functoriality of Khovanov homology, observe subtleties obstructing one of these key arguments in the spectral case, and note an idea to partly circumvent this obstruction by further localizing the problem [pun intended] which sets the scene for the rest of the paper.

The rest of the paper is independent of the discussion in this section.

Like all known proofs of functoriality of Khovanov homology, Khovanov's starts from a movie description of a cobordism. Each elementary movie is a cobordism between layered tangles. Several of the elementary movies are planar isotopies of tangles; the others are Reidemeister moves, births or deaths of zero-crossing unknots, and local saddles. Khovanov associates a map of bimodules to each of these elementary movies: for planar isotopies, there are obvious isomorphisms, for Reidemeister moves he associated isomorphisms when he proved invariance of the tangle bimodules, and the maps for births, deaths, and saddles come from the unit, counit, and multiplication and comultiplication maps in his Frobenius algebra. The map associated to a movie is obtained by tensoring the maps for elementary movies, on the local slices of the layered tangle, with the identity map on the rest of the tangle, and then composing these maps.

The next step is to prove that two movies representing isotopic cobordisms induce the same map on Khovanov homology, by checking that the maps are invariant under Carter-Saito's movie moves. Rather than laboriously checking each move (as Jacobsson did [Jac04]), the local description of the movie moves allows Khovanov to reduce this check to three principles and minor variants on them:

(1) Movies involving no crossings correspond to cobordisms in $\mathbb{R}^3$, and he verified earlier that the maps associated to cobordisms in $\mathbb{R}^3$ are isotopy invariants of those cobordisms [Kho02]. (In fact, they depend only on the combinatorics of the cobordism, and not even its embedding.) This principle is used for movie moves 8, 9, 10, 23(b), and 24.
(2) If Σ is a movie between invertible tangles inducing a quasi-isomorphism on the Khovanov complex of bimodules (e.g., because each piece is a Reidemeister move) then Σ corresponds to a unit in $\mathrm{HH}^{0,0}(\mathcal{C}(n)) = \mathrm{RHom}^{0,0}_{\mathcal{C}(n)\otimes\mathcal{C}(n)^{\mathrm{op}}}(\mathcal{C}(n),\mathcal{C}(n))$, the Hochschild cohomology of $\mathcal{C}(n)$ in bigrading $(0,0)$. This Hochschild cohomology group is identified with the part of the center of $\mathcal{C}(n)$ in quantum grading 0, which in turn is isomorphic to $\mathbb{Z}$. So, the only units are ± 1. This principle is used for movie moves 6, 12, 13, 23a, and 25, and variants on it are used for moves 7, 11, 14–22, and 26–30.
(3) The map associated to the inverse of a Reidemeister move is the inverse of the map associated to a Reidemeister move (up to sign). (In fact, as Khovanov notes, this also follows from the previous principle and its variants.) This principle is used for movie moves 1–5.
(4) The tensor product is a bifunctor, i.e., $f\otimes g = (f\otimes \mathrm{Id})\circ(\mathrm{Id}\otimes g) = (\mathrm{Id}\otimes g)\circ(f\otimes\mathrm{Id})$. This principle is used for movie move 31.

To extend this argument to the Khovanov homotopy type, there are two difficulties. The first is that we have not verified that the maps associated to cobordisms in $\mathbb{R}^3$ are isotopy invariants: in constructing the homotopy refinements of Khovanov's tangle invariants, we allow only certain isotopies of surfaces. (This restriction is because of how the Khovanov-Burnside functor is defined on genus-1 surfaces with boundary [LLSb, Section 2.11].) For movies 8, 9, and 10, it is clear that the maps of homotopy types are the same, and it would not be hard to verify directly that the maps are homotopic for moves 23(b) and 24.

The second, more serious difficulty is with Point (2). The difficulty can already been seen in the case of the identity braid on two points, i.e., for $\mathcal{C}(2)\cong\mathbb{Z}[X]/(X^2)$ and its spectral refinement $\mathscr{X}(2)\simeq\mathbb{S}\vee\mathbb{S}$. Using the biprojective resolution

$$0\leftarrow \mathbb{Z}[X,Y]/(X^2,Y^2)\xleftarrow{X-Y}\mathbb{Z}[X,Y]/(X^2,Y^2)\xleftarrow{X+Y}\mathbb{Z}[X,Y]/(X^2,Y^2)\xleftarrow{X-Y}\cdots,$$

the Hochschild cohomology of $\mathcal{C}(2)$ is the homology of the complex

$$0\to\mathcal{C}(2)\xrightarrow{0}\mathcal{C}(2)\xrightarrow{2X}\mathcal{C}(2)\xrightarrow{0}\cdots.$$

The set of homotopy classes of bimodule homomorphisms from $\mathscr{X}(2)$ to itself is $\pi_0 \operatorname{THH}^*(\mathscr{X}(2))$. Since $\mathcal{C}(2)$ is flat over $\mathbb{Z}$ and $\mathscr{X}(2)$ is connective, there is a spectral sequence converging to $\pi_* \operatorname{THH}^*(\mathscr{X}(2))$ with E^1-page given by

$$0 \to [\mathcal{C}(2) \otimes \pi_*(S^0)]_0 \xrightarrow{0} [\mathcal{C}(2) \otimes \pi_*(S^0)]_1 \xrightarrow{2X} [\mathcal{C}(2) \otimes \pi_*(S^0)]_2 \xrightarrow{0} \cdots$$

where the subscripts 0, 1, 2 are labels for the different terms. (This is the spectral sequence associated to the smash product of $\mathscr{X}(2)$ with the Postnikov tower of $\mathbb{S}$.) The gradings are as follows. The homological grading of $a \otimes \zeta \in [\mathcal{C}(2) \otimes \pi_j(S^0)]_i$ is $\operatorname{gr}_h(a) + j - i$, so the differential decreases the homological grading by 1. The quantum grading of $a \otimes \zeta \in [\mathcal{C}(2) \otimes \pi_j(S^0)]_i$ is $\operatorname{gr}_q(a) - 2i$, so the differential preserves the quantum grading.

Let $\eta \in \pi_1(S^0) \cong \mathbb{Z}/2\mathbb{Z}$ be the Hopf map. Then the (homological, quantum) bigrading $(0,0)$ part of the E^2-page is

$$\mathbb{Z}\langle [1 \otimes 1]_0 \rangle \oplus (\mathbb{Z}/2\mathbb{Z})\langle [X \otimes \eta]_1 \rangle.$$

Since the only elements in quantum grading 0 have the form are $[1 \otimes \zeta]_0$ and $[X \otimes \zeta]_1$, for quantum grading 0 the spectral sequence collapses at the E^2-page. Hence, $\pi_0 \operatorname{THH}^0(\mathscr{X}(2))$ fits into a short exact sequence

$$0 \to (\mathbb{Z}/2\mathbb{Z})\langle [X \otimes \eta]_1 \rangle \to \pi_0 \operatorname{THH}^0(\mathscr{X}(2)) \to \mathbb{Z}\langle [1 \otimes 1]_0 \rangle \to 0,$$

so

$$\pi_0 \operatorname{THH}^0(\mathscr{X}(2)) \cong \mathbb{Z}\langle [1 \otimes 1]_0 \rangle \oplus (\mathbb{Z}/2\mathbb{Z})\langle [X \otimes \eta]_1 \rangle.$$

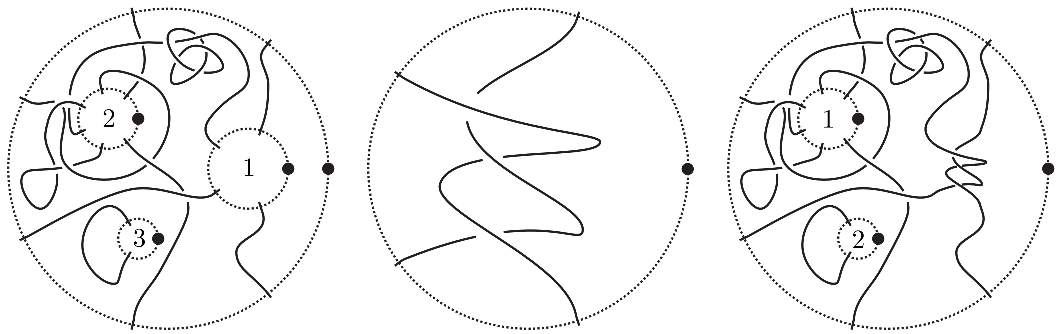
Abusing notation, we denote the generator of this $\mathbb{Z}/2\mathbb{Z}$ by $X\eta$. Then $\operatorname{Id} + X\eta$ is a nontrivial grading-preserving (derived) automorphism of the bimodule $\mathscr{X}(2)$, interfering with technique (2).

For the case $n > 2$, presumably $\mathscr{X}(n)$ has other, more complicated automorphisms, as well.

On a more optimistic note, in the case of $\mathscr{X}(2)$, the only obstruction to technique (2) was a 2-torsion class. So, if we invert 2 then Khovanov's argument would apply, to show that the homotopy classes of bimodule automorphisms are the units in $\mathbb{Z}[1/2]$. More generally, if we were only interested in $\mathscr{X}(n)$ for finitely many n, there would be a finite list of primes, corresponding to the torsion in $\pi_i(S^0)$ for i small, so that after inverting them Khovanov's argument applies. So, it is natural to adapt Khovanov's argument to be more local, so that only the $\mathscr{X}(n)$ for $n \leq 8$, say, appear. In fact, we will see that, perhaps surprisingly, this adaptation leads to a proof of naturality without inverting any primes.

4. Planar composition for Khovanov's tangle invariants and their spectral refinements

Wherein we formulate certain multicategories of tangles and tangle cobordisms, and use this language to give a minor extension of Khovanov's gluing results for bimodules over the arc algebra [Kho02], in the spirit of Bar-Natan's canopoly [Bar05, Section 8] or of Jones's planar algebras [Jon]. This material seems to be well-known to experts (see, e.g., [Rob17, Section 5.3]). We follow this with analogous extensions for the spectral tangle invariants.



T $\qquad$ S $\qquad$ $T \circ_1 S$

FIGURE 4.1. **Two diskular tangles and a composition of them.** Left: a diskular $(4,6,2;6)$ tangle $T = T^{4,6,2;6}$. Center: a diskular $(;4)$-tangle $S = S^{;4}$. Right: the composition $T \circ_1 S$.

4.1. **Multicategories of tangles.** Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$. A *round disk* in D^2 is a subset of the form $\{z \in D^2 \mid |z - z_0| \leq r\}$ for some $z_0 \in \mathring{D}^2$ and some $0 < r < 1 - |z_0|$. If D is a round disk then translation and scaling gives a canonical identification $\phi_D \colon D \to D^2$. Let $A = \{z \in D^2 \mid 1/2 \leq |z| \leq 1\}$ denote the annulus with inner radius $1/2$ and outer radius 1.

Definition 4.1. Fix non-negative, even integers $n, m_1, \dots, m_k$. A *diskular* $(m_1, \dots, m_k; n)$*-tangle* is a tangle diagram $T = T^{m_1,\dots,m_k;n}$ in $D^2 \setminus (\mathring{D}_1 \cup \dots \cup \mathring{D}_k)$, where $D_1, \dots, D_k$ are disjoint round disks in D^2, so that the boundary of T consists of

- the points $e^{2\pi i j/(n+1)}$, $j = 1, \dots, n$, in ∂D^2, and
- the points $\phi_{D_i}^{-1}(e^{2\pi i j/(m_i+1)})$, $j = 1, \dots, m_i$, in ∂D_i^2,

and T is radial near each ∂D^2 and each ∂D_i. See Figure 4.1. The disks $D_1, \dots, D_k$ are viewed as ordered.

Given diskular tangles $T^{m_1,\dots,m_k;n}$ and $S_i^{\ell_1,\dots,\ell_{j_i};m_i}$, $i = 1, \dots, k$, let

$$T \circ (S_1, \dots, S_k) = T \cup \phi_{D_1}(S_1) \cup \dots \cup \phi_{D_k}(S_k).$$

Alternatively, given an integer $1 \leq i \leq k$ and pair of tangles $T^{m_1,\dots,m_k;n}$ and $S^{\ell_1,\dots,\ell_j;m_i}$, there is a pairwise composition

$$T \circ_i S = T \cup \phi_{D_i}(S).$$

Again, see Figure 4.1. These are related by

$$T \circ (S_1, \dots, S_k) = (\cdots((T \circ_k S_k) \circ_{k-1} S_{k-1}) \circ_{k-2} \cdots) \circ_1 S_1.$$

We will call a diskular $(;n)$-tangle (i.e., a diskular tangle involving no sub-disks) simply a *diskular n-tangle.*

In Definitions 4.2 and 4.3, we define a essentially combinatorial version of cobordisms, in the spirit of movies. We give a topological interpretation of these in (and immediately preceding) Theorem 2.

Definition 4.2. Fix diskular $(m_1, \dots, m_k; n)$-tangles S and T. An *elementary cobordism* from S to T is any of the following:

(1) A planar ambient isotopy $\Phi_t \colon D^2 \to D^2$ from S to T, so that $\Phi_t|_{\mathrm{nbd}(S^1)}$ is the identity for all t and $\Phi_t|_{\mathrm{nbd}(D_i)}$ is the composition of translation and scaling for all i, t. The *support* of the ambient isotopy is the union over t of the support of Φ_t (the set of points where $\Phi_t \neq \mathrm{Id}$).
(2) A single Reidemeister move. For each type of Reidemeister move, we fix a tangle replacement in D^2 corresponding to that Reidemeister move; then a Reidemeister elementary cobordism is the image of this tangle replacement under the map ϕ_D^{-1} for some round disk D. The disk D is the *support* of the Reidemeister move.
(3) A birth or death of a 0-crossing unknot disjoint from S. Again, this is the image of a fixed standard birth in D^2 under the map ϕ_D^{-1} for some round disk D. The disk D is the *support* of the birth or death.
(4) A planar saddle. Again, this is the image of a fixed standard saddle in D^2 under the map ϕ_D^{-1} for some round disk D. The disk D is the *support* of the saddle.

We will call births, deaths, and saddles *Morse moves*.

