TOPOLOGICAL CYCLIC HOMOLOGY VIA THE NORM

VIGLEIK ANGELTVEIT, ANDREW J. BLUMBERG, TEENA GERHARDT,
MICHAEL A. HILL, TYLER LAWSON, AND MICHAEL A. MANDELL

Abstract. We describe a construction of the cyclotomic structure on topological Hochschild homology (THH) of a ring spectrum using the Hill–Hopkins–Ravenel multiplicative norm. Our analysis takes place entirely in the category of equivariant orthogonal spectra, avoiding use of the Bökstedt coherence machinery. We are also able to define versions of topological cyclic homology (TC) and TR-theory relative to an arbitrary commutative ring spectrum A. We describe spectral sequences computing the relative theory ${}_{A}TR$ in terms of TR over the sphere spectrum and vice versa. Furthermore, our construction permits a straightforward definition of the Adams operations on TR and TC.

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1. Introduction

Over the last two decades, the calculational study of algebraic K-theory has been revolutionized by the development of trace methods. In analogy with the Chern character from topological K-theory to ordinary cohomology, there exist

Angeltveit was supported in part by an NSF All-Institutes Postdoctoral Fellowship administered by the Mathematical Sciences Research Institute through its core grant DMS-0441170, NSF grant DMS-0805917, and an Australian Research Council Discovery Grant.

Blumberg was supported in part by NSF grant DMS-1151577.

Gerhardt was supported in part by NSF grants DMS-1007083 and DMS-1149408.

Hill was supported in part by NSF grant DMS-0906285, DARPA grant FA9550-07-1-0555, and the Sloan Foundation.

Lawson was supported in part by NSF grant DMS-1206008.

Mandell was supported in part by NSF grants DMS-1105255, DMS-1505579.

"trace maps" from algebraic K-theory to various more homological approximations, which also can be more computable. For a ring R, Dennis constructed a map to Hochschild homology

$$K(R) \longrightarrow HH(R)$$

that generalizes the trace of a matrix. Goodwillie lifted this trace map to negative cyclic homology

$$K(R) \longrightarrow HC^{-}(R) \longrightarrow HH(R)$$

and showed that, rationally, this map can often be used to compute $K(R)$.

In his 1990 ICM address, Goodwillie conjectured that there should be a "brave new" version of this story involving "topological" analogues of Hochschild and cyclic homology defined by changing the ground ring from $\mathbb{Z}$ to the sphere spectrum. Although the modern symmetric monoidal categories of spectra had not yet been invented, Bökstedt developed coherence machinery that enabled a definition of topological Hochschild homology (THH) along these lines. Further, he constructed a "topological" Dennis trace map [6]

$$K(R) \longrightarrow THH(R).$$

Subsequently, Bökstedt–Hsiang–Madsen [7] defined topological cyclic homology (TC) and constructed the cyclotomic trace map to TC, lifting the topological Dennis trace

$$K(R) \longrightarrow TC(R) \longrightarrow THH(R).$$

They did this in the course of resolving the K-theory Novikov conjecture for groups satisfying a mild finiteness hypothesis. Subsequently, seminal work of McCarthy [32] and Dundas [11] showed that when working at a prime p, TC often captures a great deal of information about K-theory. Hesselholt and Madsen (inter alia, [18]) then used TC to make extensive computations in K-theory, including a computational resolution of the Quillen–Lichtenbaum conjecture for certain fields.

The calculational power of trace methods depends on the ability to compute $TC(R)$, which ultimately derives from the methods of equivariant stable homotopy theory. Bökstedt's definition of $THH(R)$ closely resembles a cyclic bar construction, and as a consequence $THH(R)$ is an S^1-spectrum. Topological cyclic homology is constructed from this S^1-action on $THH(R)$, via fixed point spectra $TR^n(R) = THH(R)^{C_{p^n}}$. In fact, $THH(R)$ has a very special equivariant structure: $THH(R)$ is a *cyclotomic* spectrum, which is an S^1-spectrum equipped with additional data that models the structure of a free loop space ΛX.

The cyclic bar construction can be formed in any symmetric monoidal category $(A, \boxtimes, 1)$; we will let $N^{\mathrm{cyc}}_{\boxtimes}$ denote the resulting simplicial (or cyclic) object. Recall that in the category of spaces, for a group-like monoid M, there is a natural map

$$|N^{\mathrm{cyc}}_{\times} M| \longrightarrow \mathrm{Map}(S^1, BM) = \Lambda BM$$

(where $|\cdot|$ denotes geometric realization) that is a weak equivalence on fixed points for any finite subgroup $C_n < S^1$. Moreover, for each such C_n, the free loop space is equipped with equivalences (in fact homeomorphisms)

$$(\Lambda BM)^{C_n} \cong \Lambda BM$$

of S^1-spaces, where $(\Lambda BM)^{C_n}$ is regarded as an S^1-space (rather than an S^1/C_n-space) via pullback along the nth root isomorphism

$$\rho_n \colon S^1 \cong S^1/C_n.$$

In analogy, a cyclotomic spectrum is an S^1-spectrum equipped with compatible equivalences of S^1-spectra

$$t_n \colon \rho_n^* L\Phi^{C_n} X \longrightarrow X,$$

where $L\Phi^{C_n}$ denotes the (left derived) "geometric" fixed point functor.

The construction of the cyclotomic structure on THH has classically been one of the more subtle and mysterious parts of the construction of TC. In a modern symmetric monoidal category of spectra (e.g., symmetric spectra or EKMM S-modules), one can simply define $THH(R)$ as

$$THH(R) = |N_{\wedge}^{\mathrm{cyc}} R|,$$

but the resulting equivariant spectrum did not have the correct homotopy type. Only Bökstedt's original construction of THH seemed to produce the cyclotomic structure.

Although this situation has not impeded the calculational applications, reliance on the Bökstedt construction has limited progress in certain directions. For one thing, it does not seem to be possible to use the Bökstedt construction to define TC relative to a ground ring that is not the sphere spectrum S. Moreover, the details of the Bökstedt construction make it difficult to understand the equivariance (and therefore relevance to TC) of various additional algebraic structures that arise on THH, notably the Adams operations and the coalgebra structures.

The purpose of this paper is to introduce a new approach to the construction of the cyclotomic structure on THH using an interpretation of THH in terms of the Hill–Hopkins–Ravenel multiplicative norm. Our point of departure is the observation that the construction of the cyclotomic structure on $THH(R)$ ultimately boils down to having good models of the smash powers

$$R^{\wedge n} = \underbrace{R \wedge R \wedge \ldots \wedge R}_{n}$$

of a spectrum R as a C_n-spectrum such that there is a suitably compatible collection of diagonal equivalences

$$R \longrightarrow \Phi^{C_n} R^{\wedge n}.$$

The recent solution of the Kervaire invariant one problem involved the detailed analysis of a multiplicative norm construction in equivariant stable homotopy theory that has precisely this type of behavior. Although Hill–Hopkins–Ravenel studied the norm construction N_H^G for a finite group G and subgroup H, using the cyclic bar construction one can extend this construction to a norm $N_e^{S^1}$ on associative ring orthogonal spectra; such a construction first appeared in the thesis of Martin Stolz [37].

For the following definition, we need to introduce some notation. Let $\mathcal{S}$ denote the category of orthogonal spectra and let $\mathcal{S}_U^{S^1}$ denote the category of orthogonal S^1-spectra indexed on the complete universe U. Finally, let $\mathcal{A}ss$ denote the category of associative ring orthogonal spectra.

Definition 1.1. Define the functor

$$N_e^{S^1} \colon \mathcal{A}ss \longrightarrow \mathcal{S}_U^{S^1}$$

to be the composite functor

$$R \mapsto \mathcal{I}_{\mathbb{R}^\infty}^{U} |N_{\wedge}^{\mathrm{cyc}} R|,$$

with $|N_{\wedge}^{\mathrm{cyc}}R|$ regarded as an orthogonal S^1-spectrum indexed on the standard trivial universe $\mathbb{R}^\infty$. Here $\mathcal{I}_{\mathbb{R}^\infty}^{U}$ denotes the change of universe functor (see Definition 2.6).

Since both the cyclic bar construction and the change of universe functor preserve commutative ring orthogonal spectra, the norm above also preserves commutative ring orthogonal spectra. In the following proposition, proved in Section 4, $\mathcal{C}om$ and $\mathcal{C}om_U^{S^1}$ denote the categories of commutative ring orthogonal spectra and commutative ring orthogonal S^1-spectra, respectively.

Proposition 1.2. *$N_e^{S^1}$ restricts to a functor*

$$N_e^{S^1} : \mathcal{C}om \longrightarrow \mathcal{C}om_U^{S^1}$$

that is the left adjoint to the forgetful functor from commutative ring orthogonal S^1-spectra to commutative ring orthogonal spectra.

The forgetful functor from commutative ring orthogonal S^1-spectra to commutative ring orthogonal spectra is the composite of the change of universe functor $\mathcal{I}_U^{\mathbb{R}^\infty}$ and the functor that forgets equivariance. The proof of the above proposition identifies $N_e^{S^1} : \mathcal{C}om \to \mathcal{C}om_U^{S^1}$ as the composite functor

$$R \mapsto \mathcal{I}_{\mathbb{R}^\infty}^{U}(R \otimes S^1),$$

which is left adjoint to the forgetful functor. Here $\otimes$ denotes the tensor of a commutative ring orthogonal spectrum with an unbased space, and we regard $(-)\otimes S^1$ as a functor from commutative ring orthogonal spectra to commutative ring orthogonal spectra with an action of S^1.

The deep aspect of the Hill–Hopkins–Ravenel treatment of the norm functor is their analysis of the left derived functors of the norm. As part of this analysis they show that the norm N_H^G preserves certain weak equivalences. For our norm $N_e^{S^1}$ into $\mathcal{S}_U^{S^1}$, we work with the homotopy theory defined by the *$\mathcal{F}$-equivalences* of orthogonal S^1-spectra, where an $\mathcal{F}$-equivalence is a map that induces an isomorphism on all the homotopy groups at the fixed point spectra for the finite subgroups of S^1. We prove the following theorem in Section 4.

Proposition 1.3. *Assume that R is a cofibrant associative ring orthogonal spectrum and R' is either a cofibrant associative ring orthogonal spectrum or a cofibrant commutative ring orthogonal spectrum. If $R \to R'$ is a weak equivalence, then $N_e^{S^1}R \to N_e^{S^1}R'$ is an $\mathcal{F}$-equivalence in $\mathcal{S}_U^{S^1}$.*

Of course the conclusion holds if R is a cofibrant commutative ring orthogonal spectrum as well; the point of Proposition 1.3 is to compare cofibrant replacements in associative and commutative ring orthogonal spectra.

As a consequence we obtain the following additional observation about the adjunction in the commutative case. See Proposition 4.10 for a more precise statement.

Proposition 1.4. *The functor*

$$N_e^{S^1} : \mathcal{C}om \longrightarrow \mathcal{C}om_U^{S^1}$$

is Quillen left adjoint to the forgetful functor (for an appropriate model structure with weak equivalences the $\mathcal{F}$-equivalences on the codomain); in particular, its left derived functor exists and is left adjoint to the right derived forgetful functor.

Our first main theorem is that when R is a cofibrant associative ring orthogonal spectrum, $N_e^{S^1} R$ is a cyclotomic spectrum. To be precise, we use the point-set model of cyclotomic spectra from [5], which provides a definition entirely in terms of the category of orthogonal S^1-spectra.

Theorem 1.5. *Let R be a cofibrant associative or cofibrant commutative ring orthogonal spectrum. Then $N_e^{S^1} R$ has a natural structure of a cyclotomic spectrum.*

Experts will recognize that one can give a direct construction of the cyclotomic trace induced by the inclusion of objects in a spectral category enriched in orthogonal spectra (e.g., see [4]). We review this construction in Section 5.

Proposition 1.4, which describes $N_e^{S^1}$ as the homotopical left adjoint to the forgetful functor, suggests a generalization of our construction of THH that takes ring orthogonal C_n-spectra as input. For commutative ring orthogonal C_n-spectra, we can define $N_{C_n}^{S^1}$ as the left adjoint to the forgetful functor. However, to extend to the non-commutative case, we need an explicit construction. We give such a construction in Section 8 in terms of a cyclic bar construction, which we denote as $N_\wedge^{\mathrm{cyc},C_n} R$. Its geometric realization $|N_\wedge^{\mathrm{cyc},C_n} R|$ has an S^1-action, and by promoting it to the complete universe we obtain a genuine orthogonal S^1-spectrum that we denote as $N_{C_n}^{S^1} R$. The following proposition is a consistency check.

Proposition 1.6. *Let R be a commutative ring orthogonal C_n-spectrum. Then $N_{C_n}^{S^1} R$ is isomorphic to the left adjoint of the forgetful functor from commutative ring orthogonal S^1-spectra to commutative ring orthogonal C_n-spectra.*

Again, we can describe the left adjoint in terms of a tensor

$$N_{C_n}^{S^1} R = \mathcal{I}_{\mathbb{R}^\infty}^U (R \otimes_{C_n} S^1),$$

where the relative tensor $R \otimes_{C_n} S^1$ may be explicitly constructed as the coequalizer

$$(i^* R) \otimes C_n \otimes S^1 \rightrightarrows (i^* R) \otimes S^1$$

of the canonical action of C_n on S^1 and the action map $(i^* R) \otimes C_n \to i^* R$, where i^* denotes the change-of-group functor to the trivial group. Choosing an appropriately subdivided model of the circle produces the isomorphism between the two descriptions.

As above, by cofibrantly replacing R we can compute the left-derived functor of $N_{C_n}^{S^1}$, and in this case $N_{C_n}^{S^1} R$ is a p-cyclotomic spectrum (see Definition 3.1) provided either n is prime to p or R is "C_n-cyclotomic" (q.v. Definition 8.7 below). This leads to the obvious definition of $TC_{C_n} R$. This C_n-relative THH (and the associated constructions of TR and TC) is expected to be both interesting and comparatively easy to compute for some of the equivariant spectra that arise in Hill–Hopkins–Ravenel, in particular the real cobordism spectrum $MU_{\mathbb{R}}$.

We can also consider another kind of relative construction, namely in the situation where R is an algebra over an arbitrary commutative ring orthogonal spectrum A. Definition 1.1 can be extended to the relative setting; the equivariant indexed product can be carried out in any symmetric monoidal category, and the homotopical analysis in the case of A-modules is given in Section 6.

Definition 1.7. Let A be a cofibrant commutative ring orthogonal spectrum, and denote by A-$\mathcal{A}lg$ the category of A-algebras. We define the A-relative norm functor

$${}_A N_e^{S^1} \colon A\text{-}\mathcal{A}lg \longrightarrow A_{S^1}\text{-}\mathcal{M}od_U^{S^1}$$

by

$$R \mapsto \mathcal{I}_{\mathbb{R}^\infty}^{U} |N^{\mathrm{cyc}}_{\wedge_A} R|.$$

Here A_{S^1} denotes $\mathcal{I}_{\mathbb{R}^\infty}^{U} A$, constructed by applying the point-set change of universe functor $\mathcal{I}_{\mathbb{R}^\infty}^{U}$ to A regarded as a commutative ring orthogonal S^1-spectrum (on the universe $\mathbb{R}^\infty$) with trivial S^1-action. Then A_{S^1} is a commutative ring orthogonal S^1-spectrum (on the universe U) and A_{S^1}-$\mathcal{M}od_U^{S^1}$ denotes the category of A_{S^1}-modules in $\mathcal{S}_U^{S^1}$.

We write ${}_ATHH(R)$ for the underlying non-equivariant spectrum of ${}_AN_e^{S^1}R$; this spectrum was denoted $thh^A(R)$ in [12, IX.2.1]. When R is a commutative A-algebra, ${}_AN_e^{S^1}R$ is naturally a commutative A_{S^1}-algebra. The functor

$${}_AN_e^{S^1} \colon A\text{-}\mathcal{C}om \longrightarrow A_{S^1}\text{-}\mathcal{C}om_U^{S^1}$$

is again left adjoint to the forgetful functor. However, due to the subtleties of the behavior of $\mathcal{I}_{\mathbb{R}^\infty}^{U}$ when applied to cofibrant commutative ring orthogonal spectra regarded as S^1-spectra with trivial action, A_{S^1} is in general not cyclotomic and so neither is ${}_AN_e^{S^1}R$. In particular, we always have $A_{S^1} \cong {}_ATHH(A)$ even though A may not extend to a cyclotomic spectrum. (See also Examples 6.3 and 7.6 below.) Instead, we must settle for the following weaker analogue of Theorem 1.5. Here an op-p-precyclotomic structure asks merely for an analogue of the cyclotomic structure map (in the opposite direction) that is not necessarily an equivalence. We prove the following theorem in Section 7.

Theorem 1.8. *Let R be an A-algebra. Then ${}_AN_e^{S^1}R$ is an op-p-precyclotomic spectrum with structure map in the category of A_{S^1}-modules.*

Nonetheless, we can define the A-relative topological cyclic homology ${}^{op}_{A}TC(R)$ as a homotopy limit over the Frobenius and wrong-way restriction maps. The relative topological cyclic homology is the target of an A-relative cyclotomic trace $K(R) \to {}^{op}_{A}TC(R)$, factoring though the usual cyclotomic trace $K(R) \to TC(R)$, essentially by construction.

Theorem 1.9. *Let R be a cofibrant associative A-algebra or a cofibrant commutative A-algebra. There is an A-relative cyclotomic trace map $K(R) \to {}^{op}_{A}TC(R)$ making the following diagram commute in the stable category*

$$\begin{tikzcd} K(R) \arrow[r] \arrow[dr] & TC(R) \arrow[r] \arrow[d] & THH(R) \arrow[d] \\ & {}^{op}_{A}TC(R) \arrow[r] & {}_ATHH(R). \end{tikzcd}$$

Using the identification $N_e^{S^1}A \cong \mathcal{I}_{\mathbb{R}^\infty}^{U}(A \otimes S^1)$ in the commutative context, the map $S^1 \to *$ induces a map of equivariant commutative ring orthogonal spectra $N_e^{S^1}A \to A_{S^1}$. Just as in the non-equivariant case, we can identify ${}_AN_e^{S^1}R$ as extension of scalars along this map.

Proposition 1.10. *Let R be an associative A-algebra. There is a natural isomorphism*

$${}_AN_e^{S^1}R \cong N_e^{S^1}R \wedge_{N_e^{S^1}A} A_{S^1}.$$

When R is a cofibrant associative A-algebra or cofibrant commutative A-algebra, this induces a natural isomorphism in the stable category

$$ {}_A N_e^{S^1} R \cong N_e^{S^1} R \wedge^{\mathbf{L}}_{N_e^{S^1} A} A_{S^1}. $$

The equivariant homotopy groups $\pi_*^{C_n}(N_e^{S^1} R)$ are the TR-groups $TR_*^n(R)$ and so $\pi_*^{C_n}({}_A N_e^{S^1} R)$ are by definition the relative TR-groups ${}_A TR_*^n(R)$. The Künneth spectral sequence of [23] can be combined with the previous theorem to compute the relative TR-groups from the absolute TR-groups and Mackey functor $\underline{\mathrm{Tor}}$. More often we expect to use the relative theory to compute the absolute theory. Non-equivariantly, the isomorphism

$$ THH(R) \wedge A \cong {}_A THH(R \wedge A) \tag{1.11} $$

gives rise to a Künneth spectral sequence

$$ \mathrm{Tor}_{*,*}^{A_*(R \wedge_S R^{\mathrm{op}})}(A_*(R), A_*(R)) \implies A_*(THH(R)). $$

An Adams spectral sequence can then in theory be used to compute the homotopy groups of $THH(R)$. For formal reasons, the isomorphism (1.11) still holds equivariantly, but now we have three different versions of the non-equivariant Künneth spectral sequence (none of which have quite as elegant an E^2-term) which we use in conjunction with equation (1.11). We discuss these in Section 9.

A further application of our model of THH and TC is a construction, when R is commutative, of Adams operations on $N_e^{S^1} R$ and ${}_A N_e^{S^1} R$ that are compatible (in the absolute case) with the cyclotomic structure. McCarthy explained how Adams operations can be constructed on any cyclic object that, when viewed as a functor from the cyclic category, factors through the category of finite sets (and all maps). As a consequence, it is possible to construct Adams operations on THH of a commutative monoid object in any symmetric monoidal category of spectra. An advantage of our formulation is that we can easily verify the equivariance of these operations and in particular show they descend to TC. We prove the following theorem in Section 10.

Theorem 1.12. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. There are Adams operations $\psi^r \colon {}_A N_e^{S^1} R \to {}_A N_e^{S^1} R$. When r is prime to p, the operation ψ^r is compatible with the restriction and Frobenius maps on the p-cyclotomic spectrum $THH(R)$ and so induces a corresponding operation on $TR(R)$ and $TC(R)$.*

We have organized the paper to contain a brief review with references to much of the background needed here. Section 2 is mostly review of [29] and [20, App. B], and Section 3 is in part a review of [5, §4]. In addition, the main results in Section 4 overlap significantly with [37], although our treatment is very different: we rely on [20] to study the absolute S^1-norm whereas [37] directly analyzes the construction by using a somewhat different model structure and focuses on the case of commutative ring orthogonal spectra.