Definition 4.3. Given an elementary cobordism Σ from S to T, let

$$P(\Sigma) = \begin{cases} 1 & \text{if } \Sigma \text{ is an R1 move creating a positive crossing or an R2 move creating crossings} \\ -1 & \text{if } \Sigma \text{ is an R1 move removing a positive crossing or an R2 move removing crossings} \\ 0 & \text{otherwise.} \end{cases}$$

$$\chi'(\Sigma) = \begin{cases} -1 & \text{if } \Sigma \text{ is a saddle} \\ 1 & \text{if } \Sigma \text{ is a birth or death} \\ 0 & \text{otherwise.} \end{cases}$$

Given a sequence $\Sigma = (\Sigma_1, \dots, \Sigma_k)$ of elementary cobordisms starting at some tangle S and ending at a tangle T, and an integer P_S, define

$$P(\Sigma) = \sum_i P(\Sigma_i)$$
$$\chi'(\Sigma) = \sum_i \chi'(\Sigma_i).$$

We say that Σ *goes from* (S, P_S) to $(T, P_S + P(\Sigma))$.

An alternative definition of χ' for a cobordism $\Sigma \colon S \to T$ (viewed as a surface) is the Euler characteristic of Σ minus half the number of endpoints of S. (This works for both elementary cobordisms and compositions of elementary cobordisms.)

Definition 4.4. The *tangle movie multicategory* $\mathbb{T}$ is the multicategory enriched in categories defined as follows. The objects of $\mathbb{T}$ are the non-negative, even integers n. Given objects $m_1, \dots, m_k$

and n, an object of $\mathrm{Hom}_{\mathbb{T}}(m_1,\dots,m_k;n) = \mathbb{T}(m_1,\dots,m_k;n)$ is a diskular tangle $T^{m_1,\dots,m_k;n}$, together with an integer P. In the special case $k=1$ and $m_1=n$, we also include an identity map of n as a morphism. Composition of morphism objects is given by composition of diskular tangles as in Definition 4.1 and adding the integers P. Given objects (S,P_S), $(T,P_T)\in\mathbb{T}(m_1,\dots,m_k;n)$, a morphism from (S,P_S) to (T,P_T) is a finite sequence of elementary cobordisms which goes from (S,P_S) to (T,P_T), modulo the (transitive closure of the) the following relations:

- (D1) Elementary cobordisms with disjoint supports commute.
- (D2) Isotopic ambient isotopies are equal.
- (D3) The formal composition of two ambient isotopies is equal to the composition of the two ambient isotopies in the usual sense.
- (D4) Applying a Reidemeister move or Morse move and then an ambient isotopy is equivalent to performing the ambient isotopy first and then the corresponding Reidemeister move or Morse move. Here, the diagram must have a disk which has exactly the form of the model Reidemeister or Morse move both before and after the ambient isotopy.

Given multi-morphism morphisms (2-morphisms) $f\colon (S,P_S)\to(S',P'_{S'})$ and $g_i\colon (T_i,P_{T_i})\to(T'_i,P'_{T'_i})$ so that $(S,P_s)\circ\big((T_1,P_{T_1}),\cdots,(T_\ell,P_{T_\ell})\big)$ is sensible, the multi-composition $f\circ(g_1,\dots,g_\ell)$ is defined by scaling down the elementary cobordisms in the g_i and inserting them in the corresponding disks for f.

Lemma 4.7 below states that this does, in fact, define a multicategory.

Example 4.5. Given an oriented diskular tangle T, there is a corresponding multi-morphism object $(T,P(T))$ in the tangle movie multicategory where $P(T)$ is the number of positive crossings in T. (See also Remark 4.12.)

Consider Carter-Saito's movie moves [CS93, Figures 23–38], as listed by Khovanov [Kho06, Figures 5–9][(i)] Each is a move of layered (m,n)-tangles. There are two kinds of moves. Moves 8–22, 24, and 31 correspond to composing planar isotopies, or to commuting a planar isotopy past a Reidemeister move or Morse move. The remaining moves (moves 1–7, 23(a,b), 25–30) correspond to nontrivial sequences of Morse moves and Reidemeister moves, at least on one side. We will call the first class of moves *Type I movie moves*, and the second class *Type II movie moves*. Each Type II movie move has a *main piece*, drawn in the figure, and an identity braid to the left and right. Identify the square with D^2, so the main piece of each movie move consists of diskular n-tangles. We will call the main pieces of the Type II movie moves, viewed this way, *diskular movie moves*.

Definition 4.6. The tangle multicategory $\mathscr{T}$ is the same as the tangle movie multicategory $\mathbb{T}$ except that we quotient the 2-morphisms by (the transitive closure of) the following relations:

- (D5) If two sequences of Morse moves and Reidemeister moves are related by a diskular movie move then we declare them to be equal. More generally, if there is a round disk so that over that disk the two sequences differ by a movie move, and away from the disk they are the same, then we declare the two sequences to be equal.

[(i)] The only differences are that some of Khovanov's moves are rotated by $\pi/2$ from Carter-Saito's, and Khovanov arranges that all strands end on the top or bottom.

Lemma 4.7. *The tangle movie multicategory and tangle multicategory are, in fact, multicategories.*

Proof. In both cases, we must check that:

(MC-1) Horizontal composition (of 1-morphisms and of 2-morphisms) is well-defined.
(MC-2) Horizontal composition is associative.
(MC-3) Vertical composition (of 2-morphisms) is well-defined.
(MC-4) Vertical composition is associative.
(MC-5) Vertical composition commutes with horizontal composition.

For the tangle movie multicategory, Point (MC-1) is obvious for 1-morphisms, and for 2-morphisms it follows from the fact that we imposed the relations that elementary cobordisms with disjoint supports commute and elementary cobordisms commute with planar isotopies. Points (MC-2), (MC-3), and (MC-4) are obvious. Point (MC-5) again uses the facts that elementary cobordisms with disjoint supports commute and elementary cobordisms commute with planar isotopies.

For the tangle multicategory, we must check Points (MC-1) and (MC-3); then the others follow from the previous case. But both of these points are still obvious: both horizontal and vertical gluing respect the equivalence relation in Definition 4.6. ☐

Given a diskular tangle T, we can view T as a 1-manifold-with-boundary inside $D^2 \times \mathbb{R}$, with the boundary contained in $D^2 \times \{0\}$ or, more specifically, $S^1 \times \{0\} \cup \bigcup_i \partial D_i \times \{0\}$. Given diskular tangles S, T, a *genuine cobordism* from S to T consists of:

- A smoothly-varying family $D_{i,t}$ of round disks inside D^2, $t \in [0,1]$, disjoint for each $t \in [0,1]$ and so that the $D_{i,0}$ are the disks corresponding to S and the $D_{i,1}$ are the disks corresponding to T.
- A smoothly embedded surface
$$\Sigma \subset \big([0,1] \times D^2 \times \mathbb{R}\big) \setminus \big(\bigcup_{i,t} \{t\} \times D_{i,t} \times \mathbb{R}\big)$$
with boundary $\{0\} \times S$, $\{1\} \times T$, and the points on $(t,p,0) \in [0,1] \times \partial D_{i,t} \times \mathbb{R}$ which are the images of ∂S under the translation and scaling that sends D_i to $D_{i,t}$.

In particular, if S and T are links, this reduces to the usual definition of a link cobordism. (More generally, this is also the standard notion of a tangle cobordism when there are no sub-disks D_i.)

Fixing topological models for the elementary tangle cobordisms (mapping cylinders or traces for types 1 and 2, and elementary Morse cobordisms for types 3 and 4), any sequence of elementary cobordisms gives rise to a genuine cobordism between diskular tangles. The following is essentially due to Carter-Saito [CS93]:

Theorem 2. *Every isotopy class of genuine cobordisms is represented by a sequence of elementary tangle cobordisms. Further, two sequences of elementary tangle cobordisms represent isotopic genuine cobordisms if and only if they represent the same 2-morphism in the tangle multicategory* $\mathscr{T}$.

Proof. The first statement, that every isotopy class of genuine cobordisms is represented by a sequence of elementary tangle cobordisms, is clear: one can isotope the cobordism to be a sequence of isotopies (in which the boundary disks are also allowed to move) and Morse moves, and then perturb the isotopy steps so each consists of a sequence of planar isotopies and model Reidemeister moves. For each of the planar isotopies, one also chooses an ambient isotopy covering it.

It is also clear that each of the moves (D1)–(D5) induces an isotopy of genuine cobordisms.

We reduce the rest of the theorem to Carter-Saito's result by using the following canonical factorization of genuine cobordisms. Call a genuine cobordism $(\Sigma, \{D_{i,t}\})$ from S to T *classical* if the family of disks $D_{i,t}$ is constant (independent of t), and *braid-like* if there is an ambient isotopy ψ_t of D^2 extending the isotopy of the D_i and so that $\Sigma \cap (\{t\} \times \mathbb{R}^3) = \psi_t(S)$ for all $t \in [0,1]$. For braid-like cobordisms, we consider the ambient isotopy map ψ_t part of the data.

Given a genuine cobordism $(\Sigma, \{D_{i,t}\})$, there is a ambient ψ_t of D^2 extending the isotopy $\{D_{i,t}\}$ (with $\psi_0 = \mathrm{Id}$). Let $\Psi\colon [0,1] \times D^2 \times \mathbb{R} \to [0,1] \times D^2 \times \mathbb{R}$ be the trace of ψ_t, and a 1-parameter family of isotopies can be lifted to a 1-parameter family of ambient isotopies. Given ψ_t, there is a canonical isotopy from $(\Sigma, \{D_{i,t}\})$ to

$$\big(\Psi([0,1] \times \psi_1^{-1}(T)), \{D_{i,t}\}\big) \circ (\Psi^{-1}(\Sigma), \{D_{i,0}\}).$$

Call this a *braid-classical factorization* of $(\Sigma, \{D_{i,t}\})$ into the composition of a classical cobordism and a braid-like cobordism. Given a 1-parameter family of cobordisms, there is a corresponding 1-parameter family of braid-classical factorizations.

Now, suppose M and M' are two sequences of elementary tangle cobordisms representing isotopic genuine cobordisms. We want to show that M and M' are related by a sequence of moves of type (D1)–(D5). By applying a sequence of moves of type (D4), we can assume that M and M' consist of a classical cobordism followed by a braid-like cobordism. Further, since the braid-classical factorization applies in 1-parameter families, the resulting classical cobordisms are isotopic through classical cobordisms, and the braid-like cobordisms are isotopic through braid-like cobordisms. Hence, the the braid-like cobordisms are related by move (D2). By Carter-Saito's theorem [CS93] (or rather, its folklore extension to tangles with fixed ends, as used by Khovanov [Kho06] and Bar-Natan [Bar05]), the classical cobordisms differ by a sequence of movie moves. Every movie move is either a diskular movie move or a move of type (D1), (D2), (D3), or (D4). Hence, M and M' differ by a sequence of moves of types (D1)–(D5), as desired. ☐

There is an enlargement of these categories which is also useful:

Definition 4.8. [LLSb, Section 2.4.1] Given a multicategory $\mathscr{C}$ enriched in categories, the *canonical enlargement* $\widetilde{\mathscr{C}}$ of $\mathscr{C}$ has:

- $\mathrm{Ob}(\widetilde{\mathscr{C}}) = \mathrm{Ob}(\mathscr{C})$
- An object of $\widetilde{\mathscr{C}}(x_1, \dots, x_n; y)$ is a planar, rooted tree Y with n distinguished leaves called inputs, together with a labeling of each m-input vertex of Y in layer ℓ by an m-input multi-morphism $f_{k,\ell}$ of $\mathscr{C}$, so that:
  - All inputs of Y are at the same layer,
  - The target of the last morphism (the one closest to the root) of Y is y,
  - The sources of the first layer of morphisms $f_{k_1,1}, \dots, f_{k_j,1}$ are $x_1, \dots, x_n$, and
  - Successive layers of morphisms are composable, i.e., if $f_{k,\ell}$ is the i^{th} input to $f_{k+1,\ell'}$ in Y then $f_{k+1,\ell'} \circ_i f_{k,\ell}$ is defined.

  (Note that 0-input vertices can appear at any layer of Y.)

Given a labeled tree $(Y,\{f_{k,\ell}\})$, let $\circ(Y,\{f_{k,\ell}\})$ be the result of composing the multi-morphisms $f_{k,\ell}$ according to Y.
- Multi-composition of morphism objects is induced by composition of trees.
- Given morphism objects $(Y,\{f_{k,\ell}\})$ and $(Z,\{g_{k',\ell'}\})$, the morphisms in $\widetilde{\mathscr{C}}$ from $(Y,\{f_{k,\ell}\})$ to $(Z,\{g_{k',\ell'}\})$ are the morphisms in $\mathscr{C}$ from $\circ(Y,\{f_{k,\ell}\})$ to $\circ(Z,\{g_{k',\ell'}\})$.
- Multi-composition of morphism morphisms in $\widetilde{\mathscr{C}}$ is induced by multi-composition of morphism morphisms in $\mathscr{C}$.

There is a canonical quotient map $q\colon \widetilde{\mathscr{C}} \to \mathscr{C}$ which is the identity on objects and sends $(Y,\{f_{k,\ell}\})$ to $\circ(Y,\{f_{k,\ell}\})$.

Convention 4.9. For the rest of the paper, the word *tree* means a planar rooted tree.

Remark 4.10. We can visualize $\widetilde{\mathbb{T}}$ as follows. Consider a morphism $(Y,\{T_{k,\ell}\})$ in $\widetilde{\mathbb{T}}$ from $m_1,\dots,m_n$ to m'. (So, Y is a tree and the $T_{k,\ell}$ are diskular tangles. We are suppressing the integer P from this discussion.) There is an associated diskular tangle $T=\circ(Y,\{T_{k,\ell}\})$ in $D^2\setminus(D^2_1\cup\cdots\cup D^2_n)$. There is also a collection of disjoint, embedded, round circles $Z_{k,\ell}$ in $D^2\setminus(D^2_1\cup\cdots\cup D^2_n)$: the images of the outer boundaries of the $T_{k,\ell}$ under the composition maps. Conversely, given T and the round circles $Z_{k,\ell}$, one can reconstruct $(Y,\{T_{k,\ell}\})$ uniquely. These circles must satisfy a condition on their nesting depth. A 2-morphism is a sequence of elementary tangle cobordisms, paying no regard to the extra round circles.

The following lemma will be useful for constructing multifunctors below:

Lemma 4.11. *Given a multicategory $\mathscr{C}$ enriched in categories and a morphism object $(Y,\{f_{k,\ell}\})\in\widetilde{\mathscr{C}}(a_1,\dots,a_n;b)$ let*
$$\widetilde{\mathrm{Id}}\colon (Y,\{f_{k,\ell}\})\to\circ(Y,\{f_{k,\ell}\})$$
be the morphism morphism corresponding to the identity map of $\circ(Y,\{f_{k,\ell}\})$. Given another morphism object $(Y',\{f'_{k',\ell'}\})$, any morphism morphism $\alpha\colon (Y,\{f_{k,\ell}\})\to(Y',\{f'_{k',\ell'}\})$ can be factored uniquely as
$$\alpha=\widetilde{\mathrm{Id}}^{-1}\circ\alpha'\circ\widetilde{\mathrm{Id}}$$
where α' is a morphism from $\circ(Y,\{f_{k,\ell}\})$ to $\circ(Y',\{f'_{k',\ell'}\})$.

Proof. This is immediate from the definitions: α' is just the morphism inducing α in the definition of $\widetilde{\mathscr{C}}$. ☐

Remark 4.12. There is an *oriented tangle multicategory* with an object given by an even integer m and a function $\{1,\dots,m\}\to\{\pm1\}$, or equivalently an orientation on $\{e^{2\pi ij/(m+1)}\mid j=1,\dots,m\}$, a 1-morphism given by a diskular tangle together with an orientation of its components, compatible with the orientations of the points on its boundary, and a 2-morphism an oriented tangle cobordism. There is a forgetful functor from the oriented tangle multicategory to the tangle multicategory, sending an oriented tangle $\vec{T}$ to the pair (T,P) where T is the underlying unoriented tangle and P is the number of positive crossings of $\vec{T}$. The composition of the Khovanov multifunctor defined below with this forgetful functor gives an invariant of oriented tangles. While this is arguably a more natural invariant to study from the point of view of topology (e.g., it is clearer what Khovanov homology is an invariant of in this setting), we find it more convenient (and also slightly more general) to work at the level of the tangle multicategory.

4.2. **Arc algebra multi-modules and gluing.** The goal of this section is to prove that Khovanov's arc algebras and bimodules extend to give a functor from $\mathbb{T}$, as a warm-up for the spectral case. None of the ideas involved are new.

4.2.1. *The target multicategory.* To be parallel with the spectral situation, we give a somewhat elaborate multicategory as the target of Khovanov's arc algebra functor. See Remark 4.19 for a simpler option which is, however, not parallel to the spectral case.

Let $A_1, \dots, A_n$ and B be graded linear categories (or, less generally, rings). A *multi-module* over $A_1, \dots, A_n$ and B is just a *dg* $(A_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A_n, B)$-bimodule. More generally, the *derived category of multi-modules* over $A_1, \dots, A_n$ and B is the derived category of *dg* $(A_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A_n, B)$-bimodules. For multi-modules, however, we can form more tensor products: given algebras C, $B_1, \dots, B_n$, and $A_{i,1}, \dots, A_{i,m_i}$ $(i = 1, \dots, m)$, multi-modules M_i over $A_{i,1}, \dots, A_{i,m_i}$ and B_i, and a multi-module N over $B_1, \dots, B_n$ and C, we can form the tensor product

$$N \otimes_{B_1,\dots,B_n} (M_1, \dots, M_n) = N \otimes_{B_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} B_n} (M_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} M_n).$$

We can also form the derived tensor product of multi-modules by first replacing N and/or the M_i by projective (or flat) multi-modules and then taking the tensor product.

It is convenient to have a model of the derived category of multi-modules so that the derived tensor product is strictly associative, strictly functorial, and has a strict unit. There are standard ways to do this; here is one. First, fix a functor from the category of multimodules over $A_1, \dots, A_n$ and B to the category of projective multimodules, for each collection of linear categories $A_1, \dots, A_n$ and B (for example, the bar resolution if $A_1, \dots, A_n$ and B are finitely generated and free over $\mathbb{Z}$). Then instead of the usual derived category consider the category with objects planar, rooted trees with $n + 1$ leaves, together with a labeling of each edge by an algebra and each internal vertex by a multi-module over the algebras associated to the edges incident to it, so that the edge adjacent to the root is labeled by B and the edges associated to the other leaves are labeled by $A_1, \dots, A_n$ (in that order). The morphism set between two objects is obtained by taking the (chosen) projective resolution of the module associated to each internal vertex, tensoring the results together according to the edges, and then taking homotopy classes of *dg* module homomorphisms. Tensor product of objects is formal: it is just given by composition of trees. This tensor product is automatically associative. The identity elements correspond to the tree with two leaves and no internal vertices. It is straightforward to verify that this extends to a strictly associative tensor product of morphisms as well. For each tuple $A_1, \dots, A_n, B$, taking projective resolutions and then tensoring according to the tree gives a functor from this derived category to the usual derived category of $(A_1 \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} A_n, B)$-bimodules. This map is fully faithful and essentially surjective (i.e., an equivalence) by definition.

Fix this or any other model for the derived category of multimodules, with a strictly associative, unital derived tensor product. Then, the target of the arc algebra multifunctor is the following:

Definition 4.13. Let Bim be the multicategory enriched in categories with

(1) Objects finite, graded linear categories in which each morphism space is free as a $\mathbb{Z}$-module.

(2) Morphisms $\mathrm{Hom}_{\mathsf{Bim}}(A_1,\dots,A_n;B)=\mathsf{Bim}(A_1,\dots,A_n;B)$ the derived category of graded multi-modules over $A_1,\dots,A_n$ and B.
(3) Multi-composition of morphisms given by the derived tensor product of multi-modules.

Lemma 4.14. *The definitions above make* Bim *into a multicategory.*

Proof. This is immediate from our hypotheses on the derived tensor product. ☐

Definition 4.15. The *projectivization* of Bim is the result of quotienting each set of 2-morphisms by the relation $f\sim -f$. (Note that multi-composition respects this equivalence relation. Note also that, after quotienting, the 2-morphism sets are no longer abelian groups.) A *projective functor* to Bim is a functor to the projectivization of Bim.

4.2.2. *The arc algebra multifunctor.* Given an even integer n, consider the n points $e^{2\pi ij/(n+1)}$, $j=1,\dots,n$, in S^1. Identifying $S^1\setminus\{1\}$ with $(0,1)$, we can view an element $a\in\mathsf{B}(n)$ as a crossingless matching of $\{e^{2\pi ij/(n+1)}\}$ and hence as a flat tangle in $A\subset D^2$ with boundary $\{e^{2\pi ij/(n+1)}\}$. For definiteness, choose this embedding of a in A to be disjoint from the line segment $\{re^{0i}\mid r\in[1/2,1]\}$. Abusing notation, we continue to denote this flat tangle by a. Reflecting a in the radial direction of the annulus (i.e., reflecting across the mid-circle) gives a flat tangle $\widehat{a}$, with boundary $\{\frac{1}{2}e^{2\pi ij/(n+1)}\}$ (which corresponds, under the embedding, to the previous definition of $\widehat{a}$). Using the standard homeomorphism

$$A\cup_{S^1\times\{1\}\sim S^1\times\{1/2\}}A\cong A,$$

we can view $\widehat{a}\amalg a$ as lying in A, and the standard saddle cobordism $\widehat{a}\amalg a\to\mathrm{Id}$ as lying in $[0,1]\times A$. In particular, for any other crossingless matching b, there is an induced cobordism $b\widehat{a}a\to b$ inside $A\subset D^2$.

We extend Khovanov's arc algebra modules to associate multi-modules to diskular tangles. Given a diskular $(m_1,\dots,m_k;n)$-tangle and an integer P, as well as crossingless matchings $a_i\in\mathsf{B}(m_i)$ and $b\in\mathsf{B}(n)$, define

$$\mathcal{C}(T,P)(a_1,\dots,a_k;b)=\mathcal{C}(\widehat{b}\circ T\circ(a_1,\dots,a_k),P)\{0,n/2\}.$$

The right-hand side is the Khovanov complex of a link diagram in $\mathbb{R}^2$, with a grading shift, and $\circ$ denotes gluing tangles. This has an action of $\mathcal{C}(n)$ and $\mathcal{C}(m_i)$ by the standard saddle cobordisms.

Next, we note that Khovanov's theorem that gluing tangles corresponds to the tensor product of arc algebra bimodules [Kho02, Proposition 13] extends to this setting. Given a diskular $(m_1,\dots,m_k;n_i)$-tangle S and a diskular $(n_1,\dots,n_\ell;p)$-tangle T, construct a *gluing map*

$$\mathcal{C}(T,P_T)\otimes_{\mathbb{Z}}\mathcal{C}(S,P_S)\to\mathcal{C}(T\circ_i S,P_S+P_T)$$

as follows. Given crossingless matchings $(a_1,\dots,a_k)$, $(b_1,\dots,b_\ell)$, and c, and resolutions S_v of S and T_w of T, the canonical saddle cobordism $\widehat{b_i}\amalg b_i\to\mathrm{Id}$ gives a cobordism

$$[\widehat{c}\circ T_w\circ(b_1,\dots,b_\ell)]\circ_i[\widehat{b_i}\circ S_v\circ(a_1,\dots,a_k)]\to[\widehat{c}\circ(T_w\circ_i S_v)\circ(b_1,\dots,b_{i-1},a_1,\dots,a_k,b_{i+1},\dots,b_\ell)]. \tag{4.1}$$

The flat tangle on the left of Formula (4.1) is the disjoint union of the closed 1-manifolds $[\widehat{c}\circ T_w\circ(b_1,\dots,b_\ell)]$ and $[\widehat{b_i}\circ S_v\circ(a_1,\dots,a_k)]$, so

$$V([\widehat{c}\circ T_w\circ(b_1,\dots,b_\ell)]\circ_i[\widehat{b_i}\circ S_v\circ(a_1,\dots,a_k)])=V([\widehat{c}\circ T_w\circ(b_1,\dots,b_\ell)])\otimes_{\mathbb{Z}}V([\widehat{b_i}\circ S_v\circ(a_1,\dots,a_k)]).$$

Hence, applying V to this cobordism gives a map

$$\begin{aligned}V([\widehat{c}\circ T_w\circ(b_1,\dots,b_\ell)])\otimes_{\mathbb{Z}} V([\widehat{b_i}\circ S_v\circ(a_1,\dots,a_k)])\\ \to V([\widehat{c}\circ(T_w\circ_i S_v)\circ(b_1,\dots,b_{i-1},a_1,\dots,a_k,b_{i+1},\dots,b_\ell)]),\end{aligned}$$

shifting the quantum grading by $|b_i|$, the number of arcs in b_i (half the number of endpionts). Far-commutativity of the saddle maps implies that these gluing maps commute with the edge maps in the cube of resolutions. Hence, they induce maps of iterated mapping cones

$$\begin{aligned}\mathcal{C}([\widehat{c}\circ T\circ(b_1,\dots,b_\ell)],P_T)\otimes_{\mathbb{Z}}\mathcal{C}([\widehat{b_i}\circ S\circ(a_1,\dots,a_k)],P_S)\\ \to\mathcal{C}([\widehat{c}\circ(T\circ_i S)\circ(b_1,\dots,b_{i-1},a_1,\dots,a_k,b_{i+1},\dots,b_\ell)],P_S+P_T).\end{aligned}$$

Lemma 4.16. *This gluing map is a map of* $(\mathcal{C}(n_1),\dots,\mathcal{C}(n_{i-1}),\mathcal{C}(m_1),\dots,\mathcal{C}(m_k),\mathcal{C}(n_{i+1}),\dots,\mathcal{C}(n_\ell);\mathcal{C}(p))$*-multi-modules. Further, it descends to an isomorphism*

$$\mathcal{C}(T,P_T)\otimes_{\mathcal{C}(n_i)}\mathcal{C}(S,P_S)\to\mathcal{C}(T\circ_i S,P_S+P_T)$$

and hence to a homotopy equivalence

$$\mathcal{C}(S,P_S)\circ_i\mathcal{C}(T,P_T)\to\mathcal{C}(T\circ_i S,P_S+P_T),$$

where the left side is composition in the tangle movie multicategory $\mathbb{T}$*.*

Proof. Both the fact that the gluing map respects the multi-module structure and that it descends to the tensor product over $\mathcal{C}(n_i)$ follow from far-commutation of disjoint saddles: the multi-module structure correspond to saddles away from the gluing region, while the gluing map is induced by saddles in the gluing region; and similarly descending to the tensor product corresponds to commuting saddles in different parts of the gluing region. The fact that the map is an isomorphism follows from the fact that it is an isomorphism for the case of flat diskular tangles (tangles with no crossings), which is proved by the argument given by Khovanov [Kho02, Theorem 1]. The last statement follows from the second and the fact that $\mathcal{C}(S,P_S)$ is a complex of projective modules over $\mathcal{C}(n_i)$. ☐

Definition 4.17. Define $\mathcal{C}\colon\widetilde{\mathbb{T}}\to\mathsf{Bim}$ as follows:

(1) On an object $n\in 2\mathbb{Z}$, $\mathcal{C}(n)$ is the Khovanov arc algebra on n points.
(2) Given an elementary morphism object (1-morphism) (T,P) of $\mathbb{T}$, $\mathcal{C}(T,P)$ is the multi-module defined above.
(3) For a general morphism object, which is a formal composition of elementary morphism objects, $\mathcal{C}$ is the corresponding composition of its value on the elementary morphism objects.
(4) Given an elementary cobordism Σ from (T_0,P_0) to (T_1,P_1), the map $\mathcal{C}(\Sigma)\colon\mathcal{C}(T_0,P_0)\to\mathcal{C}(T_1,P_1)\{0,\chi'(\Sigma)\}$ is defined in the expected way. That is:
   (a) If Σ is a planar isotopy Φ_t then $\mathcal{C}(\Sigma)$ is the isomorphism obtained by applying Φ_1 to each resolution.
   (b) If Σ is a Reidemeister move then $\mathcal{C}(\Sigma)$ is the quasi-isomorphism coming from Khovanov's proof of invariance of Khovanov homology for tangles [Kho02, Section 4].

   (c) If Σ is a birth then $\mathcal{C}(\Sigma)$ is the inclusion induced by labeling the new circle by the unit, and if Σ is a death then $\mathcal{C}(\Sigma)$ is the projection induced by applying the counit to the new circle.
   (d) If Σ is a planar saddle then $\mathcal{C}(\Sigma)$ is the result of applying a merge or split map to each resolution.
(5) For the morphism morphism $\widetilde{\mathrm{Id}}$ from Lemma 4.11 from the formal tree composition of elementary morphisms to the honest composition, $\mathcal{C}(\widetilde{\mathrm{Id}})$ is the gluing quasi-isomorphism from Lemma 4.16.
(6) On a general morphism morphism, $\mathcal{C}$ is induced from points (4) and (5) via Lemma 4.11.