Acknowledgments. The authors would like to thank Lars Hesselholt, Mike Hopkins, and Peter May for many helpful conversations. We thank Aaron Royer and Ernie Fontes for helping to identify a serious error in a previous draft. We thank Cary Malkiewich for many helpful suggestions regarding the previous draft. This project was made possible by the hospitality of AIM, the IMA, MSRI, and the Hausdorff Research Institute for Mathematics at the University of Bonn.

2. Background on equivariant stable homotopy theory

In this section, we briefly review necessary details about the category of orthogonal G-spectra and the geometric fixed point and norm functors. Our primary sources for this material are the monograph of Mandell-May [29] and the appendices to Hill–Hopkins–Ravenel [20]. See also [5, §2] for a review of some of these details. We begin with two subsections discussing the point-set theory followed by two subsections on homotopy theory and derived functors.

2.1. **The point-set theory of equivariant orthogonal spectra.** Let G be a compact Lie group. We denote by $\mathcal{T}^G$ the category of based G-spaces and based G-maps. The smash product of G-spaces makes this a closed symmetric monoidal category, with function object $F(X,Y)$ the based space of (non-equivariant) maps from X to Y with the conjugation G-action. In particular, $\mathcal{T}^G$ is enriched over G-spaces. We will denote by U a fixed universe of G-representations [29, §II.1.1], by which we mean a countable dimensional vector space with linear G-action and G-fixed inner product that contains $\mathbb{R}^\infty$, is the sum of finite dimensional G-representations, and that has the property that any G-representation that occurs in U occurs infinitely often. Let $\mathcal{V}^G(U)$ denote the set of finite dimensional G-inner product spaces which are isomorphic to a G-vector subspace of U. Except in this section, we always assume that U is a *complete* G-universe, meaning that all finite dimensional irreducible G-representations are in U. For V, W in $\mathcal{V}^G(U)$, denote by $\mathscr{I}_G(V,W)$ the space of (non-equivariant) isometric isomorphisms $V \to W$, regarded as a G-space via conjugation. Let $\mathscr{I}_G^U$ be the category enriched in G-spaces with $\mathcal{V}^G(U)$ as its objects and $\mathscr{I}_G(V,W)$ as its morphism G-spaces; we write just $\mathscr{I}_G$ when U is understood.

Definition 2.1 ([29, II.2.6])**.** An orthogonal G-spectrum is a G-equivariant continuous functor $X\colon \mathscr{I}_G \to \mathcal{T}^G$ equipped with a structure map

$$\sigma_{V,W}\colon X(V) \wedge S^W \longrightarrow X(V \oplus W)$$

that is a natural transformation of enriched functors $\mathscr{I}_G \times \mathscr{I}_G \to \mathcal{T}^G$ and that is associative and unital in the obvious sense. A map of orthogonal G-spectra $X \to X'$ is a natural transformation that commutes with the structure map.

We denote the category of orthogonal G-spectra by $\mathcal{S}^G$. When necessary to specify the universe U, we include it in the notation as $\mathcal{S}^G_U$.

The category of orthogonal G-spectra is enriched over based G-spaces, where the G-space of maps consists of all natural transformations (not just the equivariant ones). Tensors and cotensors are computed levelwise. The category of orthogonal G-spectra is a closed symmetric monoidal category with unit the equivariant sphere spectrum S_G (with $S_G(V) = S^V$).

For technical reasons, it is often convenient to give an equivalent formulation of orthogonal G-spectra as diagram spaces. Following [29, §II.4], we consider the category $\mathscr{J}_G$ which has the same objects as $\mathscr{I}_G$ but morphisms from V to W given by the Thom space of the complement bundle of linear isometries from V to W.

Proposition 2.2 ([29, II.4.3])**.** *The category $\mathcal{S}^G$ of orthogonal G-spectra is equivalent to the category of $\mathscr{J}_G$-spaces, i.e., the continuous equivariant functors from $\mathscr{J}_G$ to $\mathcal{T}_G$. The symmetric monoidal structure is given by the Day convolution.*

This description provides simple formulas for suspension spectra and desuspension spectra in orthogonal G-spectra.

Definition 2.3 ([29, II.4.6])**.** For any finite-dimensional G-inner product space V we have the shift desuspension spectrum functor

$$F_V\colon \mathcal{T}^G \longrightarrow \mathcal{S}^G$$

defined by

$$(F_V A)(W) = \mathscr{J}_G(V, W) \wedge A.$$

This is the left adjoint to the evaluation functor which evaluates an orthogonal G-spectrum at V.

Remark 2.4. In [20], the desuspension spectrum $F_V S^0$ is denoted as S^{-V} and $F_0 A$ is denoted as $\Sigma^\infty A$ in a nod to the classical notation. (They write $S^{-V} \wedge A$ for $F_V A \cong F_V S^0 \wedge A$.)

Since the category $\mathcal{S}^G_U$ is symmetric monoidal under the smash product, we have categories of associative and commutative monoids, i.e., algebras over the monads $\mathbb{T}$ and $\mathbb{P}$ that create associative and commutative monoids in symmetric monoidal categories (e.g., see [12, §II.4] for a discussion).

Notation 2.5. Let $\mathcal{A}ss^G$ and $\mathcal{C}om^G$ denote the categories of associative and commutative ring orthogonal G-spectra.

For a fixed object A in $\mathcal{C}om^G$, there is an associated symmetric monoidal category A-$\mathcal{M}od^G$ of A-modules in orthogonal G-spectra, with product the A-relative smash product $\wedge_A$. As in Notation 2.5, there are categories A-$\mathcal{A}lg^G$ of A-algebras, and A-$\mathcal{C}om^G$ of commutative A-algebras [29, III.7.6].

We now turn to the description of various useful functors on orthogonal G-spectra. We begin by reviewing the change of universe functors. In contrast to the classical framework of "coordinate-free" equivariant spectra [26], orthogonal G-spectra disentangle the point-set and homotopical roles of the universe U. A first manifestation of this occurs in the behavior of the point-set "change of universe" functors.

Definition 2.6 ([29, V.1.2])**.** For any pair of universes U and U', the point-set change of universe functor

$$\mathcal{I}_U^{U'}\colon \mathcal{S}^G_U \longrightarrow \mathcal{S}^G_{U'}$$

is defined by $\mathcal{I}_U^{U'} X(V) = \mathscr{J}(\mathbb{R}^n, V) \wedge_{O(n)} X(\mathbb{R}^n)$ for V in $\mathcal{V}^G(U')$, where $n = \dim V$.

These functors are strong symmetric monoidal equivalences of categories:

Proposition 2.7 ([29, V.1.1,V.1.5])**.** *Given universes U, U', U'',*

1. *$\mathcal{I}_U^U$ is naturally isomorphic to the identity.*
2. *$\mathcal{I}_{U'}^{U''} \circ \mathcal{I}_U^{U'}$ is naturally isomorphic to $\mathcal{I}_U^{U''}$.*
3. *$\mathcal{I}_U^{U'}$ is strong symmetric monoidal.*

We are particularly interested in the change of universe functors associated to the universes U and U^G. The latter of these universes is isomorphic to the standard trivial universe $\mathbb{R}^\infty$. Note that the category of orthogonal G-spectra on $\mathbb{R}^\infty$ is just the category of orthogonal spectra with G-actions.

Given a subgroup $H < G$, we can regard a G-space $X(V)$ as an H-space $\iota_H^* X(V)$. The space-level construction gives rise to a spectrum-level change-of-group functor.

Definition 2.8 ([29, V.2.1])**.** For a subgroup $H < G$, define the functor

$$\iota_H^* \colon \mathcal{S}_U^G \longrightarrow \mathcal{S}_{\iota_H^* U}^H$$

by

$$(\iota_H^* X)(V) = \mathscr{J}_H(\mathbb{R}^n, V) \wedge_{O(n)} \iota_H^*(X(\mathbb{R}^n))$$

for V in $\mathcal{V}^H(\iota_H^* U)$, where $n = \dim(V)$.

As observed in [29, V.2.1, V.1.10], for V in $\mathcal{V}^G(U)$,

$$(\iota_H^* X)(\iota_H^* V) \cong \iota_H^*(X(V)).$$

In contrast to the category of G-spaces, there are two reasonable constructions of fixed-point functors: the "categorical" fixed points, which are based on the description of fixed points as G-equivariant maps out of G/H, and the "geometric" fixed points, which commute with suspension and the smash product (on the level of the homotopy category). Again, the description of orthogonal G-spectra as $\mathscr{J}_G$-spaces in Proposition 2.2 provides the easiest way to construct the categorical and geometric fixed point functors [29, §V].

For any normal $H \lhd G$, let $\mathscr{J}_G^H(U, V)$ denote the G/H-space of H-fixed points of $\mathscr{J}_G(U, V)$. Given any orthogonal spectrum X, the collection of fixed points $\{X(V)^H\}$ forms a $\mathscr{J}_G^H$-space. We can turn this collection into a $\mathscr{J}_{G/H}$-space in two ways. There is a functor $q\colon \mathscr{J}_{G/H} \to \mathscr{J}_G^H$ induced by regarding G/H-representations as H-trivial G-representations via the quotient map $G \to G/H$.

Definition 2.9 ([29, §V.3])**.** For H a normal subgroup of G, the categorical fixed point functor

$$(-)^H \colon \mathcal{S}_U^G \longrightarrow \mathcal{S}_{U^H}^{G/H}$$

is computed by regarding the $\mathscr{J}_G^H$-space $\{X(V)^H\}$ as a $\mathscr{J}_{G/H}$-space via q.

On the other hand, there is an equivariant continuous functor $\phi\colon \mathscr{J}_G^H \to \mathscr{J}_{G/H}$ induced by taking a G-representation V to the G/H-representation V^H.

Definition 2.10 ([29, §V.4])**.** For H a normal subgroup of G, let Fix^H denote the functor from orthogonal G-spectra ($=\mathscr{J}_G$-spaces) to $\mathscr{J}_G^H$-spaces defined by $(\mathrm{Fix}^H X)(V) = (X(V))^H$. The geometric fixed point functor

$$\Phi^H(-) \colon \mathcal{S}_U^G \longrightarrow \mathcal{S}_{U^H}^{G/H}$$

is constructed by taking $\Phi^H(X)$ to be the left Kan extension of the $\mathscr{J}_G^H$-space $\mathrm{Fix}^H X$ along ϕ.

Remark 2.11. Hill–Hopkins–Ravenel [20, B.190] call the point-set geometric fixed point functor "the monoidal geometric fixed point functor" and define it using the coequalizer

$$\bigvee_{V,W<U} \mathscr{J}_G^H(V,W) \wedge F_{W^H} S^0 \wedge (X(V))^H \rightrightarrows \bigvee_{V<U} F_{V^H} S^0 \wedge (X(V))^H,$$

derived from applying the geometric fixed point functor above to the "tautological presentation" of X:

$$\bigvee_{V,W<U} \mathscr{J}_G(V,W) \wedge F_W S^0 \wedge X(V) \rightrightarrows \bigvee_{V<U} F_V S^0 \wedge X(V),$$

noting that $\Phi^H F_V A \cong F_{V^H} A^H$ for a G-space A. Although Φ^H does not preserve coequalizers in general, it does preserve the coequalizers preserved by Fix^H, and Fix^H preserves the canonical coequalizer diagram since it is levelwise split. Thus, the definition above agrees with the definition in [20, B.190].

Both fixed-point functors are lax symmetric monoidal [29, V.3.8, V.4.7] and so descend to categories of associative and commutative ring orthogonal G-spectra.

Proposition 2.12. *Let $H \lhd G$ be a normal subgroup. Let X and Y be orthogonal G-spectra. There are natural maps*

$$\Phi^H X \wedge \Phi^H Y \longrightarrow \Phi^H(X \wedge Y) \qquad \text{and} \qquad X^H \wedge Y^H \longrightarrow (X \wedge Y)^H$$

that exhibit Φ^H and $(-)^H$ as lax symmetric monoidal functors.

Therefore, there are induced functors

$$\Phi^H, (-)^H \colon \mathcal{A}ss^G \longrightarrow \mathcal{A}ss^{G/H}$$

and

$$\Phi^H, (-)^H \colon \mathcal{C}om^G \longrightarrow \mathcal{C}om^{G/H}.$$

For a commutative ring orthogonal G-spectrum A, a corollary of Proposition 2.12 is that the fixed-point functors interact well with the category of A-modules.

Corollary 2.13. *Let A be a commutative ring orthogonal G-spectrum. The fixed-point functors restrict to functors*

$$\Phi^H \colon A\text{-}\mathcal{M}od^G \longrightarrow (\Phi^H A)\text{-}\mathcal{M}od^{G/H}$$

and

$$(-)^H \colon A\text{-}\mathcal{M}od^G \longrightarrow A^H\text{-}\mathcal{M}od^{G/H}.$$

Remark 2.14. We can extend these constructions to subgroups $H < G$ that are not normal by considering the normalizer NH and quotient $WH = NH/H$. However, since we do not need this generality herein, we do not discuss it further.

Let $z \in G$ be an element in the center of G. Then multiplication by z is a natural automorphism on objects of $\mathcal{S}^G_{\mathbb{R}^\infty}$ or on objects of $A\text{-}\mathcal{M}od^G_{\mathbb{R}^\infty}$. In particular, it will induce a natural automorphism $\mathcal{I}^U_{\mathbb{R}^\infty} z$ of $N^G_H X$ or of ${}_A N^G_H X$, as described in Sections 4 and 7.

Proposition 2.15. *Let z be an element in the center of G, and K a normal subgroup. Then for any $X \in \mathcal{S}^G_{\mathbb{R}^\infty}$, we have an identification*

$$\Phi^K(\mathcal{I}^U_{\mathbb{R}^\infty} z) = \mathcal{I}^{U^K}_{\mathbb{R}^\infty} \bar{z}$$

where $\bar{z} = zK \in G/K$. In particular, for $z \in K$ the map $\Phi^K(\mathcal{I}^U_{\mathbb{R}^\infty} z)$ is the identity.

Proof. Using the tautological presentation of $\mathcal{I}^U_{\mathbb{R}^\infty} X$ and naturality, it suffices to verify this identity on orthogonal spectra of the form $F_V Y$ for a G-representation $V \in \mathcal{V}^G(U)$; on such spectra, the map $\mathcal{I}^U_{\mathbb{R}^\infty} z \colon F_V Y \to F_V Y$ is given by $f \wedge y \mapsto (f \circ z^{-1}) \wedge (z \cdot y)$. The result follows from the fact the fixed point functor $(-)^K$

takes multiplication by z to multiplication by $\bar{z}$, and the functor $\mathscr{I}_G^K \to \mathscr{I}_{G/K}$ induces maps $\mathscr{I}_G^K(V,V) \to \mathscr{I}_{G/K}(V^K,V^K)$ taking z to $\bar{z}$. $\square$

2.2. **The point-set theory of the norm.** Central to our work is the realization by Hill, Hopkins, and Ravenel [20] that a tractable model for the "correct" equivariant homotopy type of a smash power can be formed as a point-set construction using the point-set change of universe functors. It is "correct" insofar as there is a diagonal map which induces an equivalence onto the geometric fixed points (see Section 2.3 below). They refer to this construction as the norm after the norm map of Greenlees-May [16], which in turn is named for the norm map of Evens in group cohomology [13, Chapter 6].

The point of departure for the construction of the norm is the use of the change-of-universe equivalences to regard orthogonal G-spectra on any universe U as G-objects in orthogonal spectra. (Good explicit discussions of the interrelationship can be found in [29, §V.1] and [36, 2.7].) We now give a point-set description of the norm following [36] and [10]; these descriptions are equivalent to the description of [20, §A.3] by the work of [10].

For the construction of the norm, it is convenient to use $\mathcal{B}G$ to denote the category with one object, whose monoid of endomorphisms is the finite group G. The category $\mathcal{S}^{\mathcal{B}G}$ of functors from $\mathcal{B}G$ to the category $\mathcal{S}$ of (non-equivariant) orthogonal spectra indexed on the universe $\mathbb{R}^\infty$ is isomorphic to the category $\mathcal{S}^G_{\mathbb{R}^\infty}$ of orthogonal G-spectra indexed on the universe $\mathbb{R}^\infty$. We can then use the change of universe functor $\mathcal{I}^U_{\mathbb{R}^\infty}$ to give an equivalence of $\mathcal{S}^{\mathcal{B}G}$ with the category $\mathcal{S}^G_U$ of orthogonal G-spectra indexed on U.

Definition 2.16. Let G be a finite group and $H < G$ be a finite index subgroup with index n. Fix an ordered set of coset representatives $(g_1, \ldots, g_n)$, and let $\alpha\colon G \to \Sigma_n \wr H$ be the homomorphism

$$\alpha(g) = (\sigma, h_1, \ldots, h_n)$$

defined by the relation $gg_i = g_{\sigma(i)}h_i$. The indexed smash-power functor

$$\wedge_H^G\colon \mathcal{S}^{\mathcal{B}H} \longrightarrow \mathcal{S}^{\mathcal{B}G}$$

is defined as the composite

$$\mathcal{S}^{\mathcal{B}H} \xrightarrow{\wedge^n} \mathcal{S}^{\mathcal{B}(\Sigma_n \wr H)} \xrightarrow{\alpha^*} \mathcal{S}^{\mathcal{B}G}.$$

The norm functor

$$N_H^G\colon \mathcal{S}^H_U \longrightarrow \mathcal{S}^G_{U'}$$

is defined to be the composite

$$X \mapsto \mathcal{I}^{U'}_{\mathbb{R}^\infty}(\wedge_H^G(\mathcal{I}^{\mathbb{R}^\infty}_U X)).$$

This definition depends on the choice of coset representatives; however, any other choice gives a canonically naturally isomorphic functor (the isomorphism induced by permuting factors and multiplying each factor by the appropriate element of H). As observed in [20, A.4], in fact it is possible to give a description of the norm which is independent of any choices and is determined instead by the universal property of the left Kan extension. Alternatively, Schwede [36, 9.3] gives another way of avoiding the choice above, using the set $\langle G : H\rangle$ of all choices of ordered sets of coset representatives; $\langle G : H\rangle$ is a free transitive $\Sigma_n \wr H$-set and the inclusion of $(g_1, \ldots, g_n)$ in $\langle G : H\rangle$ induces an isomorphism

$$\wedge_H^G X \cong \langle G : H\rangle_+ \wedge_{\Sigma_n \wr H} X^{\wedge n}.$$

In our work, G will be the cyclic group $C_{nr} < S^1$ and $H = C_r$ (usually for $r = 1$), and we have the obvious choice of coset representatives $g_k = e^{2\pi(k-1)i/nr}$, letting us take advantage of the explicit formulas. In the case $r = 1$, we have the following.

Proposition 2.17. *Let G be a finite group and U a complete G-universe. The norm functor*

$$N_e^G \colon \mathcal{S} \longrightarrow \mathcal{S}_U^G$$

is given by the composite

$$X \mapsto \mathcal{I}_{\mathbb{R}^\infty}^U X^{\wedge G},$$

where $X^{\wedge G}$ denotes the smash power indexed on the set G.

When dealing with commutative ring orthogonal G-spectra, the norm has a particularly attractive formal description [20, A.56], which is a consequence of the fact that the norm is a symmetric monoidal functor.

Theorem 2.18. *Let G be a finite group and let H be a subgroup of G. The norm restricts to the left adjoint in the adjunction*

$$N_H^G \colon \mathcal{C}om^H \leftrightarrows \mathcal{C}om^G \colon \iota_H^*,$$

where ι_H^* denotes the change of group functor along $H < G$.

The relationship of the norm with the geometric fixed point functor is encoded in the diagonal map [20, B.209].

Proposition 2.19. *Let G be a finite group, $H < G$ a subgroup, and $K \lhd G$ a normal subgroup. Let X be an orthogonal H-spectrum. Then there is a natural diagonal map of orthogonal G/K-spectra*

$$\Delta \colon N_{HK/K}^{G/K} \Phi^{H\cap K} X \longrightarrow \Phi^K N_H^G X.$$

(Here we suppress the isomorphism $H/H \cap K \cong HK/K$ from the notation.)

Proof. The construction of Δ is the same as [20, Proposition B.209] after generalizing the corresponding space-level diagonal. To do this, first note that for any based H-space A, there is a natural isomorphism

$$\Delta \colon N_{HK/K}^{G/K} A^{H\cap K} \xrightarrow{\cong} (N_H^G A)^K.$$

For this, it is convenient to model the space-level norm as follows. The space $N_H^G A$ is isomorphic to the subspace of tuples $a = (a_g)_{g\in G} \in \bigwedge_{g\in G} A$ such that $a_{hg} = ha_g$. The left G-action is given by $(k \cdot a)_g = a_{gk}$.

Under this identification, $N_{HK/K}^{G/K} A^{H\cap K}$ consists of tuples $b = (b_{[g]})_{[g]\in G/K}$ of elements in $A^{H\cap K}$ such that $b_{[hg]} = hb_{[g]}$ for $h \in H$. Similarly, $(N_H^G A)^K$ consists of tuples $a = (a_g)_{g\in G}$ such that $a_{hg} = ha_g$ for $h \in H$ and $a_{gk} = a_g$ for $k \in K$. This allows us to define the bijection Δ by $(\Delta b)_g = b_{[g]}$. □

For any particular commutative ring orthogonal spectrum A, the indexed smash-power construction of Definition 2.16 can be carried out in the symmetric monoidal category A-$\mathcal{M}od$. Denote the A-relative indexed smash-power by $(\wedge_A)_H^G$. For X an A-module with H-action, we understand $(\wedge_A)_H^G X$ to be

$$(\wedge_A)_H^G X := \alpha^* X^{\wedge n},$$

where the nth smash power is over A and α^* is as in Definition 2.16. This is an A-module (in $\mathcal{S}^G_{\mathbb{R}^\infty}$). We then have the following definition of the A-relative norm functor:

Definition 2.20. Let A be a commutative ring orthogonal spectrum. Write A_H for the commutative ring orthogonal H-spectrum $\mathcal{I}_{\mathbb{R}^\infty}^{U} A$ obtained by regarding A (with trivial H-action) as an object of $\mathcal{S}^{\mathcal{B}H}$ and applying the change of universe functor, and similarly for A_G. The A-relative norm functor

$${}_AN_H^G \colon A_H\text{-}\mathcal{M}od_U^H \longrightarrow A_G\text{-}\mathcal{M}od_{U'}^G$$

is defined to be the composite

$$X \mapsto \mathcal{I}_{\mathbb{R}^\infty}^{U'}((\wedge_A)_H^G(\mathcal{I}_U^{\mathbb{R}^\infty} X)).$$

The theory of the diagonal map in the A-relative context is somewhat more complicated than in the absolute setting; we explain the details in Section 7.

2.3. **Homotopy theory of orthogonal spectra.** We now review the homotopy theory of orthogonal G-spectra with a focus on discussing the derived functors associated to the point-set constructions of the preceding section. We begin by reviewing the various model structures on orthogonal G-spectra. All of these model structures are ultimately derived from the standard model structure on $\mathcal{T}^G$ (the category of based G-spaces), which we begin by reviewing.

Following the notational conventions of [29], we start with the sets of maps

$$I = \{(G/H \times S^{n-1})_+ \longrightarrow (G/H \times D^n)_+\}$$

and

$$J = \{(G/H \times D^n)_+ \longrightarrow (G/H \times (D^n \times I))_+\},$$

where $n \geq 0$ and H varies over the closed subgroups of G. Recall that there is a compactly generated model structure on the category $\mathcal{T}^G$ in which I and J are the generating cofibrations and generating acyclic cofibrations (e.g., [29, III.1.8]). The weak equivalences and fibrations are the maps $X \to Y$ such that $X^H \to Y^H$ is a weak equivalence or fibration for each closed $H < G$. Transporting this structure levelwise in $\mathcal{V}^G(U)$, we get the level model structure in orthogonal G-spectra.

Proposition 2.21 ([29, III.2.4])**.** *Fix a G-universe U. There is a compactly generated model structure on $\mathcal{S}_U^G$ in which the weak equivalences and fibrations are the maps $X \to Y$ such that each map $X(V) \to Y(V)$ is a weak equivalence or fibration of G-spaces. The sets of generating cofibrations and acyclic cofibrations are given by $I_G^U = \{F_V i \mid i \in I\}$ and $J_G^U = \{F_V j \mid j \in J\}$, where V varies over $\mathcal{V}^G(U)$.*

The level model structure is primarily scaffolding to construct the stable model structures. In order to specify the weak equivalences in the stable model structures, we need to define equivariant homotopy groups.

Definition 2.22. Fix a G-universe U. The homotopy groups of an orthogonal G-spectrum X are defined for a subgroup $H < G$ and an integer q as

$$\pi_q^H(X) = \begin{cases} \operatorname*{colim}_{V<U} \pi_q((\Omega^V X(V))^H) & q \geq 0 \\ \operatorname*{colim}_{\mathbb{R}^{-q}<V<U} \pi_0((\Omega^{V-\mathbb{R}^{-q}} X(V))^H) & q < 0, \end{cases}$$

(see [29, §III.3.2]).

These are the homotopy groups of the underlying G-prespectrum associated to X (via the forgetful functor from orthogonal G-spectra to prespectra). We define the stable equivalences to be the maps $X \to Y$ that induce isomorphisms on all homotopy groups.

Proposition 2.23 ([29, III.4.2])**.** *Fix a G-universe U. The standard stable model structure on $\mathcal{S}_U^G$ is the compactly generated symmetric monoidal model structure with the cofibrations given by the level cofibrations, the weak equivalences the stable equivalences, and the fibrations determined by the right lifting property. The generating cofibrations are given by I_G^U as above, and the generating acyclic cofibrations K are the union of J_G^U and certain additional maps described in* [29, III.4.3].

This model structure lifts to a model structure on the category $\mathcal{A}ss_U^G$ of associative monoids in orthogonal G-spectra.

Theorem 2.24 ([29, III.7.6.(iv)])**.** *Fix a G-universe U. There are compactly generated model structures on $\mathcal{A}ss_U^G$ in which the weak equivalences are the stable equivalences of underlying orthogonal G-spectra indexed on U, the fibrations are the maps which are stable fibrations of underlying orthogonal G-spectra indexed on U, and the cofibrations are determined by the left lifting property.*

To obtain a model structure on commutative ring orthogonal spectra, we also need the "positive" variant of the stable model structure. We define the positive level model structures in terms of the generating cofibrations $(I_G^U)^+ \subset I_G^U$ and $(J_G^U)^+ \subset J_G^U$, consisting of those maps $F_V i$ and $F_V j$ such that the representation V contains a nonzero trivial representation; these also extend to a positive stable model structure.

Theorem 2.25 ([29, III.5.3])**.** *Fix a G-universe U. There are compactly generated model structures on $\mathcal{C}om_U^G$ in which the weak equivalences are the stable equivalences of the underlying orthogonal G-spectra, the fibrations are the maps which are positive stable fibrations of underlying orthogonal G-spectra indexed on U, and the cofibrations are determined by the left lifting property.*

We will also use a variant of the standard stable model structure that can be more convenient when working with the derived functors of the norm. We refer to this as the positive complete stable model structure. See [20, §B.4] for a comprehensive discussion of this model structure, and [39, §A] for a brief review. In order to describe this, denote by $(I_H^{\iota_H^* U})^+$ and $(J_H^{\iota_H^* U})^+$ generating cofibrations for the positive stable model structure on orthogonal H-spectra indexed on the universe $\iota_H^* U$.

Theorem 2.26 ([20, B.63])**.** *Fix a G-universe U. There is a compactly generated symmetric monoidal model structure on $\mathcal{S}^G$ with generating cofibrations and acyclic* *cofibrations the sets* $\{G_+ \wedge_H i \mid i \in (I_H^{\iota_H^* U})^+,\ H < G\}$ *and* $\{G_+ \wedge_H j \mid j \in (J_H^{\iota_H^* U})^+,\ H < G\}$ *respectively. The weak equivalences are the stable equivalences, and the fibrations are determined by the right lifting property.*

We then have corresponding positive complete model structures for $\mathcal{C}om^G$ and $\mathcal{A}ss^G$.

Theorem 2.27 ([20, B.130], [20, B.136 (0908.3724v3)])**.** *Fix a G-universe U. There are compactly generated model structures on $\mathcal{A}ss_U^G$ and $\mathcal{C}om_U^G$ in which the weak equivalences are the stable equivalences of the underlying orthogonal G-spectra, the fibrations are the maps which are positive complete stable fibrations of underlying orthogonal G-spectra indexed on U, and the cofibrations are determined by the left-lifting property.*

For a fixed object A in $\mathcal{C}om_U^G$, there are also lifted model structures on the categories A-$\mathcal{M}od_U^G$ of A-modules, A-$\mathcal{A}lg_U^G$ of A-algebras, and A-$\mathcal{C}om_U^G$ of commutative A-algebras in both the stable and positive complete stable model structures ([29, III.7.6] and [20, B.137]). There are also lifted model structures on the category A-$\mathcal{M}od_U^G$ of A-modules when A is an object of $\mathcal{A}ss_U^G$, but we will not need these. Part of the following is [20, B.137]; the rest follows by standard arguments.

Theorem 2.28. *Fix a G-universe U. Let A be a commutative ring orthogonal G-spectrum indexed on U. There are compactly generated model structures on the categories A-$\mathcal{M}od_U^G$ and A-$\mathcal{A}lg_U^G$ in which the fibrations and weak equivalences are created by the forgetful functors to the stable, complete stable, and positive complete stable model structures on $\mathcal{S}_U^G$. There are compactly generated model structures on A-$\mathcal{C}om_U^G$ in which the fibrations and weak equivalences are created by the forgetful functors to the positive stable and positive complete stable model structures on A-$\mathcal{M}od_U^G$.*

Finally, when dealing with cyclotomic spectra, we need to use variants of these model structures where the stable equivalences are determined by a family of subgroups of G. Recall from [29, IV.6.1] the definition of a family: a family $\mathcal{F}$ is a collection of closed subgroups of G that is closed under taking closed subgroups (and conjugation). We say a map $X \to Y$ is an $\mathcal{F}$-equivalence if it induces an isomorphism on homotopy groups π_*^H for all H in $\mathcal{F}$. All of the model structures described above have analogues with respect to the $\mathcal{F}$-equivalences (e.g., see [29, IV.6.5]), which are built from sets I and J where the cells $(G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$ and $(G/H \times D^n)_+ \to (G/H \times (D^n \times I))_+$ are restricted to $H \in \mathcal{F}$. We record the situation in the following omnibus theorem.

Theorem 2.29. *There are stable, positive stable, and positive complete stable compactly generated model structures on the categories $\mathcal{S}_U^G$ and $\mathcal{A}ss_U^G$ where the weak equivalences are the $\mathcal{F}$-equivalences. There are positive stable and positive complete stable compactly generated model structures on the category $\mathcal{C}om_U^G$ where the weak equivalences are the $\mathcal{F}$-equivalences.*

Let A be a commutative ring orthogonal G-spectrum. There are stable, positive stable, and positive complete stable compactly generated model structures on the categories A-$\mathcal{M}od_U^G$, A-$\mathcal{A}lg_U^G$ where the weak equivalences are the $\mathcal{F}$-equivalences. There are positive stable and positive complete stable compactly generated model structures on A-$\mathcal{C}om_U^G$ where the weak equivalences are the $\mathcal{F}$-equivalences.

We are most interested in case of $G = S^1$ and the families $\mathcal{F}_{\mathrm{Fin}}$ of finite subgroups of S^1 and $\mathcal{F}_p$ of p-subgroups $\{C_{p^n}\}$ of S^1 for a fixed prime p.

2.4. **Derived functors of fixed points and the norm.** We now discuss the use of the model structures described in the previous section to construct the derived functors of the categorical fixed point, geometric fixed point, and norm functors. We begin with the categorical fixed point functor. Since this is a right adjoint, we have right-derived functors computed using fibrant replacement (in any of our available stable model structures):

Theorem 2.30. *Let $H \lhd G$ be a normal subgroup. Then the categorical fixed point functor $(-)^H \colon \mathcal{S}_U^G \to \mathcal{S}_{U^H}^{G/H}$ is a Quillen right adjoint; in particular, it preserves fibrations and weak equivalences between fibrant objects in the stable and positive complete stable model structures on $\mathcal{S}_U^G$.*

As the fibrant objects in the model structures on associative and commutative ring orthogonal spectra are fibrant in the underlying model structures on orthogonal G-spectra, we can derive the categorical fixed points by fibrant replacement in any of the settings in which we work.

In contrast, the geometric fixed point functor admits a Quillen left derived functor (see [29, V.4.5] and [20, B.197]).

Theorem 2.31. *Let H be a normal subgroup of G. The functor $\Phi^H(-)$ preserves cofibrations and weak equivalences between cofibrant objects in the stable, positive stable, and positive complete stable model structures on $\mathcal{S}_U^G$.*

Since the cofibrant objects in the lifted model structures on $\mathcal{A}ss_U^G$ are cofibrant when regarded as objects in $\mathcal{S}_U^G$ [29, III.7.6], an immediate corollary of Theorem 2.31 is that we can derive Φ^H by cofibrant replacement when working with associative ring orthogonal G-spectra. In contrast, the underlying orthogonal G-spectra associated to cofibrant objects in $\mathcal{C}om^G$, in either of the model structures we study, are essentially never cofibrant and the point-set functor Φ^G does not always agree on these with the geometric fixed point functor on the equivariant stable category.

The first part of the following theorem is [20, B.104]; the statement in the case of A-modules is similar and discussed in Section 6.

Theorem 2.32. *The norm $N_H^G(-)$ preserves weak equivalences between cofibrant objects in any of the various stable model structures on $\mathcal{S}^H$, $\mathcal{A}ss^H$, and $\mathcal{C}om^H$.*

Let A be a commutative ring orthogonal spectrum. Then the A-relative norm ${}_AN_e^G(-)$ preserves weak equivalences between cofibrant objects in any of the various stable model structures on A-$\mathcal{M}od$, A-$\mathcal{A}lg$, and A-$\mathcal{C}om$.

The utility of the positive complete model structure is the following homotopical version of Theorem 2.18 [20, B.135].

Theorem 2.33. *Let H be a subgroup of G. The adjunction*

$$N_H^G \colon \mathcal{C}om^H \leftrightarrows \mathcal{C}om^G \colon \iota_H^*$$

is a Quillen adjunction for the positive complete stable structures.

Finally, we have the following result about the derived version of the diagonal map [20, B.209]. We note the strength of the conclusion: the diagonal map is an isomorphism on cofibrant objects, not just a weak equivalence.

Theorem 2.34 ([20, B.209])**.** *Let H be a normal subgroup of G. The diagonal map*

$$\Delta\colon \Phi^H X \longrightarrow \Phi^G N_H^G X$$

is an isomorphism of orthogonal spectra (and in particular a weak equivalence) when X is cofibrant in any of the stable model structures on $\mathcal{S}^H$, or when X is a cofibrant object in $\mathcal{A}ss^H$.

Along the lines of Proposition 2.19, we also need the following more general statement, which essentially follows from the argument of [20, B.209] using the isomorphism given in the proof of Proposition 2.19 to start the induction.

Theorem 2.35. *Let G be a finite group, $H < G$ a subgroup, and $K \lhd G$ a normal subgroup. Let X be an orthogonal H-spectrum. The diagonal map of orthogonal G/K-spectra*

$$\Delta\colon N_{HK/K}^{G/K} \Phi^{H\cap K} X \longrightarrow \Phi^K N_H^G X.$$

is an isomorphism of orthogonal spectra (and in particular a weak equivalence) when X is cofibrant in any of the stable model structures on $\mathcal{S}^H$ or when X is a cofibrant object in $\mathcal{A}ss^H$.

We also need the commutative ring orthogonal spectrum version of Theorem 2.34.

Theorem 2.36. *The diagonal map*

$$\Delta\colon X \longrightarrow \Phi^G N_e^G X$$

is an isomorphism of orthogonal spectra when X is a cofibrant commutative ring orthogonal spectrum.

Proof. The induction in [20, B.209] and monoidality of both sides reduces the statement to the case when $X = (F_V B_+)^{(m)}/\Sigma_m$ where V is a finite-dimensional (non-equivariant) inner product space and B is the disk D^n or sphere S^{n-1}—in particular, when B is a compact Hausdorff space. In general, for a (non-equivariant) orthogonal spectrum T the diagonal map is constructed as follows: for every (non-equivariant) inner product space Z, the universal property of the indexed smash product gives a map of based G-spaces $N_e^G(T(Z)) \to (N_e^G T)(\mathrm{Ind}_e^G Z)$, which restricts on the diagonal to a map

$$T(Z) \longrightarrow (N_e^G T(\mathrm{Ind}_e^G Z))^G = (\mathrm{Fix}^G(N_e^G T))(\mathrm{Ind}_e^G Z), \tag{2.37}$$

and then (passing to the left Kan extension $\mathbb{P}_\phi$ along the fixed point functor $\phi\colon \mathscr{J}_G^G \to \mathscr{J}_e$ on the right) induces a map

$$T(Z) \longrightarrow (\Phi^G(N_e^G T))((\mathrm{Ind}_e^G Z)^G) = (\Phi^G(N_e^G T))(Z).$$

When T is a cell of the form $F_V B_+$, the map in (2.37) factors as

$$\begin{aligned} T(Z) = \mathscr{J}_e(V, Z) \wedge B_+ &\longrightarrow \mathscr{J}_G^G(\mathrm{Ind}_e^G V, \mathrm{Ind}_e^G Z) \wedge B_+ \longrightarrow \\ &(\mathscr{J}_G(\mathrm{Ind}_e^G V, \mathrm{Ind}_e^G Z) \wedge N_e^G(B)_+)^G = (\mathrm{Fix}^G(N_e^G T))(\mathrm{Ind}_e^G Z). \end{aligned}$$

The first map $T(Z) = \mathscr{J}_e(V, Z) \wedge B_+ \to \mathscr{J}_G^G(\mathrm{Ind}_e^G V, \mathrm{Ind}_e^G Z) \wedge B_+$ induces an isomorphism

$$T \longrightarrow \mathbb{P}_\phi(\mathscr{J}_G^G(\mathrm{Ind}_e^G V, -) \wedge B_+) \cong \mathscr{J}_e((\mathrm{Ind}_e^G V)^G, -) \wedge B_+.$$

By passing to quotients, we see that likewise in the case of interest,

$$T = X = (F_V B_+)^{(m)}/\Sigma_m \cong F_{V^m} B_+^m/\Sigma_m,$$

the diagonal map factors as an isomorphism

$$X \longrightarrow \mathbb{P}_\phi(\mathscr{I}_G^G(\operatorname{Ind}_e^G V^m, -) \wedge_{\Sigma_m} B_+^m) \cong \mathscr{I}_e((\operatorname{Ind}_e^G V^m)^G, -) \wedge_{\Sigma_m} B_+^m$$

followed by a map

$$\mathbb{P}_\phi(\mathscr{I}_G^G(\operatorname{Ind}_e^G V^m, -) \wedge_{\Sigma_m} B_+^m) \longrightarrow \Phi^G(N_e^G X)$$

that is the induced map on left Kan extension from a map of $\mathscr{I}_G^G$-spaces

$$\mathscr{I}_G^G(\operatorname{Ind}_e^G V^m, -) \wedge_{\Sigma_m} B_+^m \longrightarrow (\mathscr{I}_G(\operatorname{Ind}_e^G V^m, -) \wedge_{\Sigma_m^{\times G}} N_e^G(B^m)_+)^G.$$

Thus, it suffices to show that the latter map is an isomorphism. This amounts to showing that for each G-inner product space W, the map

$$\mathscr{I}_G^G(\operatorname{Ind}_e^G V^m, W) \wedge_{\Sigma_m} B_+^m \longrightarrow (\mathscr{I}_G(\operatorname{Ind}_e^G V^m, W) \wedge_{\Sigma_m^{\times G}} N_e^G(B^m)_+)^G$$

is a homeomorphism, but since both sides are compact Hausdorff spaces, it amounts to showing that the map is a bijection. The map is clearly an injection. To see that it is a surjection, we note that any non-basepoint x of $\mathscr{I}_G(\operatorname{Ind}_e^G V^m, W) \wedge_{N_e^G \Sigma_m} N_e^G(B^m)_+$ is represented by a collection of points $\vec{b}_h \in B^m$ (indexed on $h \in G$) and isometries $\phi_h \colon V^m \to W$ (indexed on $h \in G$) such that $\bigoplus_h \phi_h \colon \operatorname{Ind}_e^G V^m \to W$ is injective. The point x is G-fixed if for every $g \in G$, there exist an element $\sigma(g)$ in $N_e^G \Sigma_m$ such that

$$g \cdot ((\phi_h), (\vec{b}_h)) = ((\phi_h) \circ \sigma(g)^{-1}, \sigma(g) \cdot (\vec{b}_h)). \tag{2.38}$$

If we write $\sigma(g)$ also in coordinates $\sigma(g) = (\sigma_h(g))$, where

$$(\phi_h) \circ \sigma(g)^{-1} = (\phi_h \circ \sigma_h(g)^{-1}) \qquad \text{and} \qquad \sigma(g) \cdot (\vec{b}_h) = (\sigma_h(g) \cdot \vec{b}_h),$$

then (2.38) becomes

$$\begin{aligned} g \circ \phi_{g^{-1}h} &= \phi_h \circ \sigma_h(g)^{-1} \\ \vec{b}_{g^{-1}h} &= \sigma_h(g) \vec{b}_h. \end{aligned}$$

for all $g, h \in G$, where we have written $h \circ (-)$ to denote the action of h on W (and likewise we use $(-) \circ h$ below to denote the action of h on $\operatorname{Ind}_e^G V^m$). Let

$$\begin{aligned} \phi'_h &= h \circ \phi_1 = \phi_h \circ \sigma_h(h)^{-1} \\ \vec{b}'_h &= \sigma_h(h) \cdot \vec{b}_h = \vec{b}_1, \end{aligned}$$

Then $((\phi'_h), (\vec{b}'_h))$ also represents the element x, with $(\vec{b}'_h)$ clearly a diagonal element. Since

$$\begin{aligned} (g \cdot \phi')_h &= (g \circ \phi' \circ g^{-1})_h \\ &= g \circ \phi'_{g^{-1}h} = g \circ g^{-1} h \circ \phi_1 \\ &= h \circ \phi_1 = \phi'_h, \end{aligned}$$

we also have (ϕ'_h) in the image of $\mathscr{I}_G^G(\operatorname{Ind}_e^G V^m, W)$. $\square$

3. Cyclotomic spectra and topological cyclic homology

In this section, we review the details of the category of p-cyclotomic spectra and the construction of topological cyclic homology (TC). The diagonal maps that naturally arise in the context of the norm go in the opposite direction to the usual cyclotomic structure maps, and so we also explain how to construct TC from these "op"-cyclotomic spectra. In the following, fix a prime p and a complete S^1-universe U.