The planar composition property of the arc algebra modules is contained in the following:

Proposition 4.18. *Definition 4.17 defines a projective multifunctor.*

Proof. We must verify:

(PMF-1) $\mathcal{C}$ respects multi-composition of morphism objects.
(PMF-2) $\mathcal{C}(\widetilde{\mathrm{Id}})$ is invertible. (This is needed since invertibility of $\mathcal{C}(\widetilde{\mathrm{Id}})$ is used to define $\mathcal{C}$ of arbitrary morphism morphisms.)
(PMF-3) $\mathcal{C}$ respects the equivalence relation we imposed on morphism morphisms.
(PMF-4) $\mathcal{C}$ respects multi-composition of morphism morphisms.
(PMF-5) $\mathcal{C}$ respects the far-commutation relation that we imposed on elementary cobordisms.
(PMF-6) $\mathcal{C}$ respects 2-composition of morphism morphisms.

Point (PMF-1) is immediate from the definitions.

Point (PMF-2) follows from Lemma 4.16.

For Point (PMF-3), invariance of $\mathcal{C}$ under type (D2) and (D3) moves is obvious. Invariance under type (D4) moves follows from the definitions of the Reidemeister and birth, death, and saddle maps: none of these maps depend on the location of the tangle in the plane.

For Point (PMF-4), we need to check two basic cases: that the gluing map $\mathcal{C}(\widetilde{\mathrm{Id}})$ is associative, in the sense that given three tangles R, S, T and integers P_R, P_S, P_T, the following diagram commutes

$$\begin{tikzcd}
\begin{array}{l}\mathcal{C}(T,P_T)\circ_i\big(\mathcal{C}(S,P_S)\circ_{j'}\mathcal{C}(R,P_R)\big)\\ \quad=\big(\mathcal{C}(T,P_T)\circ_i\mathcal{C}(S,P_S)\big)\circ_j\mathcal{C}(R,P_R)\end{array} \arrow[r, "\mathrm{Id}_{\mathcal{C}(T,P_T)}\otimes\mathcal{C}(\widetilde{\mathrm{Id}})"] \arrow[d, "\mathcal{C}(\widetilde{\mathrm{Id}})\otimes\mathrm{Id}_{\mathcal{C}(R,P_R)}"'] & \mathcal{C}(T,P_T)\circ_i\mathcal{C}(S\circ_{j'}R,P_R+P_S) \arrow[d, "\mathcal{C}(\widetilde{\mathrm{Id}})"] \\
\mathcal{C}(T\circ_i S,P_S+P_T)\circ_j\mathcal{C}(R,P_R) \arrow[r, "\mathcal{C}(\widetilde{\mathrm{Id}})"'] & \begin{array}{l}\mathcal{C}\big(T\circ_i(S\circ_{j'}R),P_R+P_S+P_T\big)\\ \quad=\mathcal{C}\big((T\circ_i S)\circ_j R,P_R+P_S+P_T\big)\end{array}
\end{tikzcd} \tag{4.2}$$

and that the gluing map commutes with the maps associated to elementary cobordisms, in the sense that given tangles R, S, T and an elementary cobordism Σ from R to S, the following diagram and its analogue where T is pre-composed instead of post-composed commute

$$\begin{tikzcd}
\mathcal{C}(T,P_T)\circ_i\mathcal{C}(R,P_R) \arrow[r, "\mathcal{C}(\widetilde{\mathrm{Id}})"] \arrow[d, "\mathrm{Id}_{\mathcal{C}(T,P_T)}\otimes\mathcal{C}(\Sigma)"'] & \mathcal{C}(T\circ_i R, P_T+P_R) \arrow[d, "\mathcal{C}(\mathrm{Id}\circ_i\Sigma)"] \\
\mathcal{C}(T,P_T)\circ_i\mathcal{C}(S,P_S)\{0,\chi'(\Sigma)\} \arrow[r, "\mathcal{C}(\widetilde{\mathrm{Id}})"'] & \mathcal{C}(T\circ_i S)\{0,\chi'(\Sigma)\}.
\end{tikzcd} \tag{4.3}$$

Commutativity of Diagram (4.2) follows from far-commutativity of the saddle maps. Commutativity of Diagram (4.3) is immediate from the local nature of the definition of $\mathcal{C}(\Sigma)$.

Point (PMF-5) is immediate from Point (PMF-4).

For Point (PMF-6), it suffices to prove the result for morphisms between trees with a single internal vertex, i.e., cobordisms between tangles in $\mathbb{T}$. This is then immediate from the construction of $\mathcal{C}$, as the composition of its value on the elementary cobordisms. ☐

Remark 4.19. Since the arc algebra multi-modules are projective over $\mathcal{C}(n)$ and the maps associated to births, deaths, and Reidemeister moves are homotopy equivalences rather than just quasi-isomorphisms, we do not need to include taking resolutions in the composition maps for the target of $\mathcal{C}$. That is, we could define the multi-composition to be the ordinary tensor product of multi-modules, and 2-morphisms to be homotopy classes of chain maps of multi-modules. In the spectral case, we do not have an analogue of this stricter approach.

4.3. **Spectral refinements.** The target category for the spectral Khovanov multifunctor is the spectral analogue of Bim. First, given spectral algebras or categories $\mathscr{A}_1,\dots,\mathscr{A}_n$ and $\mathscr{B}$ there is a notion of a *spectral multi-module* over $\mathscr{A}_1,\dots,\mathscr{A}_n$ and $\mathscr{B}$: a functor $(\mathscr{A}_1\times\cdots\times\mathscr{A}_n)^{\mathrm{op}}\times\mathscr{B}\to\mathscr{S}$ or, equivalently, a spectrum with commuting actions of $\mathscr{A}_1,\dots,\mathscr{A}_n,\mathscr{B}$. (This is a simple extension of the notion of a bimodule from, e.g., [BM12, Section 2].) For each $\mathscr{A}_1,\dots,\mathscr{A}_n$ and $\mathscr{B}$, choose a cofibrant replacement functor (the analogue of a functorial projective resolution) for the category of spectral multi-modules. Then, define a derived category spectral multi-modules with a strictly associative tensor (or smash) product as in Section 4.2.1. Then the target multi-category is the following adaptation of Definition 4.13:

Definition 4.20. Let SBim be the multicategory enriched in spectral categories with

(1) Objects finite, graded spectral categories.
(2) Multi-morphisms $\mathsf{SBim}(\mathscr{A}_1,\dots,\mathscr{A}_n;\mathscr{B})$ given by the derived category of multi-modules over $\mathscr{A}_1,\cdots,\mathscr{A}_n$ and $\mathscr{B}$.
(3) Multi-composition given by the derived smash product.

Lemma 4.21. *The definitions above make* SBim *into a multicategory.*

Proof. The proof is left to the reader. ☐

We start constructing the spectral Khovanov multifunctor $\mathscr{X}$ by defining it on objects of $\mathbb{T}$, i.e., on pairs (T,P) of a diskular $(m_1,\dots,m_k;n)$-tangle T with N crossings and an integer P. The construction is essentially the same as for (m,n)-tangles in our previous paper [LLSb] (see also Section 2.3). There is a *tangle shape multicategory* $\mathcal{T}_{m_1,\dots,m_k;n}$ with an object $(a_1,\dots,a_k;a'_1,\dots,a'_k)$

for each pair of k-tuples of crossingless matchings $a_i, a_i' \in \mathsf{B}(m_i)$, an object (b, b') for each pair of crossingless matchings $b, b' \in \mathsf{B}(n)$, and an object $(a_1, \dots, a_k, T, b)$ for a tuple of crossingless matchings $a_i \in \mathsf{B}(m_i)$ and $b \in \mathsf{B}(n)$. Let $\vec{a}$ denote a k-tuple of crossingless matchings $a_i \in \mathsf{B}(m_i)$. The multicategory $\mathcal{T}_{m_1,\dots,m_i;n}$ has a unique morphism of each of the following forms:

$$\begin{aligned}
(\vec{a}^1, \vec{a}^2), (\vec{a}^2, \vec{a}^3), \dots, (\vec{a}^{\alpha-1}, \vec{a}^\alpha) &\to (\vec{a}^1, \vec{a}^\alpha) \\
(b_1, b_2), (b_2, b_3), \dots, (b_{\beta-1}, b_\ell) &\to (b_1, b_\beta) \\
(\vec{a}^1, \vec{a}^2), \dots, (\vec{a}^{\alpha-1}, \vec{a}^\alpha), (\vec{a}^\alpha, T, b_1), (b_1, b_2), \dots, (b_{\beta-1}, b_\beta) &\to (\vec{a}^1, T, b_\beta).
\end{aligned}$$

There is an associated multicategory $\underline{2}^{\mathfrak{C}} \tilde{\times} \widetilde{\mathcal{T}}_{m_1,\dots,m_k;n}$ enriched in groupoids [LLSb, Section 3.2.4].

Recall that we introduced a category of divided cobordisms, in Definition 2.4. To construct the tangle invariants, we will take the quotient of this category by certain diffeomorphisms:

Definition 4.22. The *divided cobordism category of the annulus*, $\mathsf{Cob}_d(A)$, is the result of quotienting the divided cobordism category from Definition 2.4 by radial rescaling. That is, identifying A with $[1/2, 1] \times S^1$, we declare two objects of $\mathsf{Cob}_d(A)$ to be equal if they differ by an orientation-preserving diffeomorphism of $[1/2, 1]$ which is the identity near $\{1/2, 1\}$, and declare two morphisms to be equal if they differ by a diffeomorphism of $[0,1] \times [1/2, 1]$ which is invariant in the $[0,1]$-direction near $\{0,1\} \times [1/2, 1]$ and is the identity near $[0,1] \times \{1/2, 1\}$. Composition descends to this quotient in an obvious way.

Given a finite collection of disjoint disks $\{D_i\} \subset D^2$, define $\mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i)$ as follows. Glue the annulus A to each boundary component ∂D_i by using the maps ϕ_{D_i}, and glue the annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ to ∂D^2. The result is a region $V \subset \mathbb{C}$ containing $(D^2 \setminus \bigcup_i D_i)$ in its interior. Then $\mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i)$ is $\mathsf{Cob}_d(V)$ modulo radial rescaling of each of the annuli we glued in.

There are associative multi-composition maps

$$\begin{aligned}
\mathsf{Cob}_d(A) \times \cdots \times \mathsf{Cob}_d(A) \times \mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i) &\to \mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i) \\
(Y_1, \dots, Y_k, Z) &\mapsto Z \circ (Y_1, \dots, Y_k)
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i) \times \mathsf{Cob}_d(A) &\to \mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i) \\
(Z, Y) &\mapsto Y \circ Z.
\end{aligned} \tag{4.5}$$

We can arrange the data of $\mathsf{Cob}_d(A)$ and $\mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i)$ into a multicategory with objects

$$\mathrm{Ob}\big(\mathsf{Cob}_d(A) \times \cdots \times \mathsf{Cob}_d(A)\big) \amalg \mathrm{Ob}\big(\mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i)\big) \amalg \mathrm{Ob}\big(\mathsf{Cob}_d(A)\big)$$

and three types of multi-morphisms, analogous to the three cases in $\mathcal{T}_{m_1,\dots,m_i;n}$, but using the composition maps from Formulas (4.4) and (4.5). We will abuse notation and denote this multicategory $\mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i)$.

The crossing change cobordisms and canonical saddle cobordisms induce a multifunctor

$$\underline{2}^{\mathfrak{C}}\tilde{\times}\widetilde{\mathcal{T}}_{m_1,\dots,m_k;n} \to \mathsf{Cob}_d(D^2 \setminus \bigcup_i D_i).$$

(As mentioned earlier, to define this multifunctor one needs to choose *pox* on the tangle, as in [LLSb, Definition 3.10]. The functor is, however, independent of the choice of pox.) Composing with the Khovanov-Burnside functor gives a multifunctor $\underline{\mathsf{MB}}_T\colon \underline{2}^{\mathfrak{C}}\tilde{\times}\widetilde{\mathcal{T}}_{m_1,\dots,m_k;n} \to \mathscr{B}$. Applying Elmendorf-Mandell's K-theory [EM06] and then rectifying gives a multifunctor

$$\underline{2}^{\mathfrak{C}} \times \mathcal{T}_{m_1,\dots,m_k;n} \to \mathscr{S}.$$

Desuspending P times and taking iterated mapping cones gives a functor $\mathcal{T}_{m_1,\dots,m_k;n} \to \mathscr{S}$. This functor can be reinterpreted (analogously to [LLSb]) as a spectral multi-module $\mathscr{X}(T,P)$. The constructions decompose along quantum gradings, so

$$\mathscr{X}(n) = \bigvee_j \mathscr{X}^j(n)$$
$$\mathscr{X}(T,P) = \bigvee_j \mathscr{X}^j(T,P).$$

Here, we use the same quantum grading shifts as in the combinatorial case (Sections 2.4 and 4.2.2), so that

$$C_i(\mathscr{X}^j(T,P)) \simeq \mathcal{C}_{i,j}(T,P)$$

The quantum grading shift is formal, just changing the indexing in the decomposition along quantum gradings.

The next step in constructing the multifunctor $\mathscr{X}$ is to define the maps associated to elementary cobordisms.

Consider the model diskular tangles T_0 and T_1 for a Reidemeister move. For a Reidemeister 1 move, these are 2-tangles, for a Reidemeister 2 move, these are 4-tangles, and for a Reidemeister 3 move these are 6-tangles. Let $p = 1$ for a Reidemeister 1 move introducing a positive crossing or a Reidemeister 2 move, and 0 otherwise. In our previous paper [LLSb, Proof of Theorem 4], we associated a zig-zag of weak equivalences between $\mathscr{X}(T_0,P)$ and $\mathscr{X}(T_1,P+p)$. From the definition of the derived category this zig-zag gives an equivalence $\mathscr{X}(T_0,P) \to \mathscr{X}(T_1,P+p)$. (It follows from Lemma 6.3 below that this equivalence is, in fact, unique up to sign.)

Given any two diskular tangles (T,P) and (T',P') related by a Reidemeister move Σ, tensoring the equivalence from the previous paragraph with the identity map of the multi-module associated to the rest of the diskular tangle gives a map $\mathscr{X}(\Sigma)\colon \mathscr{X}(T,P) \to \mathscr{X}(T',P')$.