3.1. **Background on p-cyclotomic spectra.** In this section, we briefly review the point-set description of p-cyclotomic spectra from [5, §4]; we refer the reader to that paper for more detailed discussion.

Definition 3.1 ([5, 4.5])**.** A *p-precyclotomic spectrum* X consists of an orthogonal S^1-spectrum X together with a map of orthogonal S^1-spectra

$$t_p\colon \rho_p^*\Phi^{C_p}X \longrightarrow X.$$

Here ρ_p denotes the p-th root isomorphism $S^1 \to S^1/C_p$. A p-precyclotomic spectrum is a *p-cyclotomic spectrum* when the induced map on the derived functor $\rho_p^* L\Phi^{C_p}X \to X$ is an $\mathcal{F}_p$-equivalence. A morphism of p-cyclotomic spectra consists of a map of orthogonal S^1-spectra $X \to Y$ such that the diagram

$$\begin{array}{ccc} \rho_p^*\Phi^{C_p}X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \rho_p^*\Phi^{C_p}Y & \longrightarrow & Y \end{array}$$

commutes.

Remark 3.2. A *cyclotomic spectrum* is an orthogonal spectrum with p-cyclotomic structures for all primes p satisfying certain compatibility relations; see [5, 4.7–8] for details.

Following [5, 5.4–5], we have the following weak equivalences for p-precyclotomic spectra.

Definition 3.3. A map of p-precyclotomic spectra is a *weak equivalence* when it is an $\mathcal{F}_p$-equivalence of the underlying orthogonal S^1-spectra.

Proposition 3.4 ([5, 5.5])**.** *A map of p-cyclotomic spectra is a weak equivalence if and only if is a weak equivalence of the underlying (non-equivariant) orthogonal spectra.*

3.2. **Constructing TR and TC from a cyclotomic spectrum.** In this section, we give a very rapid review of the definition of TR and TC in terms of the point-set category of cyclotomic spectra described above. The interested reader is referred to the excellent treatment in Madsen's CDM notes [28] for more details on the construction in terms of the classical (homotopical) definition of a cyclotomic spectrum.

For a p-precyclotomic spectrum X, the collection $\{X^{C_{p^n}}\}$ of (point-set) categorical fixed points is equipped with functors

$$F, R\colon X^{C_{p^n}} \longrightarrow X^{C_{p^{n-1}}}$$

for all n, defined as follows. The Frobenius maps F are simply the obvious inclusions of fixed points, and the restriction maps R are constructed as the composites

$$X^{C_{p^n}} \cong (\rho_p^* X^{C_p})^{C_{p^{n-1}}} \xrightarrow{(\rho_p^*\omega)^{C_{p^{n-1}}}} (\rho_p^* \Phi^{C_p} X)^{C_{p^{n-1}}} \xrightarrow{(t_p)^{C_{p^{n-1}}}} X^{C_{p^{n-1}}},$$

where the map ω is the usual map from categorical to geometric fixed points [29, V.4.3]. The Frobenius and restriction maps satisfy the identity $F \circ R = R \circ F$. When X is fibrant in the $\mathcal{F}_p$-model structure (of Theorem 2.29), we then define

$$TR(X) = \operatorname{holim}_R X^{C_{p^n}} \qquad \text{and} \qquad TC(X) = \operatorname{holim}_{R,F} X^{C_{p^n}}.$$

In general, we define TR and TC using a fibrant replacement that preserves the p-precyclotomic structure; such a functor is provided by the main theorems of [5, §5], which construct model structures on p-precyclotomic and p-cyclotomic spectra where the fibrations are the fibrations of the underlying orthogonal S^1-spectra in the $\mathcal{F}_p$-model structure. Alternatively, an explicit construction of a fibrant replacement functor on orthogonal spectra that preserves precyclotomic structures is given in [3, 4.6–7].

Proposition 3.5 (cf. [5, 1.4])**.** *A weak equivalence $X \to Y$ of p-precyclotomic spectra induces weak equivalences $TR(X_f) \to TR(Y_f)$ and $TC(X_f) \to TC(Y_f)$ of orthogonal spectra, where $(-)_f$ denotes any fibrant replacement functor in p-cyclotomic spectra.*

Remark 3.6. We do not yet have an abstract homotopy theory for multiplicative objects in cyclotomic spectra, and the explicit fibrant replacement functor $Q^{\mathcal{I}}$ of [3, 4.6] is lax monoidal but not lax symmetric monoidal. As a consequence, at present we do not know how to convert a p-cyclotomic spectrum which is also a commutative ring orthogonal S^1-spectrum into a cyclotomic spectrum that is a fibrant commutative ring orthogonal S^1-spectrum.

3.3. **Op-precyclotomic spectra.** For our construction of THH based on the norm (in the next section), the diagonal map $X \to \Phi^G N_e^G X$ is in the opposite direction of the cyclotomic structure map needed in the definition of a cyclotomic spectrum. In the case when X is cofibrant (or a cofibrant ring or cofibrant commutative ring orthogonal spectrum), the diagonal map is an isomorphism and so presents no difficulty; in the case when X is just of the homotopy type of a cofibrant orthogonal spectrum, the fact that the structure map goes the wrong way necessitates some technical maneuvering in order to construct TR and TC.

Definition 3.7. An *op-p-precyclotomic spectrum* X consists of an orthogonal S^1-spectrum X together with a map of orthogonal S^1-spectra

$$\gamma \colon X \longrightarrow \rho_p^* \Phi^{C_p} X.$$

An *op-p-cyclotomic spectrum* is an op-p-precyclotomic spectrum where the structure map is an $\mathcal{F}_p$-equivalence. A map of op-p-precyclotomic spectra is a map of orthogonal S^1-spectra that commutes with the structure map. A map of op-p-precyclotomic spectra is a weak equivalence when it is an $\mathcal{F}_p$-equivalence of the underlying orthogonal S^1-spectra.

Note that the definition above uses a condition on the point-set geometric fixed point functor rather than the derived geometric fixed point functor. Such a definition works well when we restrict to those op-p-cyclotomic spectra X where the canonical map in the S^1-equivariant stable category $\rho_p^* L\Phi^{C_p} X \to \rho_p^*\Phi^{C_p} X$ is an $\mathcal{F}_p$-equivalence. For op-p-cyclotomic spectra in this subcategory, a map is a weak equivalence if and only if it is a weak equivalence of the underlying (non-equivariant) orthogonal spectra.

Rather than study the category of op-p-precyclotomic spectra in detail, we simply explain an approach to constructing TR and TC from this data. In what follows, let $(-)_f$ denote a fibrant replacement functor in the $\mathcal{F}_p$-model structure on orthogonal S^1-spectra; to be clear, we assume the given natural transformation $X \to X_f$ is always an acyclic cofibration. Then for an op-p-precyclotomic spectrum X, we get a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\gamma"] \arrow[d, tail, "\simeq"'] & \rho_p^*\Phi^{C_p} X \arrow[d, tail, "\simeq"'] & \\
X_f \arrow[r, "\gamma_f"'] & (\rho_p^*\Phi^{C_p} X)_f \arrow[r, "\simeq"'] & (\rho_p^*\Phi^{C_p}(X_f))_f
\end{tikzcd}$$

where the bottom right horizontal map is a weak equivalence because ρ_p^* and Φ^{C_p} preserve acyclic cofibrations. In place of the restriction map R, we have a zigzag

$$R\colon (X_f)^{C_{p^n}} \longrightarrow ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} \longleftarrow (X_f)^{C_{p^{n-1}}}$$

constructed as the following composite

$$\begin{tikzcd}
(X_f)^{C_{p^n}} \arrow[rr, "\cong"] & & (\rho_p^*(X_f)^{C_p})^{C_{p^{n-1}}} \arrow[r, "\simeq"] & ((\rho_p^*(X_f)^{C_p})_f)^{C_{p^{n-1}}} \arrow[dlll] \\
((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} & ((\rho_p^*\Phi^{C_p} X)_f)^{C_{p^{n-1}}} \arrow[l, "\simeq"] & & (X_f)^{C_{p^{n-1}}}. \arrow[ll]
\end{tikzcd}$$

We can use this as an analogue of TR.

Definition 3.8. Define ${}^{op}TR(X)$ as the homotopy limit of the diagram

$$\cdots \longleftarrow (X_f)^{C_{p^n}} \longrightarrow ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} \longleftarrow (X_f)^{C_{p^{n-1}}} \longrightarrow \cdots$$
$$\cdots \longleftarrow (X_f)^{C_p} \longrightarrow (\rho_p^*\Phi^{C_p}(X_f))_f \longleftarrow X_f.$$

The zigzags R are compatible with the inclusion maps

$$F\colon (X_f)^{C_{p^n}} \longrightarrow (X_f)^{C_{p^{n-1}}}$$

in the sense that the following diagram commutes:

$$\begin{tikzcd}
(X_f)^{C_{p^{n+1}}} \arrow[r] \arrow[dr, "F"'] & ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^n}} \arrow[dr, dotted, "F"] & (X_f)^{C_{p^n}} \arrow[l] \arrow[dr, "F"] & \\
& (X_f)^{C_{p^n}} \arrow[r] & ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} & (X_f)^{C_{p^{n-1}}} \arrow[l]
\end{tikzcd}$$

We can therefore form an analogue of TC.

Definition 3.9. Define ${}^{op}TC(X)$ by taking the homotopy limit over the diagram

$$\begin{tikzcd}
\cdots \arrow[rr, bend left] \arrow[r, bend right] & (X_f)^{C_{p^n}} \arrow[l] \arrow[r] \arrow[rr, bend right] & ((\rho_p^*\Phi^{C_p}(X_f))_f)^{C_{p^{n-1}}} \arrow[rr, bend left] & (X_f)^{C_{p^{n-1}}} \arrow[l] \arrow[r] \arrow[r, bend right] & \cdots
\end{tikzcd}$$

where the middle parts are the R zigzags and the top and bottom the F maps.

This has the expected homotopy invariance property.

Proposition 3.10. *Let $X \to Y$ be a weak equivalence of op-p-precyclotomic spectra. The induced maps ${}^{op}TR(X) \to {}^{op}TR(Y)$ and ${}^{op}TC(X) \to {}^{op}TC(Y)$ are weak equivalences.*

Although we have nothing to say in general about the relationship between p-cyclotomic spectra and op-p-cyclotomic spectra or between ${}^{op}TC$ and TC, we have the following comparison result in the case when X has compatible p-cyclotomic and op-p-precyclotomic structures. This in particular applies when X has the homotopy type of a cofibrant orthogonal spectrum, as we explain in Section 4. We apply it in Section 7 to prove Theorem 1.9.

Proposition 3.11. *Let X be an op-p-precyclotomic spectrum and a p-cyclotomic spectrum and assume that the composite of the two structure maps*

$$\rho_p^* \Phi^{C_p} X \longrightarrow X \longrightarrow \rho_p^* \Phi^{C_p} X$$

is homotopic to the identity. Then there is a zig-zag of weak equivalences connecting $TR(X)$ and ${}^{op}TR(X)$ and a zig-zag of weak equivalences connecting $TC(X)$ and ${}^{op}TC(X)$.

Proof. In the case of the comparison of $TR(X)$ and ${}^{op}TR(X)$, we can use a fibrant replacement of X in the category of cyclotomic spectra to compute both $TR(X)$ and ${}^{op}TR(X)$. It follows that it suffices to show that the homotopy limits of diagrams of fibrant objects of the form

$$\dots \longleftarrow Y_n \xrightarrow{f_n} Y_n' \xleftarrow{g_n^{-1}} Y_{n-1} \longrightarrow \dots \tag{3.12}$$

and

$$\dots \longrightarrow Y_n \xrightarrow{f_n} Y_n' \xrightarrow{g_n} Y_{n-1} \longrightarrow \dots \tag{3.13}$$

are equivalent, where g_n is an equivalence and $g_n^{-1} \circ g_n$ is homotopic to the identity. This kind of rectification argument is standard, although we are not sure of a place in the literature where the precise fact we need is spelled out. We argue as follows. Choosing a homotopy H from the identity to $g_n^{-1} \circ g_n$, we get strictly commuting diagrams of the form

$$\begin{tikzcd}
Y_n \arrow[r, "f_n"] & Y_n' & Y_{n-1} \arrow[l, "g_n^{-1}"'] \arrow[r, "\mathrm{id}"] & Y_{n-1} \\
Y_n \arrow[u, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[r, "f_n \times \{0\}"] & Y_n' \times I \arrow[u, "H"'] \arrow[d, "\pi_1"] & Y_n' \arrow[l, "\mathrm{id} \times \{1\}"'] \arrow[u, "g_n"'] \arrow[d, "\mathrm{id}"] \arrow[r, "g_n"] & Y_{n-1} \arrow[u, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
Y_n \arrow[r, "f_n"] & Y_n' & Y_n' \arrow[l, "\mathrm{id}"'] \arrow[r, "g_n"] & Y_{n-1}.
\end{tikzcd}$$

Note that all the vertical maps are weak equivalences, and therefore the induced maps between the homotopy limits of the rows are both weak equivalences. The homotopy limit of the top row is weakly equivalent to the homotopy limit of (3.12) and the homotopy limit of the bottom row is weakly equivalent to the homotopy limit of (3.13). This completes the comparison of $TR(X)$ and ${}^{op}TR(X)$; the argument for comparing $TC(X)$ and ${}^{op}TC(X)$ is analogous using "ladders" in place of rows. □

Remark 3.14. The following sketches a reformulation of the above argument, showing the equivalence of homotopy limits of (3.12) and (3.13), using the more general-purpose machinery of coherent diagrams. All numbered references in the following are to [27].

As homotopy limits are invariant up to equivalence, we can assume that the objects in the diagram are cofibrant-fibrant and hence that g_n is a homotopy equivalence. If $N(\mathcal{S}^\circ)$ denotes the "simplicial nerve" [1.1.5.5] of the simplicial category of cofibrant-fibrant orthogonal spectra, homotopy limits can be computed in the quasicategory $N(\mathcal{S}^\circ)$ [4.2.4.8].

There is a simplicial set K whose 0-simplices correspond to the objects Y_n and Y'_n, whose 1-simplices correspond to the maps f_n, g_n, and g_n^{-1}, and whose 2-simplices express the composition homotopies $g_n^{-1} \circ g_n \Rightarrow \mathrm{id}$. We have a homotopy coherent diagram of orthogonal spectra indexed on K in the sense of Vogt (or [1.2.6]) expressed as follows:

$$\begin{tikzcd}
\cdots Y_{n+1} \arrow[r, "f_{n+1}"] & Y'_{n+1} \arrow[r, bend left, "g_{n+1}"] \arrow[r, phantom, "\Longleftarrow"] & Y_n \arrow[l, bend left, "g_{n+1}^{-1}"] \arrow[r, "f_n"] & Y'_n \arrow[r, bend left, "g_n"] \arrow[r, phantom, "\Longleftarrow"] & Y_{n-1} \arrow[l, bend left, "g_n^{-1}"] \arrow[r, "f_{n-1}"] & Y'_{n-1} \arrow[r, bend left, "g_{n-1}"] \arrow[r, phantom, "\Longleftarrow"] & Y_{n-2} \cdots \arrow[l, bend left, "g_{n-1}^{-1}"]
\end{tikzcd}$$

We write K^+ for the upper subcomplex containing the edges f_n and g_n, and similarly write K^- for the lower subcomplex containing the f_n and g_n^{-1}.

The inclusion $K^+ \to K$ is an iterated pushout along horn-filling maps $\Lambda_0^2 \to \Delta^2$, so this map is left anodyne [2.0.0.3] and hence final [4.1.1.3]. The restriction from K-diagrams to K^+-diagrams therefore preserves all homotopy limits [4.1.1.8].

We now consider the inclusion $K^- \to K$, which is an iterated pushout along horn-filling maps $\Lambda_2^2 \to \Delta^2$ whose last edges are g_n^{-1}. Because the maps g_n^{-1} are equivalences, the space of extensions of a diagram indexed on K^- to a diagram indexed on K is contractible because the map $\Lambda_2^2 \to \Delta^2$, with the final edge marked as an equivalence, is marked anodyne [3.1.1.1, 3.1.3.4]. In addition, the subspace of homotopy right Kan extensions is also contractible [4.2.4.8, 4.3.2.15]. Therefore, any extension of this K^--diagram to a K-diagram is a homotopy right Kan extension, and the homotopy limit of a homotopy right Kan extension is equivalent to the homotopy limit of the original diagram [4.3.2.8].

The comparison between TC and ${}^{op}TC$ follows by a similar argument. There is a diagram indexed by $K \times \Delta^1$, representing the natural transformation F on the comparison diagram for TR: we define a simplicial set L by identifying $K \times \{1\}$ with $K \times \{0\}$ after a shift. There are subcomplexes L^+ and L^-, generated by $K^+ \times \Delta^1$ and $K^- \times \Delta^1$ respectively, representing the diagrams defining TR and ${}^{op}TR$. As before, the inclusion $L^+ \to L$ is left anodyne and the inclusion $L^- \to L$ only involves extension along equivalences.

4. The construction and homotopy theory of the S^1-norm

In this section, we construct the norm from the trivial group to S^1 and study its basic point-set and homotopy properties. In particular, we prove that under mild hypotheses it gives a model for THH which is cyclotomic. Unlike norms for finite groups, the S^1-norm does not apply to arbitrary orthogonal spectra; instead we need an associative ring structure. In the case when R is commutative, we identify the S^1-norm as the left adjoint of the forgetful functor from commutative ring orthogonal S^1-spectra indexed on a complete universe to (non-equivariant) commutative ring orthogonal spectra.

Throughout this section, we fix a complete S^1-universe U. As in the definition of the norm for finite groups, the (point-set) equivalence of categories $\mathcal{I}_{\mathbb{R}^\infty}^U$ discussed in Section 2.1 will play a key technical role.

For a ring orthogonal spectrum R, let $N_\wedge^{\mathrm{cyc}} R$ denote the cyclic bar construction with respect to the smash product; i.e., the cyclic object in orthogonal spectra with k-simplices

$$[k] \longrightarrow \underbrace{R \wedge R \wedge \ldots \wedge R}_{k+1}$$

and the usual cyclic structure maps induced from the ring structure on R.

Lemma 4.1. *Let R be an object in $\mathcal{A}ss$. Then the geometric realization of the cyclic bar construction $|N_\wedge^{\mathrm{cyc}} R|$ is naturally an object in $\mathcal{S}_{\mathbb{R}^\infty}^{S^1}$.*

Proof. It is well known that the geometric realization of a cyclic space has a natural S^1-action [22, 3.1]. Since geometric realization of an orthogonal spectrum is computed levelwise, it follows by continuous naturality that the geometric realization of a cyclic object in orthogonal spectra has an S^1-action. As noted in Section 2.1, the category $\mathcal{S}_{\mathbb{R}^\infty}^{S^1}$ of orthogonal S^1-spectra indexed on $\mathbb{R}^\infty$ is isomorphic to the category of orthogonal spectra with S^1-actions. $\square$

Using the point-set change of universe functors we can regard this as indexed on the complete universe U. The following definition repeats Definition 1.1 from the introduction.

Definition 4.2. Let R be a ring orthogonal spectrum. Define the functor

$$N_e^{S^1} : \mathcal{A}ss \longrightarrow \mathcal{S}_U^{S^1}$$

to be the composite functor

$$R \mapsto \mathcal{I}_{\mathbb{R}^\infty}^U |N_\wedge^{\mathrm{cyc}} R|.$$

When R is a commutative ring orthogonal spectrum, the usual tensor homeomorphism [12, IX.3.3]

$$|N_\wedge^{\mathrm{cyc}} R| \cong R \otimes S^1$$

yields the following characterization:

Proposition 4.3. *The restriction of $N_e^{S^1}$ to $\mathcal{C}om$ lifts to a functor*

$$N_e^{S^1} : \mathcal{C}om \longrightarrow \mathcal{C}om_U^{S^1}$$

that is left adjoint to the forgetful functor

$$\iota^* : \mathcal{C}om_U^{S^1} \longrightarrow \mathcal{C}om.$$

Proof. To obtain the refinement of $N_e^{S^1}$ to a functor $\mathcal{C}om \to \mathcal{C}om_U^{S^1}$, it suffices to construct a refinement of $|N_\wedge^{\mathrm{cyc}}|$ to a functor

$$|N_\wedge^{\mathrm{cyc}}| : \mathcal{C}om \longrightarrow \mathcal{C}om_{\mathbb{R}^\infty}^{S^1}.$$

We obtain this immediately from the strong symmetric monoidal isomorphism

$$|X_\bullet| \wedge |Y_\bullet| \cong |X_\bullet \wedge Y_\bullet|$$

for simplicial objects $X_\bullet$,$Y_\bullet$ in orthogonal spectra and the easy observation that the map is S^1-equivariant for cyclic objects. Indeed, using the isomorphism

$$\mathbb{P}|X_\bullet| \cong |\mathbb{P}X_\bullet|,$$

we can identify $|N^{\mathrm{cyc}}_\wedge \mathbb{P}X|$ as $\mathbb{P}(X \wedge S^1_+)$. Now using the canonical reflexive coequalizer

$$\mathbb{P}\mathbb{P}R \rightrightarrows \mathbb{P}R \longrightarrow R$$

we can identify $|N^{\mathrm{cyc}}_\wedge R|$ as the reflexive coequalizer

$$\mathbb{P}\mathbb{P}(R \wedge S^1_+) \rightrightarrows \mathbb{P}(R \wedge S^1_+) \longrightarrow R \otimes S^1,$$

constructing the tensor of R with the unbased space S^1 in the category of commutative ring orthogonal spectra. A formal argument now identifies this as the left adjoint to the forgetful functor

$$\iota^* \colon \mathcal{C}om^{S^1}_{\mathbb{R}^\infty} \longrightarrow \mathcal{C}om$$

and it follows that $N_e^{S^1}$ is the left adjoint to the forgetful functor indicated in the statement. □

We now show that the S^1-norm $N_e^{S^1} R$ is a cyclotomic spectrum in orthogonal S^1-spectra. For this, we need to work with the C_n geometric fixed points. Since $|N^{\mathrm{cyc}}_\wedge R|$ is the geometric realization of a cyclic spectrum, the C_n-action can be computed in terms of the edgewise subdivision of the cyclic spectrum $N^{\mathrm{cyc}}_\wedge R$ [7, §1]. Specifically, the nth edgewise subdivision $\mathrm{sd}_n\, N^{\mathrm{cyc}}_\wedge R$ is a simplicial orthogonal spectrum with a simplicial C_n-action such that there is a natural isomorphism of orthogonal S^1-spectra

$$|\,\mathrm{sd}_n\, N^{\mathrm{cyc}}_\wedge R| \cong |N^{\mathrm{cyc}}_\wedge R|,$$

where the S^1-action on the left extends the C_n-action induced from the simplicial structure. For $N_e^{S^1}$ then, taking $\tilde{U} = \iota^*_{C_n} U$, a complete C_n-universe, there is an isomorphism of orthogonal C_n-spectra indexed on $\tilde{U}$

$$\iota^*_{C_n} N_e^{S^1} R \cong \mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}(\iota^*_{C_n} |N^{\mathrm{cyc}}_\wedge R|).$$

This allows us to understand the C_n-action on $N_e^{S^1} R$ in terms of the C_n-action on $|N^{\mathrm{cyc}}_\wedge R|$.

Writing this out, the orthogonal C_n-spectrum $\iota^*_{C_n} N_e^{S^1}(R)$ has a description as the geometric realization of a simplicial orthogonal C_n-spectrum having k-simplices given by norms

$$(N_e^{C_n} R)^{\wedge(k+1)} \cong \mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}(R^{\wedge n(k+1)}),$$

where C_n acts by block permutation on $R^{\wedge n(k+1)}$ and $\tilde{U} = \iota^*_{C_n} U$ (for U a complete S^1-universe). The faces are also given blockwise, with d_i for $0 \leq i \leq k-1$ the map

$$N_e^{C_n}(R^{\wedge(k+1)}) \longrightarrow N_e^{C_n}(R^{\wedge k})$$

on norms induced by the multiplication of the $(i{+}1)$st and $(i{+}2)$nd factors of R. The face map d_k is a bit more complicated and uses both an internal cyclic permutation inside the last $N_e^{C_n} R$ factor (as in Proposition 2.15) and a permutation of the $(k{+}1)$ factors of $(N_e^{C_n} R)^{\wedge(k+1)}$ together with the multiplication d_0. Writing $g = e^{2\pi i/n}$

for the canonical generator of $C_n < S^1$ and α for the natural cyclic permutation on $X^{\wedge(k+1)}$, d_k is the composite

$$(N_e^{C_n} R)^{\wedge(k+1)} \xrightarrow{\mathrm{id}^{\wedge k} \wedge \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} g} (N_e^{C_n} R)^{\wedge(k+1)} \xrightarrow{\alpha} (N_e^{C_n} R)^{\wedge(k+1)} \xrightarrow{d_0} (N_e^{C_n} R)^{\wedge k}.$$

In fact, we have the following concise description of the C_n-action in $N_e^{S^1}$-bimodule terms. We obtain a $(N_e^{C_n} R, N_e^{C_n} R)$-bimodule ${}^g N_e^{C_n} R$, using the standard right action but twisting the left action using $\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} g$. In the following statement, we use the cyclic bar construction with coefficients in a bimodule, q.v. [7, §2].

Theorem 4.4. *Let R be a ring orthogonal spectrum. For any $C_n < S^1$, there is an isomorphism of orthogonal C_n-spectra*

$$\iota_{C_n}^* N_e^{S^1}(R) \cong |N_\wedge^{\mathrm{cyc}}(N_e^{C_n} R, {}^g N_e^{C_n} R)|,$$

where the cyclic bar construction is taken in the symmetric monoidal category $\mathcal{S}_{\tilde{U}}^{C_n}$.

Next we assemble the diagonal maps into a map $N_e^{S^1} R \to \rho_n^* \Phi^{C_n} N_e^{S^1} R$ of orthogonal S^1-spectra. The following lemma (which is just a specialization of Proposition 2.15) provides the basic compatibility we need. (The lemma also follows as an immediate consequence of the much more general rigidity theorem of Malkiewich [30, §3].)

Lemma 4.5. *Let R be an orthogonal spectrum, let $H < S^1$ be a finite subgroup, and let $h \in H$. Then the map $\Phi^H(\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} h) \colon \Phi^H N_e^H R \to \Phi^H N_e^H R$ is the identity.*

We now prove the main theorem about the diagonal map cyclotomic structure.

Theorem 4.6. *Let R be a ring orthogonal spectrum. The diagonal maps*

$$\Delta_n \colon R^{\wedge(k+1)} \longrightarrow \Phi^{C_n} N_e^{C_n} R^{\wedge(k+1)}$$

assemble into natural maps of S^1-spectra

$$\tau_n \colon N_e^{S^1} R \longrightarrow \rho_n^* \Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{U} |N_\wedge^{\mathrm{cyc}} R| \cong \rho_n^* \Phi^{C_n} N_e^{S^1} R.$$

If R is cofibrant or cofibrant as a commutative ring orthogonal spectrum, then these maps are isomorphisms.

Proof. Varying k, we get a map of cyclic objects

$$N_\wedge^{\mathrm{cyc}} R \longrightarrow \Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R$$

and on realization and change of universe, a map

$$N_e^{S^1} R \longrightarrow \mathcal{I}_{\mathbb{R}^\infty}^{U} |\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R|$$

of orthogonal S^1-spectra. The map τ_n is the composite with the evident isomorphism of orthogonal S^1-spectra

$$\mathcal{I}_{\mathbb{R}^\infty}^{U} |\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R| \cong \rho_n^* \Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{U} | \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R| \cong \rho_n^* \Phi^{C_n} N_e^{S^1} R.$$

When R is cofibrant, the maps Δ_n are isomorphisms, and so therefore are the maps τ_n. □

The previous theorem establishes a precyclotomic structure. For the cyclotomic structure, we now just need to compare the pointset geometric fixed point functors with their derived functors.

Theorem 4.7. *Let R be a cofibrant ring orthogonal spectrum or a cofibrant commutative ring orthogonal spectrum. Then for any $C_n < S^1$, the point-set geometric fixed point functor on $N_e^{S^1} R$ computes the left derived geometric fixed point functor*

$$L\Phi^{C_n} N_e^{S^1} R \xrightarrow{\simeq} \Phi^{C_n} N_e^{S^1} R.$$

Moreover,

$$\Phi^{C_n} N_e^{S^1} R \cong \mathcal{I}_{\mathbb{R}^\infty}^{U} |\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R|.$$

Theorem 1.5, the assertion of the cyclotomic structure on $N_e^{S^1} R$ for R a cofibrant ring orthogonal spectrum or cofibrant commutative ring orthogonal spectrum, is now an immediate consequence of the previous theorem and Theorem 4.6. If R only has the homotopy type of a cofibrant object, application of Proposition 3.11 allows us to functorially work with ${}^{op}TR$ and ${}^{op}TC$ as models of TR and TC.

For the proof of the previous theorem, recall that a simplicial object in a category enriched in spaces is said to be *proper* when for each n the map from the kth latching object to the kth level is an h-cofibration. (Recall that an h-cofibration is a map $f\colon X \to Y$ with the homotopy extension property: Any map $\phi\colon Y \to Z$ and any path in the space of maps from X to Z starting at $\phi \circ f$ comes from the restriction of a path in the space of maps from Y to Z starting at ϕ.) The geometric realization of a proper simplicial object (in a topologically cocomplete category) is the colimit of a sequence of pushouts of h-cofibrations. This is relevant to the situation above because of the following lemma.

Lemma 4.8. *Let R be a cofibrant ring orthogonal spectrum or a cofibrant commutative ring orthogonal spectrum. Then for any $C_n < S^1$,*

$$\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R$$

is proper as a simplicial object in $\mathcal{S}_{\tilde{U}}^{C_n}$.

Proof. Since $\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}$ is a topological left adjoint, it preserves pushouts and homotopies, and therefore preserves properness. Thus, it suffices to show that

$$\operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R$$

is a proper simplicial object in $\mathcal{S}_{\mathbb{R}^\infty}^{C_n}$. In the case when R is a cofibrant ring orthogonal spectrum, each level is cofibrant as an orthogonal C_n-spectrum and the inclusion of the latching object is a cofibration. In the case when R is cofibrant as a commutative ring orthogonal spectrum, an argument similar to [12, VII.7.5] shows that the iterated pushouts that form the latching objects are h-cofibrations and the inclusion of the latching object is an h-cofibration. □

Proof of Theorem 4.7. Given the discussion above, we see that under the hypotheses of the theorem, the point-set geometric fixed point functor Φ^{C_n} commutes with geometric realization, giving us the isomorphism

$$\Phi^{C_n} N_e^{S^1} R \cong \mathcal{I}_{\mathbb{R}^\infty}^{U} |\Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R|.$$

To prove that the point-set geometric fixed point functor computes the derived geometric fixed point functor, we just need to see that it does so on each of the objects involved in the sequence of pushouts that constructs the geometric realization. This happens on the levels of $N_\bullet = \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}} \operatorname{sd}_n N_\wedge^{\mathrm{cyc}} R$ because each N_k is the smash product of copies of $N_e^{C_n} R$ and it happens on $N_e^{C_n} R$ by Theorems 2.34 and 2.36. The other pieces are the orthogonal C_n-spectra P_k defined by the pushout diagram

$$\begin{tikzcd} L_k \wedge \partial\Delta^k_+ \arrow[r] \arrow[d] & L_k \wedge \Delta^k_+ \arrow[d] \\ N_k \wedge \partial\Delta^k_+ \arrow[r] & P_k, \end{tikzcd}$$

where L_k denotes the latching object. The point-set geometric fixed point functor computes the derived geometric fixed point functor for each P_k because it does so for each N_k and for each latching object (by induction). $\square$

Finally, we turn to the question of understanding the derived functors of $N_e^{S^1}$. Recall that when dealing with cyclic sets, the S^1-fixed points do not usually carry homotopically meaningful information. As a consequence, we will work with the model structure on $\mathcal{S}_U^{S^1}$ provided by Theorem 2.29 with weak equivalences the $\mathcal{F}_{\mathrm{Fin}}$-equivalences, i.e., the maps which are isomorphisms on the homotopy groups of the (categorical or geometric) fixed point spectra for the finite subgroups of S^1 (irrespective of what happens on the fixed points for S^1). We will now write $\mathcal{S}_U^{S^1,\mathcal{F}_{\mathrm{Fin}}}$ for $\mathcal{S}_U^{S^1}$ to emphasize that we are using the $\mathcal{F}_{\mathrm{Fin}}$-equivalences. We use analogous notation for the categories of ring orthogonal S^1-spectra and commutative ring orthogonal S^1-spectra.

We now observe that $N_e^{S^1}$ admits (left) derived functors when regarded as landing in $\mathcal{S}_U^{S^1,\mathcal{F}_{\mathrm{Fin}}}$ and (in the commutative case) $\mathcal{C}om_U^{S^1,\mathcal{F}_{\mathrm{Fin}}}$. Theorems 4.6 and 4.7 have the following consequence.

Theorem 4.9. *Let $R \to R'$ be a weak equivalence of ring orthogonal spectra where R and R' is each either a cofibrant ring orthogonal spectra or a cofibrant commutative ring orthogonal spectra (four cases). Then the induced map $N_e^{S^1} R \to N_e^{S^1} R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

Proof. Since we have shown that $N_e^{S^1} R$ and $N_e^{S^1} R'$ are cyclotomic spectra and the map is a map of cyclotomic spectra, it suffices to prove that it is a weak equivalence of the underlying non-equivariant spectra, where we are looking at the map $|N_\wedge^{\mathrm{cyc}} R| \to |N_\wedge^{\mathrm{cyc}} R'|$. At each simplicial level, the map $R^{\wedge(k+1)} \to R'^{\wedge(k+1)}$is a weak equivalence and the simplicial objects are proper, so the map on geometric realizations is a weak equivalence. $\square$

In the commutative case, we have the following derived functor result.

Proposition 4.10. *Regarded as a functor on commutative ring orthogonal spectra, the functor $N_e^{S^1}$ is a left Quillen functor with respect to the positive complete model structure on $\mathcal{C}om$ and the $\mathcal{F}_{\mathrm{Fin}}$-model structure on $\mathcal{C}om_U^{S^1}$.*

Proof. The forgetful functor preserves fibrations and acyclic fibrations. $\square$

5. The cyclotomic trace

The modern importance of THH and TC derives from the application of the trace maps $K \to TC$ and $K \to TC \to THH$ to computing algebraic K-theory. In this section, we give a construction of the cyclotomic trace in terms of the norm construction of THH.

First, observe that the constructions of Section 4 and 6 generalize without modification to the setting of categories enriched in orthogonal spectra: Specifically, given a spectral category $\mathcal{C}$ we define the cyclic bar construction as the geometric realization of the cyclic orthogonal spectrum with k-simplices

$$[k] \mapsto \bigvee_{c_0,\dots c_k} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_k,c_{k-1}) \wedge \mathcal{C}(c_0,c_k).$$

This construction gives rise to an orthogonal S^1-spectrum; we have the following analogue of Lemma 4.1.

Lemma 5.1. *Let $\mathcal{C}$ be a category enriched in orthogonal spectra. Then the geometric realization of the cyclic bar construction $|N^{\mathrm{cyc}}_{\wedge}\mathcal{C}|$ is naturally an object in $\mathcal{S}^{S^1}_{\mathbb{R}^\infty}$.*

In order to obtain a cyclotomic structure, as in Theorem 1.5, we need to arrange for the mapping spectra in $\mathcal{C}$ to be cofibrant. Such a spectral category is called "pointwise cofibrant" [3, 2.5]. Following [3, 2.7], we have a cofibrant replacement functor on spectral categories with a fixed object set that in particular produces pointwise cofibrant spectral categories.

Theorem 5.2. *Let $\mathcal{C}$ be a pointwise cofibrant spectral category, then $\mathcal{I}^{U}_{\mathbb{R}^\infty}|N^{\mathrm{cyc}}_{\wedge}\mathcal{C}|$ has a natural structure of a cyclotomic spectrum.*

Proof. Much of this goes through just as in Section 4. The only real divergence is that although levelwise

$$\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}\,\mathrm{sd}_n\, N^{\mathrm{cyc}}_{\wedge}\mathcal{C}$$

is no longer given as a smash of norms, the diagonal isomorphisms

$$\bigvee_{c_0,\dots c_k} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_k,c_{k-1}) \wedge \mathcal{C}(c_0,c_k)$$
$$\longrightarrow \Phi^{C_n}\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}\Big(\bigvee_{c_0,\dots c_q} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_q,c_{q-1}) \wedge \mathcal{C}(c_0,c_q)\Big)$$

(where $q = n(k+1)-1$) arise as the composite of the diagonal isomorphism

$$\bigvee_{c_0,\dots c_k} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_k,c_{k-1}) \wedge \mathcal{C}(c_0,c_k)$$
$$\longrightarrow \Phi^{C_n}N_e^{C_n}\Big(\bigvee_{c_0,\dots c_k} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_k,c_{k-1}) \wedge \mathcal{C}(c_0,c_k)\Big)$$

and the isomorphism

$$\Phi^{C_n}N_e^{C_n}\Big(\bigvee_{c_0,\dots c_k} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_k,c_{k-1}) \wedge \mathcal{C}(c_0,c_k)\Big)$$
$$\longrightarrow \Phi^{C_n}\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty}\Big(\bigvee_{c_0,\dots c_q} \mathcal{C}(c_1,c_0) \wedge \mathcal{C}(c_2,c_1) \wedge \dots \wedge \mathcal{C}(c_q,c_{q-1}) \wedge \mathcal{C}(c_0,c_q)\Big)$$

induced by the inclusion

$$\Big(\bigvee_{c_0,\ldots c_k} \mathcal{C}(c_1,c_0)\wedge\mathcal{C}(c_2,c_1)\wedge\ldots\wedge\mathcal{C}(c_k,c_{k-1})\wedge\mathcal{C}(c_0,c_k)\Big)^{\wedge(n)}$$
$$\longrightarrow \bigvee_{c_0,\ldots c_q} \mathcal{C}(c_1,c_0)\wedge\mathcal{C}(c_2,c_1)\wedge\ldots\wedge\mathcal{C}(c_q,c_{q-1})\wedge\mathcal{C}(c_0,c_q)$$

of the summands where $c_{i(k+1)+j} = c_j$ for all $0 < i < n$, $0 \le j < k+1$. □

We simplify notation by writing $THH(\mathcal{C})$ for the orthogonal S^1-spectrum or cyclotomic spectrum $\mathcal{I}_{\mathbb{R}^\infty}^U |N_\wedge^{\mathrm{cyc}}\mathcal{C}|$. From this point, the construction of TR and TC proceeds identically with the case of ring orthogonal spectra.

We now turn to the construction of the cyclotomic trace. The trace map is induced from the inclusion of objects map

$$\mathrm{ob}(\mathcal{C}) \longrightarrow |N_\wedge^{\mathrm{cyc}}\mathcal{C}|$$

that takes x to the identity map $x \to x$ in the zero-skeleton of the cyclic bar construction. To make use of this for K-theory, we use the Waldhausen construction of K-theory as the geometric realization of the nerve of the multisimplicial spectral category $w_\bullet S_\bullet^{(n)}\mathcal{C}$ and consider the bispectrum $THH(w_\bullet S_\bullet^{(n)}\mathcal{C})$. The construction now proceeds in the usual way (e.g., see [4, 1.2.5]).

6. A description of relative THH as the relative S^1-norm

In this section, we extend the work of Section 4 to the setting of A-algebras for a commutative ring orthogonal spectrum A. The category of A-modules is a symmetric monoidal category with respect to $\wedge_A$, the smash product over A. As explained in [20, §A.3], the construction of the indexed smash product can be carried out in the symmetric monoidal category of A-modules. Our construction of relative THH will use the associated A-relative norm.