Similarly, if U is an unknot diagram (with no crossings) and $\Sigma\colon \varnothing \to U$ is the birth cobordism then we associate to Σ the inclusion

$$\mathscr{X}(\Sigma)\colon \mathscr{X}(\varnothing,0) = \mathbb{S} \hookrightarrow \mathbb{S}\{-1\} \vee \mathbb{S}\{1\} = \mathscr{X}(U,0),$$

of the summand $\mathbb{S}\{-1\}$, where the number inside braces indicates the quantum grading. This decreases the quantum grading down by 1. If $\Sigma'\colon U \to \varnothing$ is the death cobordism then we associate to Σ' the projection

$$\mathscr{X}(\Sigma')\colon \mathscr{X}(U,0) = \mathbb{S}\{-1\} \vee \mathbb{S}\{1\} \twoheadrightarrow \mathbb{S} = \mathscr{X}(\varnothing,0),$$

to the summand $\mathbb{S}\{1\}$. Again, this decreases the quantum grading by 1. (These definitions are exactly analogous to Khovanov homology, and are special cases of the maps associated to elementary cobordisms of links in our previous paper [LS14b].) As in the case of Reidemeister moves, these extend to births or deaths of unknots in arbitrary diskular tangles, by taking the tensor product with the identity map on the rest of the tangle.

Let T be a diskular 4-tangle with no closed components and a single crossing. There is an associated multifunctor $\underline{\mathsf{MB}}_T\colon \underline{2}^1\tilde{\times}\widetilde{\mathcal{T}}_{;4}\to\mathscr{B}$ [LLSb, Section 3.5]. If T_0 and T_1 denote the 0- and 1-resolutions of T, respectively, then $\underline{\mathsf{MB}}_{T_1}$ is (isomorphic to) an insular subfunctor of $\underline{\mathsf{MB}}_T$ with corresponding quotient functor (isomorphic to) $\underline{\mathsf{MB}}_{T_0}$ [LLSb, Definition 3.29]. Applying K-theory and rectifying, this gives a cofibration sequence

$$\mathscr{X}^j(T_1,P)\to\mathscr{X}^{j+1}(T,P)\to\Sigma\mathscr{X}^{j-1}(T_0,P).$$

(This also uses the fact that naturally isomorphic functors give equivalent modules; see [LLSb, Proof of Proposition 4.7].) The Puppe construction gives a map $\Sigma\mathscr{X}^{j-1}(T_0,P)\to\Sigma\mathscr{X}^j(T_1,P)$. This is the cobordism map associated to a basic saddle. For a saddle in a general link diagram, the associated map is the tensor product of this map with the identity map of the rest of the diagram.

The last ingredients in constructing the spectral Khovanov multifunctor are the gluing equivalences. These are defined using the analogue of the gluing multicategory [LLSb, Section 5]. Fix even integers $\vec{m}=(m_1,\dots,m_k)$, $\vec{n}=(n_1,\dots,n_\ell)$, and p, and an integer j with $1\le j\le\ell$. The *tangle gluing multicategory* $\mathcal{U}_{\vec{m};\vec{n};p}$ has five kinds of objects:

- Pairs $(\vec{a}^1,\vec{a}^2)$ where $a_i^1,a_i^2\in\mathsf{B}(m_i)$.
- Pairs $(\vec{b}^1,\vec{b}^2)$ where $b_i^1,b_i^2\in\mathsf{B}(n_i)$.
- Pairs (c^1,c^2) where $c^1,c^2\in\mathsf{B}(p)$.
- Triples $(\vec{a},S,b)$ where $a_i\in\mathsf{B}(m_i)$, $b\in\mathsf{B}(n_j)$, and S is a placeholder.
- Triples $(\vec{b},T,c)$ where $b_i\in\mathsf{B}(n_i)$, $c\in\mathsf{B}(p)$, and T is a placeholder.
- Quadruples $(\vec{a},S,\vec{b},T,c)$ where $a_i\in\mathsf{B}(m_i)$, $b_i\in\mathsf{B}(n_i)$, $c\in\mathsf{B}(p)$, and S and T are placeholders.

The tangle shape multicategories $\mathcal{T}_{\vec{m};n_j}$ and $\mathcal{T}_{\vec{n};p}$ are full subcategories of $\mathcal{U}_{\vec{m};\vec{n};p}$. There is also a unique multi-morphism

$$\big((\vec{a}^1,\vec{a}^2),\cdots,(\vec{a}^{\alpha-1},\vec{a}^{\alpha}),(\vec{b}^1,\vec{b}^2),\cdots,(\vec{b}^{\beta-1},\vec{b}^{\beta}),(\vec{a}^{\alpha},S,\vec{b}^{\beta},T,c^1),(c^1,c^2),\cdots,(c^{\gamma-1},c^{\gamma})\big)$$
$$\to(\vec{a}^1,S,\vec{b}^1,T,c^{\gamma}).$$

Given a finite set $\mathfrak{C}$, there is a groupoid-enriched product $\underline{2}^{\mathfrak{C}}\tilde{\times}\widetilde{\mathcal{U}}_{\vec{m};\vec{n};p}$ (a trivial adaptation of the construction in [LLSb, Section 5]). Given 1-morphisms (S,P_S) and (T,P_T) of $\mathbb{T}$, where S is a diskular $(m_1,\dots,m_k;n_j)$-tangle and T is a diskular $(n_1,\dots,n_\ell;p)$-tangle, if we let $\mathfrak{C}=\mathfrak{C}(S)\cup\mathfrak{C}(T)$ denote the set of crossings of $S\cup T$, then there is a functor

$$\underline{2}^{\mathfrak{C}}\tilde{\times}\widetilde{\mathcal{U}}_{\vec{m};\vec{n};p}\to\mathsf{Cob}_d$$

induced by the canonical saddle cobordisms $\widehat{a_i^{\alpha}}\amalg a_i^{\alpha}\to\mathrm{Id}$, $\widehat{b_i^{\beta}}\amalg b_i^{\beta}\to\mathrm{Id}$, $\widehat{c_i}\amalg c_i\to\mathrm{Id}$, and the saddles between different resolutions of S and T. Here, Cob_d is a mild generalization of the multicategory $\mathsf{Cob}_d(D^2\setminus\bigcup_i D_i)$ from Definition 4.1, allowing diskular tangles of the form of capped-off resolutions of S, capped-off resolutions of T, and capped-off resolutions of $T \circ_i S$. In the last case, in addition to quotienting by radial diffeomorphisms near the boundary circles, we also quotient by radial reparametrization near the circle where the gluing $\circ_i$ occurred.

Composing with the Khovanov-Burnside functor and Elmendorff-Mandell's K-theory gives a multifunctor $\underline{2}^{\mathfrak{C}} \tilde{\times} \widetilde{\mathcal{U}}_{\vec{m};\vec{n};p} \to \mathscr{S}$, which rectifies to a functor $\underline{2}^{\mathfrak{C}} \times \mathcal{U}_{\vec{m};\vec{n};p} \to \mathscr{S}$. Such a functor induces a map of multi-modules

$$\mathscr{X}(S, P_S) \otimes_{\mathscr{X}(n_i)} \mathscr{X}(T, P_T) \to \mathscr{X}(T \circ_i S, P_S + P_T) \tag{4.6}$$

(cf. [LLSb, Lemma 5.4]).

Lemma 4.23. *The gluing map of spectral multi-modules from Formula* (4.6) *is a weak equivalence.*

Proof. The induced map on homology agrees with the Khovanov gluing map from Section 4.2.2 (see [LLSb, Lemma 5.6]), so the result follows from Lemma 4.16 and Whitehead's theorem. ☐

The following is a straightforward adaptation of Definition 4.17 to the spectral setting:

Definition 4.24. Define $\mathscr{X} : \widetilde{\mathbb{T}} \to \mathsf{SBim}$ as follows:

(1) On an object $n \in 2\mathbb{Z}$, $\mathscr{X}(n)$ is the spectral Khovanov arc algebra on n points.
(2) Given an elementary morphism object (1-morphism) (T, P) of $\mathbb{T}$, where T is a diskular $(m_1, \dots, m_k; n)$-tangle with N crossings, $\mathscr{X}(T, P)$ is the spectral Khovanov multi-module defined above.
(3) For a general morphism object, which is a formal composition of elementary morphism objects, $\mathscr{X}$ is the corresponding composition of its value on the elementary morphism objects.
(4) Given an elementary cobordism Σ from (T_0, P_0) to (T_1, P_1), the map $\mathscr{X}(\Sigma) \colon \mathscr{X}(T_0, P_0) \to \mathscr{X}(T_1, P_1)\{0, \chi'(\Sigma)\}$ is defined in the expected way. That is:
   (a) If Σ is a planar isotopy then $\mathscr{X}(\Sigma)$ is the isomorphism obtained by applying Φ_1 to each resolution.
   (b) If Σ is a Reidemeister move then $\mathscr{X}(\Sigma)$ is the map associated above to the Reidemeister move.
   (c) If Σ is a Morse move (birth, death, or planar saddle) then $\mathscr{X}(\Sigma)$ is the map of spectral multi-modules defined above.
(5) For the morphism morphism $\widetilde{\mathrm{Id}}$ from Lemma 4.11 from the formal tree composition of elementary morphisms to the honest composition, $\mathscr{X}(\widetilde{\mathrm{Id}})$ is the gluing quasi-isomorphism from Lemma 4.23.
(6) On a general morphism morphism, $\mathscr{X}$ is induced from points (4) and (5) via Lemma 4.11.

Proposition 4.25. *Definition 4.24 defines a projective multifunctor.*

Proof. We must check the same points (PMF-1)–(PMF-6) as in the proof of Proposition 4.18. As there, Point (PMF-1) is immediate from the local definition of the multifunctor $\mathcal{C}$ on morphism objects.

Point (PMF-2) is Lemma 4.23.

Point (PMF-3) follows by the same reasoning as in the combinatorial case: the maps associated to Reidemeister moves and Morse moves is independent of the location in the plane.

For Point (PMF-4), we must check that the analogues of Diagrams (4.2) and (4.3) commute. Consider first Diagram (4.2). To keep notation simple, assume R is an $(\ell; m)$-tangle, S is an $(m; n)$-tangle, and T is an $(n; p)$-tangle; only the notation is more complicated in the general case. Construct an analogue of the gluing multicategory but for the three tangles R, S, T. There are three maps from this triple-gluing multicategory to a corresponding divided cobordism category:

- An analogue of the gluing multifunctor, merging $aR_u\widehat{b}$, $bS_v\widehat{c}$, and $cT_w\widehat{d}$ all at once.
- The composition of the gluing multifunctor merging $aR_u\widehat{b}$ and $bS_v\widehat{c}$ with the gluing multifunctor merging $aR_uS_v\widehat{c}$ and $cT_w\widehat{d}$.
- The composition of the gluing multifunctor merging $bS_v\widehat{c}$ and $cT_w\widehat{d}$ with the gluing multifunctor merging $aR_u\widehat{b}$ and $bS_vT_w\widehat{d}$.

By far-commutation of saddles in the divided cobordism category, all three of these multifunctors are naturally isomorphic. Hence, composing with the Khovanov-Burnside functor and K-theory gives three naturally isomorphic multifunctors from the triple-gluing multicategory to the homotopy category of spectra. Each of these can be reinterpreted as a map

$$\mathscr{X}(R, P_R) \otimes^L \mathscr{X}(S, P_S) \otimes^L \mathscr{X}(T, P_T) \to \mathscr{X}(T \circ S \circ R, P_R + P_S + P_T). \tag{4.7}$$

The fact that these maps are equal is the desired associativity property.

As in the combinatorial case, commutativity of the analogue of Diagram (4.3) is immediate from the local definition of $\mathscr{X}(\Sigma)$.

Again as in the combinatorial case, Point (PMF-5) is immediate from Point (PMF-4) and Point (PMF-6) is immediate from the definitions. □

5. Duality properties of Khovanov's tangle invariants and their spectral refinements

Wherein we show that the arc algebra bimodule associated to the MIRROR of a tangle T is homotopy equivalent to the ONE-SIDED DUAL of the bimodule for T, a result that is WELL-KNOWN to experts, and deduce the ANALOGOUS RESULT for the SPECTRAL REFINEMENTS.

We only need these duality results for n-tangles, but prove them in general.

5.1. Dualizability for the modules and spectra.

To verify the duality theorem for the spectral bimodules, we need a technical condition on the spectral arc algebras and modules, called dualizability. Essentially, this is a finiteness condition, like the fact that the isomorphism between a vector space and its double dual holds only for finite-dimensional vector spaces. Dualizability has a number of implications, including relating the chains on the dual with the cochains on the original spectral module.

Definition 5.1. Let A be a *dg* algebra or spectral algebra. A (*dg* or spectral) A-module X is *dualizable* if, for all A-modules Z, the natural map

$$\mathrm{Hom}_A(X, A) \otimes_A Z \to \mathrm{Hom}_A(X, Z) \tag{5.1}$$

is an isomorphism. Here Hom_A takes place in the homotopy category of left A-modules.

Given another (*dg* or spectral) algebra B, an (A, B)-bimodule X is *left-dualizable* if X is dualizable as an A-module, and *right-dualizable* if X is dualizable as a B-module.

The following properties are straightforward to verify:

Proposition 5.2. *For any (dg or spectral) algebra A, the collection of dualizable A-modules is closed under the following.*

(1) Equivalence*: if X is dualizable and $Y \simeq X$, then Y is dualizable.*
(2) Retracts*: if Y is dualizable and X is a retract of Y, then X is dualizable.*
(3) Sums*: If X and Y are dualizable, then so is the sum $X \oplus Y$.*
(4) Shifts*: If X is dualizable, then so are the shifts $\Sigma^n X$ for $n \in \mathbb{Z}$.*
(5) Cofibers*: if $f\colon X \to Y$ is a map of dualizable A-modules, then the mapping cone Cf is dualizable.*
(6) Unit*: A is dualizable.*

Further, the category of dualizable A-modules is the smallest category of A-modules with this property.

In other words, the category of dualizable A-modules is the smallest *thick subcategory* of the homotopy category of A-modules containing A.

For spectra, the homology Whitehead theorem implies that the following well-known criterion for dualizability as modules over the sphere spectrum $\mathbb{S}$.

Proposition 5.3. *A spectrum X is dualizable over $\mathbb{S}$ if and only if X is k-connective for some k and its homology*

$$H_*(X) = \bigoplus_n H_n(X; \mathbb{Z})$$

is a finitely generated abelian group.

Definition 5.4. An R-algebra A is *proper* if it is dualizable as an R-module.