We will write A_G to denote the commutative ring orthogonal G-spectrum obtained by regarding A as having trivial G-action; i.e., $A_G = \mathcal{I}_{\mathbb{R}^\infty}^U A$. This is a commutative ring orthogonal G-spectrum since $\mathcal{I}_{\mathbb{R}^\infty}^U$ is a symmetric monoidal functor. For example, if A is the sphere spectrum then A_G is the G-equivariant sphere spectrum.

Warning 6.1. Although $\mathcal{I}_{\mathbb{R}^\infty}^U$ performs the (derived) change of universe on stable categories for cofibrant orthogonal spectra, and $\mathcal{I}_{\mathbb{R}^\infty}^U$ has a left derived functor on commutative ring orthogonal spectra (Proposition 6.2 below), the underlying object in the stable category of A_G is not the derived change of universe applied to A except in rare cases like $A = S$; see Example 6.3 below. As a consequence, in the following result the comparison map between the left derived functor and the left derived functor of $\mathcal{I}_{\mathbb{R}^\infty}^U\colon \mathcal{S} \to \mathcal{S}_U^G$ is not an isomorphism.

Proposition 6.2. *The functor $\mathcal{I}_{\mathbb{R}^\infty}^U\colon \mathcal{C}om \to \mathcal{C}om_U^G$ is a Quillen left adjoint.*

Proof. The functor in question is the composite of the inclusion of $\mathcal{C}om$ in $\mathcal{C}om_{\mathbb{R}^\infty}^G$ as the objects with trivial G-action (which is Quillen left adjoint to the G-fixed point functor) and the Quillen left adjoint $\mathcal{I}_{\mathbb{R}^\infty}^U\colon \mathcal{C}om_{\mathbb{R}^\infty}^G \to \mathcal{C}om_U^G$. The Quillen right adjoint is the composite $(-)^G \circ \mathcal{I}_U^{\mathbb{R}^\infty}$. □

Example 6.3. For X a non-equivariant positive cofibrant orthogonal spectrum, $\mathbb{P}X$ is a cofibrant commutative ring orthogonal spectrum. We have that $\mathcal{I}_{\mathbb{R}^\infty}^U \mathbb{P}X = \mathbb{P}\mathcal{I}_{\mathbb{R}^\infty}^U X$, whose underlying object in the equivariant stable category is isomorphic to $\bigvee E_G\Sigma_{n+} \wedge_{\Sigma_n} \mathcal{I}_{\mathbb{R}^\infty}^U X^{\wedge n}$ by [29, III.8.4], [20, B.117]. On the other hand, the underlying object of $\mathbb{P}X$ in the non-equivariant stable category is isomorphic to $\bigvee E\Sigma_{n+} \wedge_{\Sigma_n} X^{\wedge n}$, which the derived functor on stable categories takes to $\bigvee E\Sigma_{n+} \wedge_{\Sigma_n} \mathcal{I}_{\mathbb{R}^\infty}^U X^{\wedge n}$. In general, the commutative ring derived functor is related to the stable category derived functor by change of operads along $E\Sigma_* \to E_G\Sigma_*$, cf. [2].

For an A-algebra R, let $N^{\mathrm{cyc}}_{\wedge_A} R$ denote the cyclic bar construction with respect to the smash product over A. The same proof as Lemma 4.1 implies the following.

Lemma 6.4. *Let R be an object in A-$\mathcal{A}lg$. Then the geometric realization of the cyclic bar construction $|N^{\mathrm{cyc}}_{\wedge_A} R|$ is naturally an object in A-$\mathcal{M}od^{S^1}_{\mathbb{R}^\infty}$.*

Using the point-set change of universe functors we can turn this into an orthogonal S^1-spectrum indexed on the complete universe U.

Definition 6.5. Let R be a ring orthogonal spectrum. Define the functor

$$ {}_AN_e^{S^1} \colon A\text{-}\mathcal{A}lg \longrightarrow A\text{-}\mathcal{M}od_U^{S^1} $$

as the composite

$$ {}_AN_e^{S^1} R = \mathcal{I}_{\mathbb{R}^\infty}^U |N^{\mathrm{cyc}}_{\wedge_A} R|. $$

The argument for Proposition 4.3 also proves the following relative version.

Proposition 6.6. *The restriction of ${}_AN_e^{S^1}$ to commutative A-algebras lifts to a functor*

$$ {}_AN_e^{S^1} \colon A\text{-}\mathcal{C}om \longrightarrow A_{S^1}\text{-}\mathcal{C}om_U^{S^1} $$

that is left adjoint to the forgetful functor

$$ \iota^* \colon A_{S^1}\text{-}\mathcal{C}om_U^{S^1} \longrightarrow A\text{-}\mathcal{C}om $$

We now make a non-equivariant observation about relative THH (ignoring the group action temporarily) that informs our description of the equivariant structure. Similar theorems have appeared previously in the literature, e.g., [33, §5].

Lemma 6.7. *Let R be an A-algebra in orthogonal spectra. Then there is an isomorphism*

$$ {}_STHH(R) \wedge_{{}_STHH(A)} A \cong {}_ATHH(R). $$

Proof. Commuting the smash product with geometric realization reduces the lemma to verifying the formula

$$ (R \wedge R \wedge \ldots \wedge R) \wedge_{A \wedge A \wedge \ldots \wedge A} A \cong R \wedge_A R \wedge_A \ldots \wedge_A R, $$

which is a straightforward calculation. □

We now generalize Lemma 6.7 to take advantage of the equivariant structure.

Proposition 6.8. *Let G be a finite group. Let A be a commutative ring orthogonal spectrum and M an A-module. The A-relative norm is obtained by base-change from the usual norm:*

$$ {}_AN_e^G M \cong N_e^G M \wedge_{N_e^G A} A_G $$

Proof. Since M is an A-module, we know that $N_e^G M$ is an $N_e^G A$-module (in the category $\mathcal{S}_U^G$), using the fact that the norm is a symmetric monoidal functor [20, A.53]. The right hand side is the extension of scalars along the canonical map $N_e^G A \to A_G$ obtained as the adjoint of the natural (non-equivariant) map $A \to A_G$. Because the map $N_e^G(-) \to {}_A N_e^G(-)$ is a monoidal natural transformation, we obtain a canonical map from $N_e^G M \wedge_{N_e^G A} A_G$ to ${}_A N_e^G M$; this map is an isomorphism because it is clearly an isomorphism after forgetting the equivariance. □

Extending this to S^1, if R is an A-algebra we have the following characterization of relative THH as an S^1-spectrum that follows by essentially the same argument.

Proposition 6.9. *Let R be an A-algebra in orthogonal spectra. Then we have an isomorphism*

$$ {}_A N_e^{S^1} R \cong N_e^{S^1} R \wedge_{N_e^{S^1} A} A_{S^1} $$

We now turn to the homotopical analysis of ${}_A N_e^{S^1}$. The following theorem asserts that the left derived functor of ${}_A N_e^{S^1}$ exists.

Theorem 6.10. *Let $R \to R'$ be a weak equivalence of cofibrant A-algebras. Then the induced map ${}_A N_e^{S^1} R \to {}_A N_e^{S^1} R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

To prove this theorem, it suffices to prove the following theorem, which in particular implies Proposition 1.10.

Theorem 6.11. *Let R be a cofibrant A-algebra. The smash product $N_e^{S^1} R \wedge_{N_e^{S^1} A} A_{S^1}$ represents the derived smash product in the $\mathcal{F}_{\mathrm{Fin}}$-model structure.*

Proof. Let N be a cofibrant $N_e^{S^1} A$-module approximation of $N_e^{S^1} R$; the assertion is that the map

$$ N \wedge_{N_e^{S^1} A} A_{S^1} \longrightarrow N_e^{S^1} R \wedge_{N_e^{S^1} A} A_{S^1} $$

is a $\mathcal{F}_{\mathrm{Fin}}$-equivalence. We compare to the bar construction: Let $B(N, N_e^{S^1} A, A_{S^1})$ be the geometric realization of the simplicial object with k-simplices

$$ N \wedge \underbrace{N_e^{S^1} A \wedge \cdots \wedge N_e^{S^1} A}_{k} \wedge A_{S^1}, $$

and similarly for $B(N_e^{S^1} R, N_e^{S^1} A, A_{S^1})$. Then we have a commutative diagram

$$ \begin{array}{ccc} B(N, N_e^{S^1} A, A_{S^1}) & \longrightarrow & N \wedge_{N_e^{S^1} A} A_{S^1} \\ \downarrow & & \downarrow \\ B(N_e^{S^1} R, N_e^{S^1} A, A_{S^1}) & \longrightarrow & N_e^{S^1} R \wedge_{N_e^{S^1} A} A_{S^1} \end{array} $$

We want to show that the righthand map is a $\mathcal{F}_{\mathrm{Fin}}$-equivalence; we show that the remaining three maps are $\mathcal{F}_{\mathrm{Fin}}$-equivalences. We apply the change of groups functor $\iota_{C_n}^*$ and show that they are weak equivalences of orthogonal C_n-spectra. Since $\iota_{C_n}^*$ commutes with smash product and geometric realization, we have isomorphisms

$$ \iota_{C_n}^* B(N, N_e^{S^1} A, A_{S^1}) \cong B(\iota_{C_n}^* N, \iota_{C_n}^* N_e^{S^1} A, A_{C_n}) $$
$$ \iota_{C_n}^* (N \wedge_{N_e^{S^1} A} A_{S^1}) \cong \iota_{C_n}^* N \wedge_{\iota_{C_n}^* N_e^{S^1} A} A_{C_n} $$

and similarly for $N_e^{S^1}R$ in place of N.

Before proceeding, we note that $\iota_{C_n}^* N_e^{S^1}A$ and $\iota_{C_n}^* N_e^{S^1}R$ are flat in the sense of [20, B.15]. This can be seen as follows. $N_e^{C_n}A$ is flat by [20, B.147] and $N_e^{C_n}R$ is flat being the sequential colimit of pushouts over h-cofibrations of flat objects. Likewise, $\iota_{C_n}^* N_e^{S^1}A$, $\iota_{C_n}^* N_e^{S^1}R$, and $\iota_{C_n}^* N$ are sequential colimits of pushouts over h-cofibrations of objects that are flat, q.v. Theorem 4.4 for $N_e^{S^1}A$ and $N_e^{S^1}R$. As an immediate consequence, we see that the map

$$B(N, N_e^{S^1}A, A_{S^1}) \longrightarrow B(N_e^{S^1}R, N_e^{S^1}A, A_{S^1})$$

is a $\mathcal{F}_{\mathrm{Fin}}$-equivalence as

$$B(\iota_{C_n}^* N, \iota_{C_n}^* N_e^{S^1}A, \iota_{C_n}^* A_{S^1}) \longrightarrow B(\iota_{C_n}^* N_e^{S^1}R, \iota_{C_n}^* N_e^{S^1}A, \iota_{C_n}^* A_{S^1})$$

is a weak equivalence on each simplicial level and the simplicial objects are proper.

To see that $\iota_{C_n}^* B(N, N_e^{S^1}A, A_{S^1}) \to \iota_{C_n}^*(N \wedge_{N_e^{S^1}A} A_{S^1})$ is a weak equivalence, let M be a cofibrant $N_e^{S^1}A$-module approximation of A_{S^1}. Since smash product commutes with geometric realization, we have compatible isomorphisms

$$B(N, N_e^{S^1}A, N_e^{S^1}A) \wedge_{N_e^{S^1}A} M \cong B(N, N_e^{S^1}A, M)$$
$$B(N, N_e^{S^1}A, N_e^{S^1}A) \wedge_{N_e^{S^1}A} A_{S^1} \cong B(N, N_e^{S^1}A, A_{S^1})$$

Now we have a commutative diagram

$$\begin{array}{ccccc}
B(N, N_e^{S^1}A, M) & \cong & B(N, N_e^{S^1}A, N_e^{S^1}A) \wedge_{N_e^{S^1}A} M & \longrightarrow & N \wedge_{N_e^{S^1}A} M \\
\downarrow & & \downarrow & & \downarrow \\
B(N, N_e^{S^1}A, A_{S^1}) & \cong & B(N, N_e^{S^1}A, N_e^{S^1}A) \wedge_{N_e^{S^1}A} A_{S^1} & \longrightarrow & N \wedge_{N_e^{S^1}A} A_{S^1}.
\end{array}$$

with the bottom composite map becoming the map in question after applying $\iota_{C_n}^*$. The lefthand map becomes a weak equivalence after applying $\iota_{C_n}^*$ because both $\iota_{C_n}^* N$ and $\iota_{C_n}^* N_e^{S^1}A$ are flat. The top map is a weak equivalence because $(-) \wedge_{N_e^{S^1}A} M$ preserves the weak equivalence $B(N, N_e^{S^1}A, N_e^{S^1}A) \to N$ and the righthand map is a weak equivalence because $N \wedge_{N_e S^1} (-)$ preserves the weak equivalence $M \to A_{S^1}$.

Finally, to see that the map

$$\iota_{C_n}^* B(N_e^{S^1}R, N_e^{S^1}A, A_{S^1}) \longrightarrow \iota_{C_n}^* N_e^{S^1}R \wedge_{N_e^{S^1}A} A_{S^1}$$

is a weak equivalence, we apply Theorem 4.4 to observe that it is induced by a map of simplicial objects

$$\begin{aligned}
&B(N_\wedge^{\mathrm{cyc}}(N_e^{C_n}R, {}^g N_e^{C_n}R), N_\wedge^{\mathrm{cyc}}(N_e^{C_n}A, {}^g N_e^{C_n}A), A_{C_n}) \\
&\qquad\qquad \longrightarrow N_\wedge^{\mathrm{cyc}}(N_e^{C_n}R, {}^g N_e^{C_n}R) \wedge_{N_\wedge^{\mathrm{cyc}}(N_e^{C_n}A, {}^g N_e^{C_n}A)} A_{C_n}.
\end{aligned}$$

Here at the kth level, the map is

$$\begin{aligned}
&B((N_e^{C_n}R)^{\wedge(k)} \wedge {}^g N_e^{C_n}R, (N_e^{C_n}A)^{\wedge(k)} \wedge {}^g N_e^{C_n}A, A_{C_n}) \\
&\qquad\qquad \longrightarrow ((N_e^{C_n}R)^{\wedge(k)} \wedge {}^g N_e^{C_n}R) \wedge_{(N_e^{C_n}A)^{\wedge(k)} \wedge {}^g N_e^{C_n}A} A_{C_n},
\end{aligned}$$

which is a weak equivalence since $(N_e^{C_n} R)^{\wedge(k)} \wedge {}^g N_e^{C_n} R$ is flat as a module over $(N_e^{C_n} A)^{\wedge(k)} \wedge {}^g N_e^{C_n} A$. □

Similarly, we can extend the homotopical statement of Proposition 4.10 to the relative setting.

Proposition 6.12. *Regarded as a functor on commutative A-algebras, the functor ${}_A N_e^{S^1}$ is a left Quillen functor with respect to the positive complete model structure on A-$\mathcal{C}om$ and the $\mathcal{F}_{\mathrm{Fin}}$-model structure on A_{S^1}-$\mathcal{C}om_U^{S^1}$.*

Proposition 6.13. *Let $R \to R'$ be a weak equivalence of A-algebras where R is cofibrant and R' is a cofibrant commutative A-algebra. Then the induced map ${}_A N_e^{S^1} R \to {}_A N_e^{S^1} R'$ is an $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

Proof. By Theorem 6.11,

$$ {}_A N_e^{S^1} R \cong N_e^{S^1} R \wedge_{N_e^{S^1} A} A_{S_1} $$

represents the derived smash product. Since $N_e^{S^1} R'$ is cofibrant as a commutative $N_e^{S^1} A$-algebra,

$$ {}_A N_e^{S^1} R' \cong N_e^{S^1} R' \wedge_{N_e^{S^1} A} A_{S_1} $$

also represents the derived smash product. □

7. The op-p-precyclotomic structure on ${}_A N_e^{S^1} R$

One application of the perspective of THH as the S^1-norm is the construction of TC-like functors built from ${}_A N_e^{S^1} R$, which we discuss in this section. The authors previously believed that ${}_A N_e^{S^1}$ could in general be endowed with a cyclotomic structure. However, except for very special choices for A (such as $A = S$), this is not correct due to the subtleties of the behavior of the derived functor of change of universe on commutative ring orthogonal spectra, q.v. Example 6.3 above and Example 7.6 below.

Although ${}_A N_e^{S^1}$ does not generally have a cyclotomic structure, it does naturally have an op-precyclotomic structure in A_{S^1}-modules: the geometric fixed point functor Φ^H is lax symmetric monoidal, and therefore gives rise to a functor

$$ \Phi^H \colon A_G\text{-}\mathcal{M}od_U^G \longrightarrow (\Phi^H A_G)\text{-}\mathcal{M}od_{U^H}^{G/H} $$

when H is normal in G. In the case of a finite subgroup $C_n < S^1$, for an A_{S^1}-module X, we have that $\Phi^{C_n} X$ is an orthogonal S^1/C_n-spectrum and a module over $\Phi^{C_n} A_{S^1}$. In fact, it is a module over A_{S^1/C_n}.

Proposition 7.1. *Let A be a (non-equivariant) cofibrant commutative ring orthogonal spectrum and let H be a normal subgroup of a compact lie group G. There is a natural map of commutative ring orthogonal G/H-spectra $A_{G/H} \to \Phi^H A_G$.*

Proof. Let X be an arbitrary non-equivariant orthogonal spectrum and write X_G for the application of the point-set functor $\mathcal{I}_{\mathbb{R}^\infty}^U$. We write $\Phi^H X_G$ as the coequalizer

$$ \bigvee_{V,W<U} \mathscr{J}_G^U(V,W)^H \wedge F_{W^H} S^0 \wedge (X_G(V))^H \rightrightarrows \bigvee_{V<U} F_{V^H} S^0 \wedge (X_G(V))^H $$

in orthogonal G/H-spectra. For V an H-fixed G-inner product space, we can also regard V as a G/H-inner product space, and we have

$$X_{G/H}(V) \cong X_G(V) = (X_G(V))^H.$$

Writing $X_{G/H}$ as the coequalizer

$$\bigvee_{V,W<U^H} \mathscr{J}^{U^H}_{G/H}(V,W) \wedge F_W S^0 \wedge X_{G/H}(V) \rightrightarrows \bigvee_{V<U^H} F_V S^0 \wedge X_{G/H}(V),$$

we get a canonical natural map of orthogonal G/H-spectra $\lambda\colon X_{G/H} \to \Phi^H X_G$. The symmetric monoidal transformation $\Phi^H X_G \wedge \Phi^H Y_G \to \Phi^H(X_G \wedge Y_G)$ is induced by the natural map

$$F_{V_1^H} S^0 \wedge (X_G(V_1))^H \wedge F_{V_2^H} S^0 \wedge (Y_G(V_2))^H \longrightarrow F_{(V_1\oplus V_2)^H} S^0 \wedge ((X_G \wedge Y_G)(V_1 \oplus V_2))^H,$$

and we see that λ is also lax symmetric monoidal. Applying these observations to the commutative ring orthogonal spectrum A and the multiplication map $A \wedge A \to A$, we see that λ induces a map of commutative ring orthogonal G/H-spectra $A_{G/H} \to \Phi^H A_G$, natural in the commutative ring orthogonal spectrum A. □

We now specialize this to the subgroup $C_n < S^1$ and an A_{S^1}-module X. Pulling back along the nth root isomorphism $\rho_n\colon S^1 \to S^1/C_n$ gives rise to an orthogonal S^1-spectrum $\rho_n^* \Phi^{C_n} X$ that is a module over $A_{S^1} \cong \rho_n^* A_{S^1/C_n}$.

Definition 7.2. An op-p-precyclotomic spectrum relative to A consists of an A_{S^1}-module X together with a map of A_{S^1}-modules

$$\gamma\colon X \longrightarrow \rho_p^* \Phi^{C_p} X.$$

Following the development in the absolute setting, we can now establish (using the same argument as for Theorem 4.6) the op-p-precyclotomic structure on ${}_A N_e^{S^1} R$. In order to do this, we need to construct an A-relative version of the diagonal map

$$\Delta_A\colon X \longrightarrow \Phi^G {}_A N_e^G X.$$

(a special case of the analogue of Proposition 2.19), which we can now state using Proposition 7.1.

Proposition 7.3. *Let A be a commutative ring orthogonal spectrum and let X be an A-module. For any finite group G, there is a natural diagonal map of A-modules*

$$\Delta_A\colon X \longrightarrow \Phi^G {}_A N_e^G X.$$

Proof. The map itself is constructed as the composite

$$X \xrightarrow{\Delta} \Phi^G N_e^G X \longrightarrow \Phi^G(A_G \wedge_{N_e^G A} N_e^G X) \cong \Phi^G {}_A N_e^G X,$$

where the last isomorphism is Proposition 6.8. Monoidality of Δ implies that Δ is a map of A-modules, where A acts on $\Phi^G N_e^G X$ via the map $\Delta\colon A \to \Phi^G N_e^G A$ (which is a map of commutative ring orthogonal spectra). Since the composite map of commutative ring orthogonal spectra $A \to \Phi^G N_e^G A \to \Phi^G A_G$ is the canonical map, Δ_A is also a map of A-modules. □

We also need the following analogue of Lemma 4.5.