Proposition 5.5. *Suppose that A is a proper R-algebra. Then every dualizable A-module is also a dualizable R-module.*

Proof. This follows from the fact that, given a dualizable A-module X and an R-module Z, the natural map from Equation (5.1) factors as

$$\begin{aligned}
\mathrm{Hom}_R(X, R) \otimes_R Z &\cong \mathrm{Hom}_R(X \otimes_A A, R) \otimes_R Z \cong \mathrm{Hom}_A(X, \mathrm{Hom}_R(A, R)) \otimes_R Z \\
&\cong \mathrm{Hom}_A(X, A) \otimes_A \mathrm{Hom}_R(A, R) \otimes_R Z \cong \mathrm{Hom}_A(X, A) \otimes_A \mathrm{Hom}_R(A, Z) \\
&\cong \mathrm{Hom}_A(X, \mathrm{Hom}_R(A, Z)) \cong \mathrm{Hom}_R(X \otimes_A A, Z) \\
&\cong \mathrm{Hom}_R(X, Z)
\end{aligned}$$

where the second line uses dualizability of X over A and then of A over R. ☐

Proposition 5.6. *If A a dualizable spectral algebra and X is a dualizable A-module then the natural map*

$$C_*(\mathrm{Hom}_A(X, A)) \to \mathrm{Hom}_{C_*(A)}(C_*(X), C_*(A))$$

from singular chains on the morphism spectrum to the morphism complex of singular chain complexes induces an isomorphism on homology. The same applies to one-sided Homs of left-dualizable spectral bimodules.

Proof. This follows from Proposition 5.2 and induction. Specifically, the result clearly holds for $X = A$, and the category of A-modules for which the result holds is closed under equivalences, retracts, sums, shifts, and cofibers, hence contains all dualizable A-modules. ☐

Proposition 5.7. *The arc algebra module $\mathcal{C}(T, P_T)$ associated to an (m, n)-tangle T is left-dualizable and right-dualizable.*

Proof. Elementary projective modules over $\mathcal{C}(m)$ are retracts of $\mathcal{C}(m)$. The homological grading gives a filtration of $\mathcal{C}(T, P_T)$ so that each sub-quotient is homotopy equivalent to a finite direct sum of shifts of elementary projective modules. By Proposition 5.2, the category of dualizable modules is closed under shifts, sums, and retracts, and contains the algebra, so each sub-quotient is dualizable. The fact that dualizability is preserved by mapping cones and induction then gives the result. ☐

Similarly:

Proposition 5.8. *The spectral arc algebras $\mathscr{X}(n)$ are dualizable and the spectral bimodules $\mathscr{X}(T, P)$ are left- and right-dualizable.*

Proof. The first statement follows from Proposition 5.3. The proof of the second statement is the same as the proof of Proposition 5.7: the cube induces a filtration of $\mathscr{X}(T, P)$ so that each sub-quotient is equivalent to a wedge sum of shifts of retracts of $\mathscr{X}(m)$. ☐

5.2. **Arc algebra bimodules for mirrors.** In this section, we write down the proof of a well-known duality property for Khovanov's tangle invariants, which follows from functoriality of Khovanov homology and a familiar TQFT-style argument. We use this formulation to prove functoriality of the spectral refinements. Since we are also proving functoriality of Khovanov homology itself, we also give a direct proof of the case of this duality result needed there.

The duality results in this section perhaps first appeared in the work of Clark-Morrison-Walker [CMW09, Theorem 1.3].

Given an (m, n)-tangle T in $[0, 1] \times (0, 1) \times (0, 1)$, $[0, 1] \times T$ is a tangle cobordism in $[0, 1] \times [0, 1] \times (0, 1) \times (0, 1)$. Identifying $\{0\} \times [0, 1] \cup [0, 1] \times \{1\} \cup \{1\} \times [0, 1]$ with $[0, 1]$, this cobordism can be viewed as a tangle cobordism $\Sigma_{T\widehat{T}}$ from the (m, m)-tangle $T\widehat{T}$ to the identity braid on m points. Similarly, this cobordism can be viewed as a tangle cobordism $\Sigma_{\widehat{T}T}$ from the identity braid on n points to the (n, n)-tangle $\widehat{T}T$. See Figure 5.1. (There are also similar cobordism $\widehat{T}T \to \mathrm{Id}_n$ and $\mathrm{Id}_m \to T\widehat{T}$, but we will not name or need these.) Let N be the number of crossings of T. For any integer P there are corresponding maps

$$\mathcal{C}(\Sigma_{T\widehat{T}})\colon \mathcal{C}(T, P) \otimes_{\mathcal{C}(n)} \mathcal{C}(\widehat{T}, N - P)\{0, \tfrac{m-n}{2}\} = \mathcal{C}(T\widehat{T}, N)\{0, \tfrac{m-n}{2}\} \to \mathcal{C}(\mathrm{Id}_m) = \mathcal{C}(m)$$
$$\mathcal{C}(\Sigma_{\widehat{T}T})\colon \mathcal{C}(n) = \mathcal{C}(\mathrm{Id}_n) \to \mathcal{C}(\widehat{T}T, N)\{0, \tfrac{m-n}{2}\} = \mathcal{C}(\widehat{T}, N - P) \otimes_{\mathcal{C}(m)} \mathcal{C}(T, P)\{0, \tfrac{m-n}{2}\}.$$

The cobordisms $\Sigma_{T\widehat{T}}$ and $\Sigma_{\widehat{T}T}$ satisfy that

$$(\Sigma_{T\widehat{T}} \cup \mathrm{Id}_T) \circ (\mathrm{Id}_T \cup \Sigma_{\widehat{T}T})$$

is isotopic to the obvious ambient isotopy from $T \cup \mathrm{Id}$ to $\mathrm{Id} \cup T$ and

$$(\mathrm{Id}_{\widehat{T}} \cup \Sigma_{T\widehat{T}}) \circ (\Sigma_{\widehat{T}T} \cup \mathrm{Id}_{\widehat{T}})$$

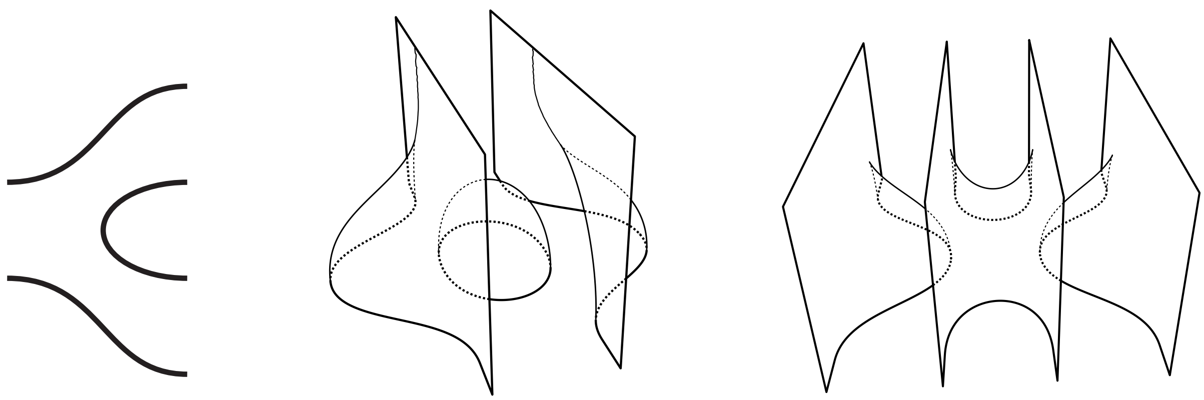


FIGURE 5.1. **The cobordisms $\Sigma_{T\widehat{T}}$ and $\Sigma_{\widehat{T}T}$.** Left: the tangle T. Center and right: the cobordisms $\Sigma_{T\widehat{T}}$ and $\Sigma_{\widehat{T}T}$. The space shown is the projection $[0,1]\times[0,1]\times(0,1)$ of the space $[0,1]\times[0,1]\times(0,1)^2$ where the tangle cobordisms lie. The cobordism direction is the first coordinate, read upwards. Anaglyph versions of this figure and Figure 5.2 can be found at `http://pages.uoregon.edu/lipshitz/CSI.html`

is isotopic to the obvious ambient isotopy from $\mathrm{Id}\cup\widehat{T}$ to $\widehat{T}\cup\mathrm{Id}$. See Figure 5.2. Hence, if we identify $\mathcal{C}(\mathrm{Id}\cup T,P)=\mathcal{C}(T,P)=\mathcal{C}(T\cup\mathrm{Id},P)$ and $\mathcal{C}(\widehat{T}\cup\mathrm{Id},N-P)=\mathcal{C}(\widehat{T},N-P)=\mathcal{C}(\mathrm{Id}\cup\widehat{T},N-P)$ via the ambient isotopy then functoriality of Khovanov homology implies that

$$(\mathcal{C}(\Sigma_{T\widehat{T}})\otimes\mathrm{Id}_{\mathcal{C}(T)})\circ(\mathrm{Id}_{\mathcal{C}(T)}\otimes\mathcal{C}(\Sigma_{\widehat{T}T}))\sim\mathrm{Id}\colon\mathcal{C}(T,P)\to\mathcal{C}(T,P)\tag{5.2}$$

$$(\mathrm{Id}_{\mathcal{C}(\widehat{T})}\otimes\mathcal{C}(\Sigma_{T\widehat{T}}))\circ(\mathcal{C}(\Sigma_{\widehat{T}T})\otimes\mathrm{Id}_{\mathcal{C}(\widehat{T})})\sim\mathrm{Id}\colon\mathcal{C}(\widehat{T},N-P)\to\mathcal{C}(\widehat{T},N-P).\tag{5.3}$$

Proposition 5.9. *Let T be an (m,n)-tangle with N crossings and $\widehat{T}$ its mirror. For any integer P the map*

$$\begin{aligned}D\colon\mathcal{C}(\widehat{T},N-P)_{h,q+\frac{n-m}{2}}&\to\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T,P),\mathcal{C}(m))_{h,q},\\ D(x)(y)&=\mathcal{C}(\Sigma_{T\widehat{T}})(y\otimes x)\end{aligned}\tag{5.4}$$

is a quasi-isomorphism. (Here, the subscripts denote the homological and quantum gradings.)

In particular, given m-tangles T_1,T_2 with N_1 and N_2 crossings, respectively, and integers P_1,P_2, we have

$$\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T_1,P_1),\mathcal{C}(T_2,P_2))_{h,q}\cong\mathcal{C}(\widehat{T_1}T_2,N_1-P_1+P_2)_{h,q-m/2}.$$

(Note that in Formula (5.4) we are taking the chain complex of left-module morphisms, not the complex of bimodule morphisms. Also, Hom denotes the chain complex of maps, not the group of chain maps. The cycles are the chain maps (of all degrees).)

Proof. Equations (5.2) and (5.3) are the statement that $\mathcal{C}(T,P)$ and $\mathcal{C}(\widehat{T},N-P)\{0,\frac{n-m}{2}\}$ are dual 1-morphisms in the bicategory of $\mathbb{Z}$-algebras, chain complexes of bimodules, and homotopy

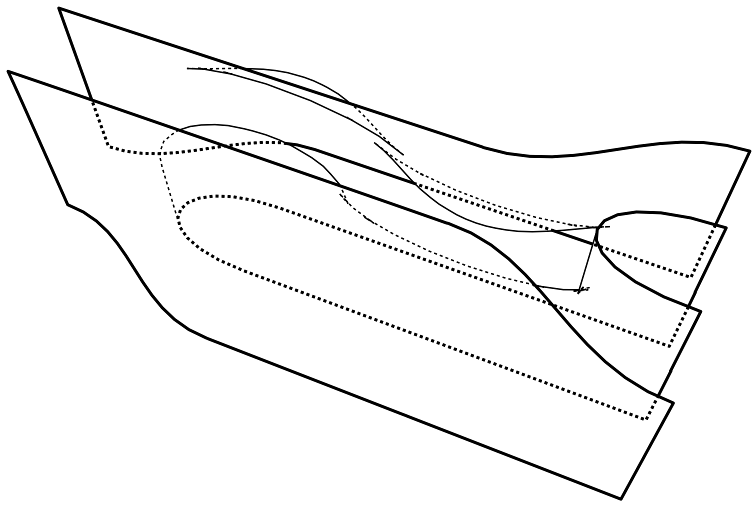


FIGURE 5.2. **Composing cobordisms to get the identity.** With the same tangles and conventions as in Figure 5.1, this is $(\Sigma_{T\widehat{T}} \cup \mathrm{Id}_T) \circ (\mathrm{Id}_T \cup \Sigma_{\widehat{T}T})$. The cobordism $(\mathrm{Id}_{\widehat{T}} \cup \Sigma_{T\widehat{T}}) \circ (\Sigma_{\widehat{T}T} \cup \mathrm{Id}_{\widehat{T}})$ is similar.

classes of chain maps [SP14, Definition 6.1]. Since $\mathcal{C}(T,P)$ and $\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T,P),\mathcal{C}(m))$ are also a dual pair, the result follows from (the proof of) uniqueness of the dual of a dualizable 1-morphism (essentially [EGNO15, Proposition 2.10.5], for instance). The second statement follows by tensoring the first statement with $\mathcal{C}(T_2,P_2)$ and then applying Proposition 5.7 and the composition theorem for the tangle invariants. □

To avoid circular reasoning, we also give a direct proof of the isomorphism in Proposition 5.9 for the special case of $(m,0)$-tangles.

Proposition 5.10. *Let T be a $(m,0)$-tangle and $\widehat{T}$ its mirror. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T,P),\mathcal{C}(m))_{h,q} \cong \mathcal{C}(\widehat{T},N-P)_{h,q-m/2}. \tag{5.5}$$

In particular, given $(m,0)$-tangles T_1,T_2, we have

$$\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T_1,P_1),\mathcal{C}(T_2,P_2))_{h,q} \cong \mathcal{C}(\widehat{T_1}T_2,N_1-P_1+P_2)_{h,q-m/2}.$$

Proof. For the first statement, suppose first that T is a flat tangle and $P=0$. Write T as the union of (the mirror of) a crossingless matching $\widehat{a}$ and k unknots. Then $\mathcal{C}(T,0)=V^{\otimes k}\otimes\mathcal{C}(\widehat{a},0)$. Since $\mathcal{C}(\widehat{a},0)$ is an elementary projective module, an element $f\in\mathrm{Hom}(\mathcal{C}(\widehat{a},0),\mathcal{C}(m))$ is determined by $f(1_a)$ (where $1_a\in V(a\widehat{a})$). Further, $f(1_a)=1_af(1_a)$, so $f(1_a)$ must be an element of $1_a\mathcal{C}(m)=\mathcal{C}(\widehat{a},0)$. The map $1_a\mapsto 1_a$ generates this $\mathcal{C}(m)$-module. The element $1_a\in\mathcal{C}(\widehat{a},0)$ has quantum grading $-m/2$, while $1_a\in\mathcal{C}(m)$ has quantum grading 0, so this map shifts the quantum grading up by $m/2$. We also have $\mathcal{C}(\widehat{T},0)\cong V^{\otimes k}\otimes\mathcal{C}(a,0)$, but here the quantum grading is shifted up by $m/2$ (so if $k=0$, 1_a would have quantum grading 0, not $-m/2$). Finally, the isomorphism $V\cong V^*$ which sends

$$1\mapsto (X\mapsto 1,\ 1\mapsto 0) \qquad\qquad X\mapsto (X\mapsto 0,\ 1\mapsto 1)$$

preserves the quantum grading. Hence, overall, the isomorphism decreases the quantum grading by $m/2$.

For the general case, we apply the isomorphism of the previous paragraph at each vertex. Rather than giving an abstract argument that these are chain maps, we simply check all the cases; see Figure 5.3.

Turning to the gradings, the isomorphism exchanges 0 and 1 resolutions, positive and negative crossings, and the generators 1 and X. Dualizing also negates the grading. Hence, given a generator of $\mathcal{C}(T,P)$ in $V(aT_v)$ with grading q, the dual generator of $\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T,P),\mathcal{C}(m))$ has grading $|v|-2N+3P-q$. The corresponding generator of $\mathcal{C}(\widehat{T},N-P)$ has grading

$$m/2-(N-|v|)+2N-3(N-P)-q=m/2+|v|-2N+3P-q,$$

which is $m/2$ higher, as claimed.

For the homological grading, every generator of $V(aT_v)$ has homological grading $N-|v|-P$, their dual generators of $\mathrm{Hom}_{\mathcal{C}(m)}(\mathcal{C}(T,P),\mathcal{C}(m))$ have homological grading $P-N+|v|=N-(N-P)-(N-|v|)$, which is the grading of the corresponding generators of $\mathcal{C}(\widehat{T},N-P)$.

As in Proposition 5.9, the second statement follows from the first, Proposition 5.7, and the composition theorem for the tangle invariants. □

5.3. **Duality for spectral modules.** We have the following spectral refinement of Proposition 5.9:

Proposition 5.11. *Let T be a (m,n)-tangle with N crossings and $\widehat{T}$ its mirror. Then there is a weak equivalence*

$$\mathrm{Hom}_{\mathscr{X}(m)}(\mathscr{X}(T,P),\mathscr{X}(m))_q\simeq \mathscr{X}^{q+\frac{n-m}{2}}(\widehat{T},N-P).$$

(This is the Hom *as left module spectra.) In particular, given m-tangles T_1,T_2 with N_1 and N_2 crossings, respectively, and integers P_1,P_2, we have*

$$\mathrm{Hom}_{\mathscr{X}(m)}(\mathscr{X}(T_1,P_1),\mathscr{X}(T_2,P_2))_q\simeq \mathscr{X}^{q-m/2}(\widehat{T_1}T_2,N_1-P_1+P_2).$$

Proof. From Section 4.3, given a tangle cobordism Σ from T to T', decomposed as a movie, there is an induced map $\mathscr{X}(\Sigma)\colon \mathscr{X}(T)\to\mathscr{X}(T')$ of spectral bimodules and a commutative diagram

$$\begin{tikzcd} C_*(\mathscr{X}(T,P)) \arrow[r, "\mathscr{X}(\Sigma)"] & C_*(\mathscr{X}(T',P'))\{0,\chi'(\Sigma)\} \\ \mathcal{C}(T,P) \arrow[u, "\simeq"'] \arrow[r, "\mathcal{C}(\Sigma)"'] & \mathcal{C}(T',P')\{0,\chi'(\Sigma)\}. \arrow[u, "\simeq"'] \end{tikzcd} \tag{5.6}$$

In particular, if $\Sigma_{T\widehat{T}}$ is the cobordism from Section 5.2 then there is an induced map of spectral bimodules

$$\mathscr{X}(\Sigma_{T\widehat{T}})\colon \mathscr{X}(T,P)\otimes^L_{\mathscr{X}(n)}\mathscr{X}(\widehat{T},-P)\to\mathscr{X}(\mathrm{Id}_m,0)\{0,n-m\}\simeq\mathscr{X}(m)\{0,\tfrac{n-m}{2}\}.$$

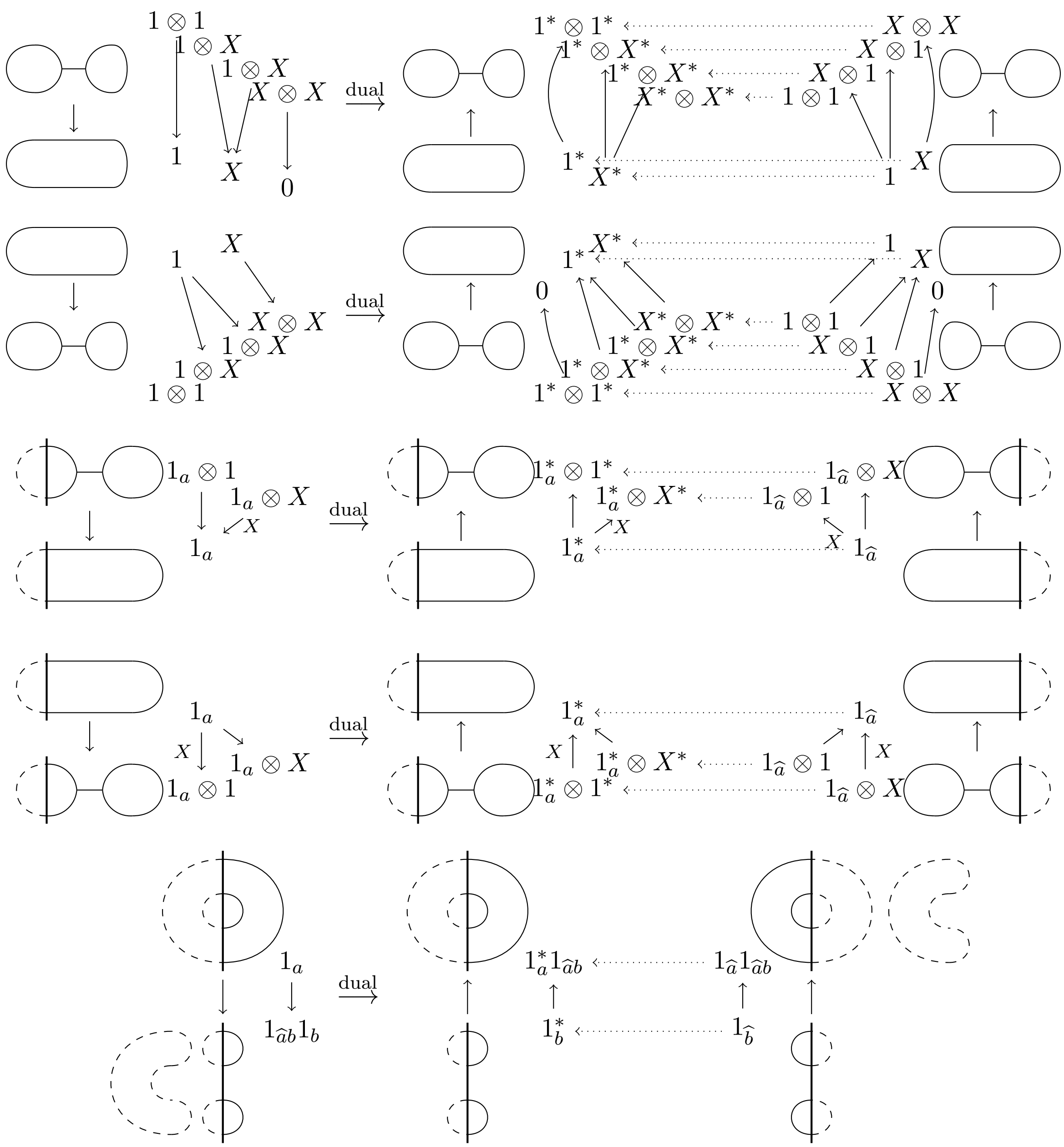


FIGURE 5.3. **Checking the duality isomorphisms induce chain maps.** We check that the duality isomorphisms commute with merge and split maps. There are cases depending on how many of the circles being merged or split are in the interior versus the boundary of the tangle diagram. Resolutions of the tangle are drawn with solid lines, and crossingless matchings capping it off are dashed. In the last case, $1_{\widehat{a}b}$ denotes labeling the circle $\widehat{a}b$ by 1.

There is an induced map

$$\mathcal{D}\colon \mathscr{X}(\widehat{T}, N-P)_{q+\frac{n-m}{2}} \to \operatorname{Hom}_{\mathscr{X}(m)}(\mathscr{X}(T,P), \mathscr{X}(m))_q.$$

By Proposition 5.9, Diagram (5.6) and Propositions 5.7 and 5.6, the map $\mathcal{D}$ induces an isomorphism on homology. Hence, D is a weak equivalence of spectral modules (cf. [LLSb, Theorem 2.18 and proof of Theorem 5]). ☐

6. Functoriality of Khovanov's tangle invariants and their spectral refinements

Wherein we prove that certain modules over the arc algebra and spectral arc algebra have NO NONTRIVIAL AUTOMORPHISMS up to sign, and use this and similar results to VERIFY FUNCTORIALITY for Khovanov homology and its spectral refinement.

6.1. Some rigidity results.

The key to Khovanov's proof of functoriality of Khovanov homology, and hence also the key to ours, is rigidity of the certain bimodules, i.e., the fact that they have no nontrivial automorphisms. For us, the relevant tangles are the following:

Definition 6.1. A *bridge tangle* is a diskular n-tangle (n even) so that the corresponding geometric tangle is isotopic to a collection of embedded arcs in $S^1 \times \mathbb{R} \subset D^2 \times \mathbb{R}$. Equivalently, a bridge tangle is a tangle with no closed components, such that every component is unknotted and there is a collection of disks in the complement of T separating the components of T.

Lemma 6.2. *Let T be a bridge tangle. Then, up to chain homotopy, the only grading-preserving chain homotopy autoequivalences of the Khovanov module $\mathcal{C}(T,P)$ associated to T are multiplication by ± 1.*

Proof. We want to show that the only units in $H_{0,0}\operatorname{Hom}_{\mathcal{C}(n)}(\mathcal{C}(T,P), \mathcal{C}(T,P))$ are ± 1. By Proposition 5.9, this group is exactly $Kh_{0,-n/2}(\widehat{T}T, N) = Kh_{0,-n/2}(U_{n/2}, 0)$, the Khovanov homology of the $n/2$-component unlink. Since $Kh_{0,-n/2}(U_{n/2}, 0) \cong \mathbb{Z}$, the result follows. ☐

Lemma 6.3. *Let T be a bridge tangle. Then, up to homotopy, the only grading-preserving automorphisms of the spectral Khovanov module $\mathscr{X}(T,P)$ associated to T are multiplication by ± 1.*

Proof. Suppose T has $n/2$ bridges. Let N be the number of crossings of T. By Proposition 5.11,

$$\operatorname{Hom}_{\mathscr{X}(n)}(\mathscr{X}(T,P), \mathscr{X}(T,P))_{0,0} \simeq \mathscr{X}^{-n/2}(\widehat{T}T, N) = \mathscr{X}^{-n/2}(U_{n/2}),$$

the Khovanov spectrum of the $n/2$-component unlink, in quantum grading $-n/2$. This space is exactly $\mathbb{S}$, the sphere spectrum. Hence, the homotopy classes of endomorphisms are $\pi_0\mathbb{S} \cong \mathbb{Z}$. The only automorphisms are ± 1. ☐

6.2. Functoriality of the arc algebra multi-modules.

In the language of Section 4, functoriality of Khovanov homology is the following:

Theorem 3. *The projective multi-functor $\mathcal{C}$ from Definition 4.17 descends to a projective multi-functor $\mathcal{C}\colon \widetilde{\mathscr{T}} \to \mathsf{Bim}$.*

Proof. We must check that the value of $\mathcal{C}$ on 2-morphisms is invariant under type (D5) moves, i.e., under the diskular movie moves. That is, we must show that the main parts of the type II movie moves, viewed as maps of m-tangles (where $0 \le m \le 8$ depends on the move), give homotopic maps of spectral bimodules (up to sign). Recall that the diskular movie moves correspond to movie moves 1, 2, 3, 4, 5, 6, 7, 23(a), 23(b), 25, 26, 27, 28, 29, and 30 in Khovanov's list.

For any cobordism between bridges consisting entirely of Reidemeister moves and planar isotopies, Lemma 6.2 implies that the two maps agree up to sign. This handles moves 1–7, 23(a), 25 and 26. The remaining movie moves are 23(b), 27, 28, 29, and 30.

Invariance under move 23b is easy to check directly. (So is invariance under lots of other moves, of course.)

For move 27, both sides are maps from the empty link to the unknot U_1 of (h,q)-bidegree $(0,-1)$. Further, both are compositions of the birth map, which maps to the unit $1 \in Kh(U_1)$, with an isomorphism. Since up to chain homotopy the only grading-preserving isomorphisms of $Kh(U_1)$ are multiplication by ± 1, these two maps agree up to sign. A similar argument applies to this movie read backwards, with a death in place of a birth.

Similarly, both movies in move 28 are (h,q)-bidegree $(0,-1)$ homomorphisms from the invariant of a single bridge B to the invariant of a bridge union an unknot, $B \cup U_1$. By Proposition 5.9, this homomorphism is an element of $Kh_{0,-2}(\widehat{B} \cup B \cup U_1) = Kh_{0,-2}(U_2)$, where U_2 denotes the 2-component unlink. This group is isomorphic to $\mathbb{Z}$. Further, since both maps are a birth followed by an isomorphism, both correspond to ± 1 in $\mathbb{Z}$. A similar argument applies to move 28 read backwards; again, the map lies in $Kh_{0,-2}(U_2)$, this time because a death map has bidegree $(0,-1)$.

Move 29 is the composition of a saddle and a Reidemeister move. The saddle has bidegree $(0,1)$, so by Proposition 5.9, both sides are represented by elements of $Kh_{0,-1}(U_1) \cong \mathbb{Z}$. Further, since there exist invertible cobordism maps containing some saddles (e.g., by move 23b), both elements must be ± 1 in this group.