Lemma 7.4. *Let R be an orthogonal A-algebra, let $H < S^1$ be a finite subgroup, and let $h \in H$. Then the map $\Phi^H(\mathcal{I}^{\tilde{U}}_{\mathbb{R}^\infty} h)\colon \Phi^H {}_A N_e^H R \to \Phi^H {}_A N_e^H R$ is the identity.*

Putting everything together, we have the following analogue of Theorem 4.6.

Theorem 7.5. *Let R be an A-algebra. The diagonal maps*

$$\Delta_n \colon R^{\wedge_A (k+1)} \longrightarrow \Phi^{C_n}\, {}_A N_e^{C_n} R^{\wedge_A (k+1)}$$

assemble into natural maps of A_{S^1}-modules

$$\tau_n \colon {}_A N_e^{S^1} R \longrightarrow \rho_n^* \Phi^{C_n} \mathcal{I}_{\mathbb{R}^\infty}^U |\operatorname{sd}_n N^{\mathrm{cyc}}_{\wedge_A} R|.$$

The following example indicates that in general we cannot do better than an op-p-precyclotomic structure.

Example 7.6. In the previous theorem, consider the case when $R = A$ and $A = \mathbb{P}F_{\mathbb{R}} S^0$ is the free commutative ring orthogonal spectrum on $F_{\mathbb{R}} S^0 \simeq S^{-1}$. When $n = 2$,

$$\begin{aligned} \operatorname{Fix}^{C_2} \mathbb{P}F_{\mathbb{R}} S^0(W) &= \bigvee_m (\mathscr{J}_{S^1}(\mathbb{R}^m, W)/\Sigma_m)^{C_2} \\ &\cong \bigvee_m \Big(\bigvee_{f \colon C_2 \to \Sigma_m} \mathscr{J}_{S^1}(f^*\mathbb{R}^m, W)^{C_2} \Big)/\Sigma_m, \end{aligned}$$

where the inner sum is over homomorphisms $f \colon C_2 \to \Sigma_m$ and Σ_m acts on the set of such f by conjugation (as well as acting on $\mathbb{R}^m$ by permuting coordinates). Here $f^*\mathbb{R}^m$ denotes $\mathbb{R}^m$ with C_2-action coming from f and the coordinate permutation action of Σ_m. We can then calculate

$$\Phi^{C_2} \mathbb{P}F_{\mathbb{R}} S^0 \cong \bigvee_m \Big(\bigvee_{f \colon C_2 \to \Sigma_m} F_{(f^*\mathbb{R}^m)^{C_2}} S^0 \Big)/\Sigma_m.$$

The summands with f the trivial map contribute a summand of A_{S^1/C_2}, but the remaining summands make non-trivial contributions of orthogonal G/H-spectra of the form $F_{(\mathbb{R}^m)^\sigma} S^0/Z(\sigma)$ where σ is an order 2 element of Σ_m, $Z(\sigma)$ is its centralizer, and $(f^*\mathbb{R}^m)^\sigma$ is its fixed points. In this case we see that the natural map of Proposition 7.1 is split, and in general it is split for free commutative ring orthogonal spectra, but the splitting is not natural and so does not extend to a splitting for arbitrary commutative ring orthogonal spectra A.

Now, using the relative analogues of Definitions 3.8 and 3.9, we obtain analogues of TR and TC which we denote ${}^{op}_{A}TR$ and ${}^{op}_{A}TC$. These constructions are functorial in the following sense. Suppose that we are given a map of commutative ring orthogonal spectra $\phi \colon A \to A'$ and an A'-algebra R. Pullback along ϕ allows us to regard R as an A-algebra, and this gives rise to induced maps on relative THH, TR, and TC.

Proposition 7.7. *Let R be a (commutative) A'-algebra and $\phi \colon A \to A'$ a map of commutative ring orthogonal spectra. Then we have a map*

$${}_A N_e^{S^1} R \longrightarrow {}_{A'} N_e^{S^1} R$$

of op-p-precyclotomic spectra that gives rise to maps

$${}^{op}_{A}TR(R) \longrightarrow {}^{op}_{A'}TR(R) \qquad \textit{and} \qquad {}^{op}_{A}TC(R) \longrightarrow {}^{op}_{A'}TC(R).$$

Proof. The natural map $R \wedge_A R \to R \wedge_{A'} R$ gives rise to a map of orthogonal S^1-spectra ${}_A N_e^{S^1} R \to {}_{A'} N_e^{S^1} R$. Since the relative diagonal map is functorial in ϕ, it follows that this is a map of op-p-precyclotomic spectra. The remaining statements now follow from the functoriality of all of the constructions involved in defining ${}^{op}TR(-)$ and ${}^{op}TC(-)$. □

Combining the previous proposition with Proposition 3.11 now proves Theorem 1.9 from the introduction.

8. THH of ring C_n-spectra

For G a finite group and $H < G$ a subgroup, the norm N_H^G provides a functor from orthogonal H-spectra to orthogonal G-spectra. In this section, we generalize this construction to a relative norm $N_{C_n}^{S^1}$, which we view as a "C_n-relative THH". We begin with an explicit construction in terms of a cyclic bar construction, which generalizes the simplicial object studied in Section 4 on the edgewise subdivision of the cyclic bar construction.

Definition 8.1. Let R be an associative ring orthogonal C_n-spectrum indexed on the trivial universe $\mathbb{R}^\infty$. Let $N_\wedge^{\mathrm{cyc},C_n} R$ denote the simplicial object that in degree q is $R^{\wedge(q+1)}$, has degeneracy s_i (for $0 \le i \le q$) induced by the inclusion of the unit in the $(i+1)$-st factor, has face maps d_i for $0 \le i < q$ induced by multiplication of the ith and $(i+1)$st factors. The last face map d_q is given as follows. Let α_q be the automorphism of $R^{\wedge(q+1)}$ that cyclically permutes the factors, putting the last factor in the zeroth position, and then acts on that factor by the generator $g = e^{2\pi i/n}$ of C_n. The last face map is $d_q = d_0 \circ \alpha_q$.

The previous definition constructs a simplicial object but not a cyclic object. Nevertheless, it does have extra structure of the same sort found on the edgewise subdivision of a cyclic object. The operator α_q in simplicial degree q is the generator of a $C_{n(q+1)}$-action (the action obtained by regarding $R^{\wedge(q+1)}$ as an indexed smash product for $C_n < C_{n(q+1)}$). The faces, degeneracies, and operators α_q satisfy the following relations in addition to the usual simplicial relations:

$$\begin{aligned}
\alpha_q^{n(q+1)} &= \mathrm{id} \\
d_0\alpha_q &= d_q \\
d_i\alpha_q &= \alpha_{q-1}d_{i-1} \text{ for } 1 \le i \le q \\
s_i\alpha_q &= \alpha_{q+1}s_{i-1} \text{ for } 1 \le i \le q \\
s_0\alpha_q &= \alpha_{q+1}^2 s_q
\end{aligned}$$

This defines a Λ_n^{op}-object in the notation of [7, 1.5]. As explained in [7, 1.6–8], the geometric realization has an S^1-action extending the C_n-action.

Definition 8.2. Let R be an associative ring orthogonal C_n-spectrum indexed on the universe $\tilde{U} = \iota_{C_n}^* U$. The relative norm $N_{C_n}^{S^1} R$ is defined as the composite functor

$$N_{C_n}^{S^1} R = \mathcal{I}_{\mathbb{R}^\infty}^{U} |N_\wedge^{\mathrm{cyc},C_n}(\mathcal{I}_{\tilde{U}}^{\mathbb{R}^\infty} R)|$$

When R is a commutative ring orthogonal C_n-spectrum, we have the following analogue of Proposition 4.3.

Proposition 8.3. *The restriction of $N_{C_n}^{S^1}$ to $\mathcal{C}om_{\tilde{U}}^{C_n}$ lifts to a functor*

$$N_{C_n}^{S^1}\colon \mathcal{C}om_{\tilde{U}}^{C_n} \longrightarrow \mathcal{C}om_{U}^{S^1}$$

that is left adjoint to the forgetful functor

$$\iota^*\colon \mathcal{C}om_{U}^{S^1} \longrightarrow \mathcal{C}om_{\tilde{U}}^{C_n}.$$

We now describe the homotopical properties of the relative norm. The following analogue of Theorem 4.9 has the same proof.

Theorem 8.4. *Let $R \to R'$ be a weak equivalence of cofibrant associative ring orthogonal C_n-spectra. Then $N_{C_n}^{S^1} R \to N_{C_n}^{S^1} R'$ is a $\mathcal{F}_{\mathrm{Fin}}$-equivalence.*

In the commutative case, we have the following analogue of Proposition 4.10 (also using an identical proof).

Theorem 8.5. *Regarded as a functor on commutative ring orthogonal C_n-spectra, the functor $N_{C_n}^{S^1}$ is a left Quillen functor with respect to the positive complete model structure on $\mathcal{C}om^{C_n}$ and the positive complete $\mathcal{F}_{\mathrm{Fin}}$-model structure on $\mathcal{C}om_U^{S^1}$.*

We now turn to the question of the cyclotomic structure.

Theorem 8.6. *Let R be a cofibrant associative ring orthogonal C_n-spectrum. If p is prime to n, then $N_{C_n}^{S^1} R$ has the natural structure of a p-cyclotomic spectrum.*

Proof. As in the proof of Theorem 4.6, we can identify $\iota_{C_{pn}}^* N_{C_n}^{S^1} R$ as the geometric realization of a simplicial orthogonal C_{pn}-spectrum of the form

$$N_{C_n}^{C_{pn}}(R^{\wedge(\bullet+1)}).$$

Since p is prime to n, by Proposition 2.19 we have a diagonal map $R^{\wedge(q+1)} \to \Phi^{C_p} N_{C_n}^{C_{pn}} R^{\wedge(q+1)}$, which again commutes with the simplicial structure and induces a diagonal map

$$\tau_p\colon N_{C_n}^{S^1} R \longrightarrow \rho_p^* \Phi^{C_p} N_{C_n}^{S^1} R.$$

Under the hypothesis that R is cofibrant as an orthogonal C_n-spectrum, Theorem 2.35 shows that the diagonal map $R^{\wedge(q+1)} \to \Phi^{C_p} N_{C_n}^{C_{pn}} R^{\wedge(q+1)}$ is an isomorphism, and it follows that τ_p is an isomorphism. The inverse gives the p-cyclotomic structure map. □

As usual, we can construct $TR_{C_n} R$ and $TC_{C_n} R$ from the cyclotomic structure on $N_{C_n}^{S^1} R$. And as before, when R only has the homotopy type of a cofibrant object, application of Proposition 3.11 allows us to work with ${}^{op}TR_{C_n}$ and ${}^{op}TC_{C_n}$.

When p divides n, the diagonal map is of the form

$$N_{C_{n/p}}^{S^1} \Phi^{C_p} R \longrightarrow \Phi^{C_p} N_{C_n}^{S^1} R,$$

and is an isomorphism when R is cofibrant as an orthogonal C_n-spectrum or as a commutative ring orthogonal C_n-spectrum. In these cases, we can get a p-cyclotomic structure map if we have one on R of the following form.

Definition 8.7. For $p \mid n$, a C_n p-cyclotomic spectrum consists of an orthogonal C_n-spectrum X together with a map of orthogonal C_n-spectra

$$t\colon N_{C_{n/p}}^{C_n} \Phi^{C_p} X \longrightarrow X$$

that induces a weak equivalence to X from the derived composite functor.

Proposition 8.8. *Assume $p \mid n$ and let R be an associative ring orthogonal C_n-spectrum with a C_n p-cyclotomic structure such that the structure map t is a ring map. Then $N_{C_n}^{S^1} R$ has the natural structure of a p-cyclotomic spectrum.*

At present, we do not know if the previous proposition is interesting. For any (non-equivariant) ring orthogonal spectrum R', $R = N_e^{C_n} R'$ satisfies the hypothesis of the previous proposition, and $N_{C_n}^{S^1} R \cong N_e^{S^1} R'$. The assumptions on t imply that such an R must be a norm from a subgroup of order relatively prime to p; however, the structure map t does not necessarily preserve it.

9. Spectral sequences for ${}_A TR$

In this section we present four spectral sequences for computing ${}_A TR$. In each case we actually have two spectral sequences, one graded over the integers and a second graded over $RO(S^1)$. We follow the modern convention of denoting an integral grading with $*$ and an $RO(S^1)$-grading with $\star$. Although the two look formally similar, they are very different computationally, for reasons explained in the introduction to [23]: the Tor terms are computed using very different notions of projective module. Specifically, for V a non-trivial representation $\underline{\pi}_*^{(-)}(\Sigma^V R)$ cannot be expected to be projective as a $\underline{\pi}_*^{(-)} R$ Mackey functor module; however, $\underline{\pi}_\star^{(-)}(\Sigma^V R)$ is of course projective as a $\underline{\pi}_\star^{(-)} R$ Mackey functor module, being just a shift of the free module $\underline{\pi}_\star^{(-)} R$.

9.1. **The absolute to relative spectral sequence.** The equivariant homotopy groups $\pi_*^{C_n}(N_e^{S^1} R)$ are the TR-groups $TR_*^n(R)$ and so $\pi_*^{C_n}({}_A N_e^{S^1} R)$ are by definition the relative TR-groups ${}_A TR_*^n(R)$.

Notation 9.1. Let

$$TR_*^{(-)}(R) = \underline{\pi}_*^{(-)}(N_e^{S^1}(R)) \qquad TR_\star^{(-)}(R) = \underline{\pi}_\star^{(-)}(N_e^{S^1}(R))$$
$${}_A TR_*^{(-)}(R) = \underline{\pi}_*^{(-)}({}_A N_e^{S^1}(R)) \qquad {}_A TR_\star^{(-)}(R) = \underline{\pi}_\star^{(-)}({}_A N_e^{S^1}(R))$$

Using the isomorphism of Proposition 6.9

$${}_A N_e^{S^1}(R) \cong N_e^{S^1}(R) \wedge_{N_e^{S^1} A} A_{S^1},$$

we can apply the Künneth spectral sequences of [23] to compute the relative TR-groups from the absolute TR-groups and Mackey functor $\underline{\mathrm{Tor}}$. Technically, to apply [23] and for ease of statement, we restrict to a finite subgroup $H < S^1$. Recall that for a commutative ring orthogonal spectrum A, A_H denotes $I_{\mathbb{R}^\infty}^{\tilde{U}} A$ where $\tilde{U}$ is the complete S^1-universe regarded as a complete H-universe, and we regard A as an H-trivial orthogonal H-spectrum.

Theorem 9.2. *Let A be a cofibrant commutative ring orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. For each finite subgroup $H < S^1$, there is a natural strongly convergent spectral sequence of H-Mackey functors*

$$\underline{\mathrm{Tor}}_{*,*}^{TR_*^{(-)}(A)}(TR_*^{(-)}(R), \underline{\pi}_*^{(-)}(A_H)) \quad \Longrightarrow \quad {}_A TR_*^{(-)}(R),$$

compatible with restriction among finite subgroups of S^1.

Compatibility with restriction among finite subgroups of S^1 refers to the fact that for $H < K$, the restriction of the K-Mackey functor $\underline{\mathrm{Tor}}$ to an H-Mackey functor is canonically isomorphic to the H-Mackey functor $\underline{\mathrm{Tor}}$ and the corresponding isomorphism on E^∞-terms induces the same filtration on $\underline{\pi}_*$. (Free K-Mackey functor modules restrict to free H-Mackey functor modules essentially because finite K-sets restrict to finite H-sets.)

We also have corresponding Künneth spectral sequences graded on $RO(H)$ for $H < S^1$ or $RO(S^1)$. We choose to state our results in terms of the $RO(S^1)$-grading because this makes the behavior of the restriction among subgroups easier to describe; the restriction maps $RO(S^1) \to RO(H)$ are surjective, and as a result $\underline{\mathrm{Tor}}$-groups calculated in $RO(H)$-graded homological algebra restrict naturally to $\underline{\mathrm{Tor}}$-groups calculated in $RO(S^1)$-graded homological algebra. In the following theorem, $\star$ denotes the $RO(S^1)$-grading.

Theorem 9.3. *Let A be a cofibrant commutative ring orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. For each finite subgroup $H < S^1$, there is a natural strongly convergent spectral sequence of H-Mackey functors*

$$\underline{\mathrm{Tor}}^{TR^{(-)}_\star(A)}_{*,\star}(TR^{(-)}_\star(R), \underline{\pi}^{(-)}_\star(A_H)) \quad \Longrightarrow \quad {}_ATR^{(-)}_\star(R),$$

compatible with restriction among finite subgroups of S^1.

9.2. **The simplicial filtration spectral sequence.** The spectral sequence of the preceding subsection essentially gives a computation of the relative theory in terms of absolute theory. More often we expect to use the relative theory to compute the absolute theory. Non-equivariantly, the isomorphism

$$THH(R) \wedge A \cong {}_ATHH(R \wedge A) \tag{9.4}$$

gives rise to a Künneth spectral sequence

$$\mathrm{Tor}^{A_*(R\wedge_S R^{\mathrm{op}})}_{*,*}(A_*(R), A_*(R)) \quad \Longrightarrow \quad A_*(THH(R)).$$

As employed by Bökstedt, an Adams spectral sequence can then in practice be used to compute the homotopy groups of $THH(R)$. For formal reasons, the isomorphism (9.4) still holds equivariantly, but now we have three different versions of the non-equivariant Künneth spectral sequence (none of which have quite as elegant an E^2-term) which we use in conjunction with equation (9.4).

The first equivariant spectral sequence generalizes the Künneth spectral sequence in the special case when π_*A is a field. Non-equivariantly, it derives from the simplicial filtration of the cyclic bar construction; equivariantly, we restrict to a finite subgroup $H < S^1$ and look at the simplicial filtration on the nth edgewise subdivision (described in the proof of Theorem 4.9).

Theorem 9.5. *Let A be a cofibrant commutative ring orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Let H be a finite subgroup of S^1.*

(1) *There is a natural spectral sequence strongly converging to the integer graded H-Mackey functor ${}_ATR^{(-)}_*(R)$ with E^1-term*

$$\underline{E}^1_{s,t} = \underline{\pi}_t({}_AN^H_e(R^{\wedge(s+1)})).$$

(2) *There is a natural spectral sequence strongly converging to the $RO(S^1)$–graded H-Mackey functor ${}_A TR^{(-)}_\star(R)$ with E^1-term*

$$\underline{E}^1_{s,\tau} = \underline{\pi}_\tau({}_A N_e^H(R^{\wedge(s+1)})).$$

The E^2-terms of both spectral sequences are compatible with restriction among finite subgroups of S^1.

To see the compatibility with restriction among subgroups, we note that for $H = C_{mn}$, the E^2-term $(E^2_{*,\tau})^{C_m}$ is the homology of the simplicial object

$$\mathrm{sd}_n\, \pi_\star^{C_m}((N_e^{C_m} A)^{\wedge(\bullet+1)}).$$

For $H < K$, the subdivision operators then induce an isomorphism on E_2-terms.

In general, we do not know how to describe the E^2-term of these spectral sequences. One can formulate box-flatness hypotheses that would permit the identification of the E^2-term as a kind of Mackey functor Hochschild homology [1]; however, such hypotheses will rarely hold in practice. On the other hand, when $A = H\mathbb{F}$ for $\mathbb{F}$ a field, for formal reasons, the E^1-term is a purely algebraic functor of the graded vector space $\pi_* R$. We conjecture that the E^2-term is a functor of the graded $\mathbb{F}$-algebra $\pi_* R$.

9.3. **The cyclic filtration spectral sequence.** We have a second spectral sequence arising from the filtration on cyclic objects constructed by Fiedorowicz and Gajda [14]. Although they work in the context of spaces, their arguments generalize to provide an $\mathcal{F}_{\mathrm{Fin}}$-equivalence

$$|EX_\bullet| \longrightarrow |X_\bullet|$$

for cyclic orthogonal spectra, where E is the evident orthogonal spectrum generalization of the construction in their Definition 1:

$$EX_\bullet = \int_{[m]\in\Lambda_{\mathrm{face}}} X_m \wedge \Lambda(\bullet, [m])_+$$

The proof of their Proposition 1 (which in fact only gives an $\mathcal{F}_{\mathrm{Fin}}$-equivalence for spaces) also applies in the orthogonal spectrum context, substituting geometric fixed points for fixed points, to prove the $\mathcal{F}_{\mathrm{Fin}}$-equivalence for orthogonal spectra. Change of universe $\mathcal{I}_{\mathbb{R}^\infty}^U$ commutes with geometric realization, and we use the coend filtration of $EX_\bullet$ for $X_\bullet = N^{\mathrm{cyc}}_{\wedge_A} R$ to obtain the following Fiedorowicz-Gajda cyclic filtration spectral sequences.