Move 30 is the composition of a saddle and a planar isotopy. Hence, the corresponding maps have bidegree $(0,1)$. By Proposition 5.9 again, both sides are represented by elements of $Kh_{0,-2}(U_2) \cong \mathbb{Z}$. This element is a generator by the same argument as for move 29. This completes the proof. ☐

Corollary 6.4. *Given an oriented link cobordism $\Sigma\colon L_0 \to L_1$ between oriented links there is an induced chain homotopy class of chain maps $\mathcal{C}(\Sigma)\colon \mathcal{C}(L_0) \to \mathcal{C}(L_1)\{0,\chi(\Sigma)\}$, well-defined up to sign. Further, given another oriented link cobordism $\Sigma'\colon L_1 \to L_2$,*

$$\mathcal{C}(\Sigma') \circ \mathcal{C}(\Sigma) = \pm\mathcal{C}(\Sigma' \circ \Sigma).$$

6.3. **Functoriality of the spectral invariants.** Functoriality of the Khovanov stable homotopy type is the following:

Theorem 4. *The projective multi-functor $\mathscr{X}$ from Definition 4.24 descends to a projective multi-functor $\mathscr{X}\colon \widetilde{\mathscr{T}} \to \mathsf{SBim}$.*

Proof. The proof is the same as the proof of Theorem 3, using Lemma 6.3 in place of Lemma 6.2 and Proposition 5.11 in place of Proposition 5.9. ☐

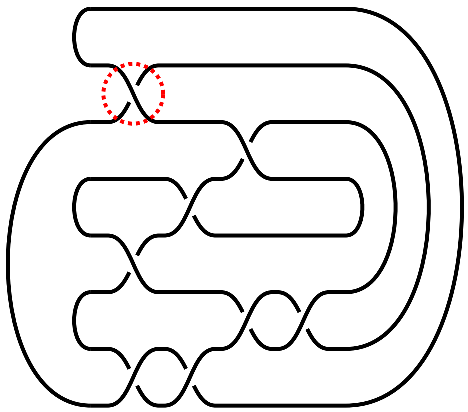


FIGURE 7.1. **The knot 8_{19}.**

Proof of Theorem 1. Given oriented link diagrams L_0, L_1 with P_0 and P_1 positive crossings and an oriented cobordism Σ from L_0 to L_1, we have $P(\Sigma) = P_1 - P_0$, so Σ goes from (L_0, P_0) to (L_1, P_1). Hence, Theorem 4 gives a well-defined homotopy class of maps $\mathscr{X}(\Sigma)\colon \mathscr{X}^j(L_0) \to \mathscr{X}^{j-\chi(\Sigma)}(L_1)$. It is immediate from that theorem that $\mathscr{X}$ is functorial in Σ.

It remains to verify that the maps associated to Reidemeister moves and elementary cobordisms agree with the maps defined in our previous papers. This is equivalent to showing that the map associated with Reidemeister moves and elementary cobordisms in our previous papers [LLSb, LS14b] commute with the gluing map for gluing tangles, up to homotopy. This is straightforward from the definitions, and is left to the reader. □

7. COMPUTATIONS AND APPLICATIONS

Wherein we describe an example of a HOPF-LIKE INVARIANT OF LINK COBORDISMS coming from naturality of the Khovanov spectrum.

Maps of spaces are much richer than maps of abelian groups. In particular, there can be non-nullhomotopic maps of spaces when the induced maps on homology vanish for grading reasons: the familiar Hopf map in $\pi_1^s(S^0) = \mathbb{Z}/2$ is an example. Another example is the Hopf-like map in $\pi_1(M(\mathbb{Z}/2)) = [S^{n+2}, \Sigma^n\mathbb{R}P^2] = \mathbb{Z}/2$ $(n \geq 2)$. For the Khovanov spectrum, this phenomenon can even occur for maps between Khovanov-thin knots, even though the Khovanov spectra for Khovanov-thin knots are wedge sums of Moore spectra [LS14a, Section 9.3] and, consequently, determined by their homology. One way to detect interesting maps is to study their mapping cones. As an example, we have the following proposition.

Proposition 7.1. *There is an orientable cobordism Σ from the knot $K_0 = 5_2$ to the link $K_1 = 5_1 \cup$ meridian so that the induced map of Khovanov spectra*

$$S^0 \vee S^1 \simeq \mathscr{X}^3(K_0) \to \mathscr{X}^4(K_1) \simeq S^0 \vee \Sigma^{-1}\mathbb{R}\mathrm{P}^2 \tag{7.1}$$

	0	1	2	3	4	5
13						$\mathbb{Z}$
11					$\mathbb{Z}_2$	
9				$\mathbb{Z}$	$\mathbb{Z}$	
7			$\mathbb{Z}\mathbb{Z}_2$			
5		$\mathbb{Z}_2$	$\mathbb{Z}$			
3	$\mathbb{Z}$	$\mathbb{Z}$				
1	$\mathbb{Z}$					

$K_0 = 5_2$

	-2	-1	0	1	2	3	4	5
14								$\mathbb{Z}$
12							$\mathbb{Z}_2$	
10						$\mathbb{Z}^2$	$\mathbb{Z}$	
8					$\mathbb{Z}_2^2$			
6				$\mathbb{Z}$	$\mathbb{Z}^2$			
4			$\mathbb{Z}\mathbb{Z}_2$					
2			$\mathbb{Z}^2$					
0	$\mathbb{Z}$							
-2	$\mathbb{Z}$							

K_1

	0	1	2	3	4	5
17						$\mathbb{Z}$
15						$\mathbb{Z}$
13				$\mathbb{Z}$	$\mathbb{Z}$	
11			$\mathbb{Z}_2$		$\mathbb{Z}$	
9			$\mathbb{Z}$			
7	$\mathbb{Z}$					
5	$\mathbb{Z}$					

$K = m(T(3,4))$

FIGURE 7.2. **Khovanov homology of $K_0 = 5_2$, K_1, and $K = m(T(3,4))$.** The symbol $\oplus$ has been suppressed in $\mathbb{Z} \oplus \mathbb{Z}/2$ in two places. The homological grading is horizontal and the quantum grading is vertical.

sends S^1 to $\Sigma^{-1}\mathbb{R}\mathrm{P}^2$ via the Hopf map. More precisely, we can choose the homotopy equivalences in Formula (7.1) so that the maps $S^0 \to \Sigma^{-1}\mathbb{R}P^2$ and $S^1 \to S^0$ are nullhomotopic, in which case the map $S^1 \to \Sigma^{-1}\mathbb{R}P^2$ is the Hopf map.

Proof. Let $K = 8_{19} = m(T(3,4))$, which is shown in Figure 7.1. The 0-resolution (respectively 1-resolution) of the circled crossing is K_0 (respectively K_1). Hence, this crossing corresponds to a single saddle cobordism from K_0 to K_1. Since K_1 has two components, this cobordism is orientable.

The Khovanov homologies of K, K_0, and K_1 are shown in Figure 7.2. These computations were extracted from the Knot Atlas and Mathematica KnotTheory packages [BM]. Knot Atlas is not consistent about the distinction between a knot and its mirror, but since K is a negative knot, with our conventions its Khovanov homology is supported in positive gradings (see Remark 2.6). For the 2-component link K_1, the KnotTheory package gives idiosyncratic gradings; we have shifted the results to agree with our conventions.

We have

$$\mathcal{C}(K)\{-2,-7\} \simeq \mathrm{Cone}(\mathcal{C}(\Sigma)\colon \mathcal{C}(K_0)\{0,1\} \to \mathcal{C}(K_1)),$$
$$\Sigma^{-2}\mathscr{X}^j(K) \simeq \mathrm{Cone}(\mathscr{X}(\Sigma)\colon \mathscr{X}^{j-8}(K_0) \to \mathscr{X}^{j-7}(K_1)).$$

One can verify the grading shift either from the diagram and grading formulas or by examining the Khovanov homologies: this is the only possibility consistent with a long exact sequence $\cdots \to Kh(K_1) \to Kh(K)\{a,b\} \to Kh(K_0) \to \cdots$.

Consider $\mathscr{X}^{11}(K)$. It was calculated previously [LS14c, JLS17] that

$$\mathscr{X}^{11}(K) \simeq \Sigma^{-1}\mathbb{R}P^5/\mathbb{R}P^2.$$

(Note our conventions are different from [LS14c].) On the other hand, since K_0 and K_1 are thin we have

$$\mathscr{X}^3(K_0) \simeq S^0 \vee S^1$$
$$\mathscr{X}^4(K_1) \simeq S^0 \vee \Sigma^{-1}\mathbb{R}P^2.$$

Write the map $S^0 \vee S^1 \to S^0 \vee \Sigma^{-1}\mathbb{R}P^2$ as $(a,b,c,d) \in \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^3$. By considering the homology of $\mathscr{X}^{11}(K)$, a must be a unit. So, we can pre-compose with an automorphism of $S^0 \vee S^1$ and post-compose with an automorphism of $S^0 \vee \Sigma^{-1}\mathbb{R}P^2$ so that $b = c = 0$. Then, considering the Steenrod squares on $\mathbb{R}P^5/\mathbb{R}P^2$, the map $\mathscr{X}^3(K_0) \supset S^1 \to \Sigma^{-1}\mathbb{R}P^2 \subset \mathscr{X}^4(K_1)$ must be the Hopf map, as claimed. ☐

Remark 7.2. The Khovanov stable homotopy type does not give an interesting invariant of closed surfaces in an obvious way. Given a closed surface Σ, viewed as a map from the empty link to the empty link, there is an induced map

$$\mathscr{X}(\Sigma)\colon \mathscr{X}^j(\varnothing) \to \mathscr{X}^{j-\chi(\Sigma)}(\varnothing).$$

Since $\mathscr{X}^j(\varnothing)$ is the sphere spectrum $\mathbb{S}$ if $j = 0$ and trivial for $j \neq 0$, the map $\mathscr{X}(\Sigma)$ can only be nontrivial if $\chi(\Sigma) = 0$. In this case, by the Hurewicz theorem, the homotopy class of the map $\mathscr{X}^j(\Sigma)$ is determined by the induced map on homology. This map $\mathbb{Z} \to \mathbb{Z}$ sends 1 to 2^{b_0} if Σ consists of b_0 tori, and 0 if Σ has any non-toroidal components [Ras, Tan06, GL].

Table of notation

Notation	Meaning
$a, b, \dots$	Crossingless matchings (4)
$\mathsf{B}(n)$	Set of crossingless matchings of n points (n even) (4)
$\widehat{a}$, $\widehat{T}$	The mirror of a tangle or crossingless matching (4)

Notation	Meaning
N	The number of crossings of a link L or tangle T
P	Auxiliary integer. Morally, number of positive crossings (8)
$\mathfrak{C}$	The set of crossings of a tangle T (3)
A	A specific annulus (12)
$T^{m_1,\dots,m_k;n}$ or T	A diskular tangle (12)
$T \circ_i S$, $T \circ (S_1, \dots, S_k)$	Composition of diskular tangles (12)
T_v	Resolution of tangle T associated to vertex v of $\underline{2}^{\mathfrak{C}}$ (4)
Σ	Cobordism of diskular tangles (13)
$P(\Sigma)$	Effect of Σ on number of positive crossings (13)
$\chi'(\Sigma)$	Modified Euler characteristic of Σ (13)
$\mathbb{T}$	The tangle cobordism movie multicategory (13)
$\mathscr{T}$	The tangle cobordism multicategory (14)
$\widetilde{\mathbb{T}}$, $\widetilde{\mathscr{T}}$	Canonical groupoid enrichments of $\mathbb{T}$, $\mathscr{T}$ (16)
Ab	(Multi-)Category of abelian groups (3)
$\mathscr{S}$	(Multi-)Category of symmetric spectra (6)
Bim	Multicategory of dg multimodules (18)
SBim	Multicategory of spectral multimodules (22)
$\mathbb{S}$	The sphere spectrum (2)
$\mathcal{C}(L)$	The Khovanov complex of a link L (3)
$Kh(L)$	Khovanov homology of a link L (3)
V	The Khovanov Frobenius algebra or TQFT (3)
$V(Z)$	The Khovanov TQFT applied to a closed 1-manifold Z (3)
$\underline{2}^{\mathfrak{C}}$	Cube category on the set $\mathfrak{C}$ (3)
$\underline{2}^{\mathfrak{C}}_+$	Result of doubling terminal object in $\underline{2}^{\mathfrak{C}}$ (3)
$\lvert v\rvert$	Height of a vertex v of $\underline{2}^{\mathfrak{C}}$ (9)
$\mathcal{C}(n)$	Khovanov's arc algebra on n points (n even) (4)
$\mathcal{C}(T)$	Khovanov's complex of bimodules associated to $(2m, 2n)$-tangle T (4)
$\mathcal{C}(\Sigma)$	Khovanov map associated to a tangle cobordism Σ (20)
$\{h, q\}$	Homological grading shift by h, quantum grading shift by q (9)
gr_q, gr_h	Quantum and homological gradings (11)
$\mathscr{X}(K)$, $\mathscr{X}^j(K)$	Khovanov spectrum of a link K, in quantum grading j (2,24)
$\mathscr{X}(n)$	Spectral arc algebra on n points (n even) (6)
$\mathscr{X}(T)$, $\mathscr{X}(T,P)$	Spectral arc algebra bimodule associated to a (m,n)-tangle or diskular tangle T (8, 24)
$\mathscr{X}(\Sigma)$	Map of Khovanov spectra associated to tangle cobordism Σ (24)
$\mathcal{S}_n$	Arc algebra shape multicategory (5)
$\mathcal{T}_{m;n}$, $\mathcal{T}_{m_1,\dots,m_k;n}$	Tangle shape multicategory (6, 22)
$\mathcal{U}_{m;n;p}$	Gluing shape multicategory (8)

Notation	Meaning
$\widetilde{\mathcal{S}}_n$, $\widetilde{\mathcal{T}}_{m;n}$, $\widetilde{\mathcal{U}}_{m;n;p}$	Groupoid enriched versions of $\mathcal{S}_n$, $\mathcal{T}_{m;n}$, $\mathcal{U}_{m;n;p}$ (6, 8, 25)
Cob_d	Divided cobordism category (6, 7, 23)
$\widetilde{\mathscr{C}}$	Canonical groupoid enrichment of $\mathscr{C}$ (7, 16)
$\widetilde{\mathrm{Id}}$	Particular morphism related to canonical groupoid enrichment (17)
$\underline{2}^{\mathfrak{C}}\tilde{\times}\widetilde{\mathcal{T}}_{m;n}$	Thickened product of $\underline{2}^{\mathfrak{C}}$ and $\widetilde{\mathcal{T}}_{m;n}$ (8)

Table 7.1. **Table of notation.** The page where each notation is introduced is noted in parentheses.

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