Theorem 9.6. *Let A be a cofibrant commutative ring orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Let H be a finite subgroup of S^1.*

(1) *There is a natural spectral sequence of integer graded H-Mackey functors strongly converging to ${}_A TR^{(-)}_*(R)$ with E^1-term*

$$\underline{E}^1_{s,t} = \underline{\pi}_t(\mathcal{I}_{\mathbb{R}^\infty}^U(S^1_+ \wedge_{C_{s+1}} R^{\wedge(s+1)})).$$

(2) *There is a natural spectral sequence of $RO(S^1)$-graded H-Mackey functors strongly converging to ${}_A TR^{(-)}_\star(R)$ with E^1-term*

$$\underline{E}^1_{s,\tau} = \underline{\pi}_\tau(\mathcal{I}_{\mathbb{R}^\infty}^U(S^1_+ \wedge_{C_{s+1}} R^{\wedge(s+1)})).$$

The E^1-terms are compatible with restriction among finite subgroups of S^1.

9.4. **The relative cyclic bar construction spectral sequence.** The third spectral sequence directly involves Mackey functor $\underline{\mathrm{Tor}}$. For an A-algebra R, let ${}^{g}_{A}N_e^{C_n}R$ denote the $({}_AN_e^{C_n}R, {}_AN_e^{C_n}R)$-bimodule obtained by twisting the left action of ${}_AN_e^{C_n}R$ on ${}_AN_e^{C_n}R$ by the generator $g = e^{2\pi i/n}$ of C_n. We can identify the C_n-homotopy type of ${}_AN_e^{S^1}R$ in terms of this bimodule,

$$ {}_AN_e^{S^1}R \cong \mathcal{I}_{\tilde{U}}^{U} N^{\mathrm{cyc}}_{\wedge_A}({}_AN_e^{C_n}R, {}^{g}_{A}N_e^{C_n}R), $$

where the cyclic bar construction on the right is taken in the symmetric monoidal category of A-modules in orthogonal C_n-spectra and $\tilde{U} = \iota^*_{C_n}U$ denotes U viewed as a complete C_n-universe. A consequence of this description is that the main theorem of [23] constructing the equivariant Künneth spectral sequence applies:

Theorem 9.7. *Let A be a cofibrant commutative ring orthogonal spectrum and let R be a cofibrant associative A-algebra or cofibrant commutative A-algebra. Fix $n > 0$.*

(1) *There is a natural strongly convergent spectral sequence of integer graded C_n-Mackey functors*

$$ E^2_{*,*} = \underline{\mathrm{Tor}}^{N_e^{C_n}(R\wedge_A R^{\mathrm{op}})}_{*,*}(\underline{\pi}_{*A}N_e^{C_n}R, \underline{\pi}_{*}{}^{g}_{A}N_e^{C_n}R) \Longrightarrow {}_ATR^{(-)}_*(R). $$

(2) *There is a natural strongly convergent spectral sequence of $RO(S^1)$-graded C_n-Mackey functors*

$$ E^2_{*,\star} = \underline{\mathrm{Tor}}^{N_e^{C_n}(R\wedge_A R^{\mathrm{op}})}_{*,\star}(\underline{\pi}_{\star A}N_e^{C_n}R, \underline{\pi}_{\star}{}^{g}_{A}N_e^{C_n}R) \Longrightarrow {}_ATR^{(-)}_\star(R). $$

We see no reason why the E^2-terms for the spectral sequences of the previous theorem should be compatible under restriction among finite subgroups of S^1.

10. Adams operations

In this section, we study the circle power operations on $THH(R)$ for a commutative ring R and on ${}_ATHH(R)$ for a commutative A-algebra R. Such operations were first defined on Hochschild homology by Loday [24] and Gerstenhaber-Schack [15] and explained by McCarthy [32] in terms of covering maps of the circle and extended to THH by [34]. Following [9, 4.5.3], we refer to these as *Adams operations* and denote as ψ^r (though in older literature [25, 4.5.16], the Adams operations differ by a factor of the operation number r). Specifically, we study how the operations interact with the equivariance, and we show that when r is prime to p, ψ^r descends to an operation on $TR(R)$, $TC(R)$, and in the commutative A-algebra context to an operation on ${}^{op}_ATR(R)$ and ${}^{op}_ATC(R)$, cf. [9, §7]. We study the effect of ψ^r on ${}^{op}_ATR_0(R)$ and ${}^{op}_ATC_0(R)$, where it is shown to be the identity on ${}^{op}_ATR_0(R)$ when R is connective.

We recall the construction of McCarthy's Adams operations, which ultimately derives from the identification of $N^{\mathrm{cyc}}_{\wedge_A}R$ as the tensor $R \otimes S^1$ in the category of commutative A-algebras. Using the standard model for the circle as the geometric realization of a simplicial set $S^1_\bullet$ (with one 0-simplex and one non-degenerate 1-simplex), the tensor identification is just observing that $N^{\mathrm{cyc}}_{\wedge_A}R$ is the simplicial object obtained by taking $S^1_\bullet$ coproduct factors of R in simplicial degree $\bullet$,

$$ N^{\mathrm{cyc}}_{\wedge_A}R = R \otimes S^1_\bullet. $$

The operation ψ^r is induced by the r-fold covering map

$$q_r \colon S^1 \longrightarrow S^1, \qquad e^{i\theta} \mapsto e^{ri\theta}.$$

after tensoring with R.

Definition 10.1. Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $r \neq 0$, the Adams operation

$$\psi^r \colon {}_A THH(R) \longrightarrow {}_A THH(R)$$

is the map of (non-equivariant) commutative A-algebras obtained as the tensor of R with the covering map $q_r \colon S^1 \to S^1$.

We will study the equivariance of ψ^r using the C_n-action that arises on the edgewise subdivision sd_n of a cyclic set. To make this section more self-contained, we again recall from [7, §1] how this works. There are natural homeomorphisms

$$\delta_n \colon |\operatorname{sd}_n X| \longrightarrow |X|$$

for the n-fold edgewise subdivision of a simplicial space or simplicial orthogonal spectrum, and canonical isomorphisms of simplicial objects $\operatorname{sd}_r \operatorname{sd}_s X \to \operatorname{sd}_{rs} X$, which together make the following diagram commute [7, 1.12]:

(10.2)
$$\begin{tikzcd}
{|\operatorname{sd}_r \operatorname{sd}_s X|} \arrow[r] \arrow[d, "\delta_r"'] & {|\operatorname{sd}_{rs} X|} \arrow[d, "\delta_{rs}"] \\
{|\operatorname{sd}_s X|} \arrow[r, "\delta_s"'] & {|X|.}
\end{tikzcd}$$

When X has a cyclic structure, $\operatorname{sd}_n X$ comes with a natural C_n-equivariant structure which on the geometric realization is the restriction to C_n of the natural S^1-action; moreover, in the diagram above, the left hand isomorphism is C_s-equivariant [7, 1.7–8].

We have a simplicial model of ψ^r by McCarthy's observation that q_r is the geometric realization of a quotient map of simplicial sets $\operatorname{sd}_r S^1_\bullet \to S^1_\bullet$. By naturality, diagram (10.2) is compatible with this quotient map.

Proposition 10.3. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $r \neq 0$ and n relatively prime to r, the restriction of q_r is the multiplication by r isomorphism $C_n \to C_n$ and the Adams operations ψ^r is a map of commutative ring orthogonal C_n-spectra*

$$\psi^r \colon \iota^*_{C_n} {}_A N_e^{S^1} R \longrightarrow q_r^* \iota^*_{C_n} {}_A N_e^{S^1} R.$$

Moreover, for s relatively prime to n, the formula

$$(q_r)^*(\psi^s) \circ \psi^r = \psi^{rs} \colon \iota^*_{C_n} {}_A N_e^{S^1} R \longrightarrow q_{rs}^* \iota^*_{C_n} {}_A N_e^{S^1} R.$$

holds.

Proof. As above, the r-fold covering map defining the Adams operations becomes a C_n-equivariant map

$$\operatorname{sd}_n(\operatorname{sd}_r S^1) \longrightarrow (q_r \,|_{C_n})^*(\operatorname{sd}_n S^1).$$

Tensoring levelwise and applying $\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}$, we obtain a map of simplicial commutative A-algebras

$$\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}(R \otimes (\operatorname{sd}_n \operatorname{sd}_r S^1)) \longrightarrow q_r^* \mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}(R \otimes \operatorname{sd}_n S^1).$$

The result now follows from diagram (10.2) and its compatibility with the covering projections q_r. $\square$

In the case when $p \nmid r$, the previous proposition shows that in particular the operation ψ^r should pass to categorical C_{p^n}-fixed points (in the derived category of A). Taking fibrant replacements, we get a map (of non-equivariant A-modules)

$$\psi^r\colon ({}_AN_e^{S^1}R)_f^{C_{p^n}} \longrightarrow ({}_AN_e^{S^1}R)_f^{C_{p^n}}$$

making the diagram

$$\begin{tikzcd} ({}_AN_e^{S^1}R)_f^{C_{p^{n+1}}} \arrow[r, "\psi^r"] \arrow[d, "F"'] & ({}_AN_e^{S^1}R)_f^{C_{p^{n+1}}} \arrow[d, "F"] \\ ({}_AN_e^{S^1}R)_f^{C_{p^n}} \arrow[r, "\psi^r"'] & ({}_AN_e^{S^1}R)_f^{C_{p^n}} \end{tikzcd}$$

commute, where F is the natural inclusion of fixed-points. Passing to the homotopy limit, we get an Adams operation ψ^r on ${}_ATF(R)$.

We next argue that for $p \nmid r$, the Adams operation ψ^r descends to ${}^{op}_ATR(R)$ and ${}^{op}_ATC(R)$.

Theorem 10.4. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. For $p \nmid r$, the Adams operation ψ^r induces maps*

$$\psi^r\colon {}^{op}_ATR(R) \longrightarrow {}^{op}_ATR(R)$$

and

$$\psi^r\colon {}^{op}_ATC(R) \longrightarrow {}^{op}_ATC(R)$$

natural in the derived category of A.

Proof. It suffices to consider the case when R is cofibrant as a commutative A-algebra and to show that ψ^r commutes with the op-p-cyclotomic structure map

$$\gamma = \tau_p\colon {}_AN_e^{S^1}R \longrightarrow \rho_p^*\Phi^{C_p}\mathcal{I}_{\mathbb{R}^\infty}^{\tilde{U}}|\,\mathrm{sd}_p\, N^{\mathrm{cyc}}_{\wedge_A}R|.$$

This is clear from the naturality of (10.2). $\square$

Finally, we provide the following computation for the action of the Adams operations on ${}^{op}_ATR_0$ and ${}^{op}_ATC_0$.

Theorem 10.5. *Let A be a cofibrant commutative ring orthogonal spectrum and R a cofibrant commutative A-algebra. Assume that A and R are connective. Then for $p \nmid r$, the Adams operation ψ^r acts by the identity on ${}_ATR_0(R)$.*

Proof. Using Proposition 6.9, Theorem 6.11, and edge of the Künneth spectral sequence from [23], we find that there is an isomorphism

$$\underline{\pi}_0({}_AN_e^{S^1}R) \cong (\underline{\pi}_0 N_e^{S^1}R) \underset{\underline{\pi}_0 N_e^{S^1}A}{\square} (\underline{\pi}_0(A_{S^1}))$$

of Mackey functors when restricted to any finite subgroup of S^1. The left-hand factor in the box product is $\underline{\pi}_0THH(R)$, and the right-hand factor is $\underline{\pi}_0({}_ATHH(A))$, and so by naturality it suffices to check these two cases separately.

The simplicial spectrum ${}_ATHH(A)$ is naturally *isomorphic* to the constant cyclic spectrum with value A, and both the maps defining the cyclic structure and the maps defining the Adams operations are carried to the identity maps under this identification. Therefore, the Adams operations act trivially on $\underline{\pi}_*({}_ATHH(A))$.

We now verify that the operations act trivially on $TR_0(R)$. Writing $R_0 = \pi_0 R$, the hypothesis of connectivity implies that

$$\pi_0 TR(R) \cong \pi_0 TR(R_0),$$

and so it suffices to consider the case when $R = HR_0$. By [17, Addendum 3.3], we have a canonical isomorphism of $TR_0(R)$ with the p-typical Witt ring $W(R_0)$ and canonical isomorphisms of $\pi_0^{C_{p^n}} N_e^{S^1} R$ with $W_{n+1}(R_0)$, the p-typical Witt vectors of length $n+1$. Letting R_0 vary over all commutative rings, ψ^r then restricts to natural transformations ψ^r_{n+1} of rings $W_{n+1}(-) \to W_{n+1}(-)$, compatible with the restriction maps. We complete the proof by arguing that such a natural transformation must be the identity.

Since W_{n+1} is representable, it suffices to prove that ψ^r_{n+1} is the identity when R_0 is the representing object $\mathbb{Z}[x_0, \ldots, x_n]$, or, since this is torsion free, when $R_0 = \mathbb{Q}[x_0, \ldots, x_n]$. *A fortiori*, it suffices to prove ψ^r_{n+1} is the identity when R_0 is a $\mathbb{Q}$-algebra. Since for a $\mathbb{Q}$-algebra $W_{n+1}(R_0)$ is isomorphic as a ring to the Cartesian product of $n+1$ copies of R_0 via the ghost coordinates, the only possible natural ring endomorphisms of W_{n+1} are the maps that permute the factors. Since ψ^r commutes with the restriction map R on $TR(R)$, and on the ghost coordinates the restriction map induces the projection onto the first n factors, it follows by induction that ψ^r_{n+1} is the identity. $\square$

Since a connective commutative A-algebra R admits a canonical map of commutative A-algebras $R \to H\pi_0 R$, we obtain the following corollary of the previous theorem and its proof.

Corollary 10.6. *Let A be a commutative ring orthogonal spectrum and R a commutative A-algebra. Assume that R is connective and that $p \nmid r$. Then ${}^{op}_{A}TC_0(R)$ has the Frobenius invariants of $W(\pi_0 R)$ as a quotient and the action of ψ^r descends to the identity map on this quotient.*

Example 10.7. When we take $A = R = S$ to be the sphere spectrum, [7, §5] identifies $TC(S)^\wedge_p$ as $(S \vee \Sigma\mathbb{C}P^\infty_{-1})^\wedge_p$, where $\mathbb{C}P^\infty_{-1}$ denotes the Thom spectrum of the virtual bundle $-L$, where L denotes the tautological line bundle. More to the point, $\Sigma\mathbb{C}P^\infty_{-1}$ is the homotopy fiber of the S^1-transfer $\Sigma\Sigma^\infty_+\mathbb{C}P^\infty \to S$. The tom Dieck splitting identifies

$$TR^n(S)^\wedge_p \simeq \prod_{0 \le m \le n} (\Sigma^\infty_+ B(C_{p^n}/C_{p^m}))^\wedge_p \cong \prod_{0 \le k \le n} (\Sigma^\infty_+ B(C_{p^k}))^\wedge_p.$$

For $p \nmid r$, ψ^r acts on $THH(S)$ as the identity (on the point set level), and so acts on the C_{p^n}-fixed points via the multiplication by r map $C_{p^n} \to C_{p^n}$. It therefore induces the corresponding multiplication by r map on each classifying space $B(C_{p^n}/C_{p^m})$ in each factor in $TR^n(S)$; note that multiplication by r on C_{p^n}/C_{p^m} is multiplication by r on C_{p^k} (under the canonical isomorphism). This allows us to determine the action of ψ^r on $TC(S)$. The computation of $TC(S)$ in [7, §5] and [28, §4.4] uses a weak equivalence

$$(\Sigma\Sigma^\infty_+\mathbb{C}P^\infty)^\wedge_p \simeq \operatorname{holim}(\Sigma^\infty_+ BC_{p^k})^\wedge_p,$$

and the action of ψ^r on BC_{p^k} is compatible with the action of ψ^r on $(\Sigma\Sigma^\infty_+\mathbb{C}P^\infty)^\wedge_p$ given by multiplication by r on the suspension and the action on $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$

induced by the multiplication by r on $\mathbb{Z}$. The fiber sequence

$$\Sigma\mathbb{C}P^{\infty}_{-1} \longrightarrow \Sigma\Sigma^{\infty}_{+}\mathbb{C}P^{\infty} \longrightarrow S$$

has a consistent action of ψ^r (where we use the trivial action on S). After p-completion, the action of $\{r \mid p \nmid r\}$ extends to an action of the units of $\mathbb{Z}^{\wedge}_{p}$. The Teichmüller character then gives an action of $(\mathbb{Z}/p)^{\times}$ and (since $p-1$ is invertible in $\mathbb{Z}^{\wedge}_{p}$) a splitting into $p-1$ "eigenspectra" wedge summands. This decomposition of $TC(S)^{\wedge}_{p}$ is well-known and plays a role in Rognes' cohomological analysis of $Wh(*)^{\wedge}_{p}$ at regular primes [35, §5].

11. Madsen's remarks

In his CDM notes [28, p. 218], Madsen describes the restriction map, and notes that the inverse is not as readily accessible even in the algebraic setting since "$\Delta(r) = r \otimes \cdots \otimes r$ is not linear". Yet in our framework, we naturally get the inverse to the cyclotomic structure map, rather than the cyclotomic structure map itself. At first blush, this seems to pose a curious contradiction. The answer arises from the transfer: $v \mapsto v^{\otimes p}$ is linear modulo the ideal generated by the transfer, and this is exactly the ideal killed by $L\Phi^{H}$.

The observation that the ideal killed by $L\Phi^{H}$ coincides with the ideal generated by the transfer is essentially a formal consequence of the definition of the derived geometric fixed point functor: $L\Phi^{H}(X) = (X \wedge \widetilde{E\mathcal{P}})^{H}$ is a composite of the categorical fixed points with the localization killing cells of the form S^1/K for K a proper subgroup of H. Computationally, this means that all transfers from proper subgroups of H are killed.

The observation that the algebraic diagonal map is linear modulo the transfer is more interesting. In particular, this question highlights the issue of constructing an algebraic model of the norm functor that correctly reflects the homotopy theory. We first consider the naive smash power which is simply the C_p-module $(\mathbb{Z}\{x,y\})^{\otimes p}$, where $\mathbb{Z}\{x,y\}$ is the free abelian group on the set $\{x,y\}$. Inside is the element $(x+y)^{\otimes p}$, which is obviously in the fixed points of the C_p-action. In this context, Madsen's remark boils down to the fact that $(x+y)^{\otimes p}$ is not $x^{\otimes p} + y^{\otimes p}$. We can expand $(x+y)^{\otimes p}$ using a non-commutative version of the binomial theorem as follows. Observing that the full symmetric group Σ_p acts on the tensor power (and the C_p-action is just the obvious restriction), if we group all terms with i tensor factors of x and $p-i$ tensor factors of y, then we see that the symmetric group permutes these and a subgroup conjugate to $\Sigma_i \times \Sigma_{p-i}$ stabilizes each element. We therefore see that the sum of all of such terms for a fixed i can be expressed as the transfer

$$\mathrm{Tr}_{\Sigma_i \times \Sigma_{p-i}}^{\Sigma_p} x^{\otimes i} \otimes y^{\otimes (p-i)}.$$

Letting i vary and summing the terms (and then restricting back to C_p) shows that

$$(x+y)^{\otimes p} = x^{\otimes p} + y^{\otimes p} + \mathrm{Res}_{C_p}^{\Sigma_p}\big(\sum_{i=1}^{p-1} \mathrm{Tr}_{\Sigma_i \times \Sigma_{p-i}}^{\Sigma_p} x^{\otimes i} y^{\otimes (p-i)}\big).$$

All of the terms involving transfers are in the ideal generated by transfers by definition, and so we conclude that the p^{th} power map is linear modulo these.

However, this algebraic model is not the correct analogue of the norm. First, when we reduce modulo the transfer from proper subgroups in the pth tensor power of a ring, then we also kill the transfer of the element 1. This then takes us from

$\mathbb{Z}$-modules to $\mathbb{Z}/p$-modules. Second, the fixed point Mackey functor associated to the pth tensor power functor is not the right algebraic version of the norm.

There are now several constructions of a norm functor in the category of Mackey functors that exhibit the correct homotopy-theoretic behavior. Mazur describes one for cyclic p-groups [31], Hill-Hopkins gives one for a general finite group by stepping through the norm in spectra [19], and subsequently Hoyer gave a purely algebraic definition for all finite groups and showed it to be equivalent to the others [21]. One of the basic properties of the algebraic norm is that the norm from H-Mackey functors to G-Mackey functors is the functor underlying the left adjoint to the forgetful functor from G-Tambara functors to H-Tambara functors. In particular, since $\pi_0(R)$ for R a commutative ring G-spectrum is a G-Tambara functor [8], the algebraic norm precisely mirrors the multiplicative behavior of the norm in spectra. A more detailed exposition of the connection between the algebraic norm and THH will appear in [1].

In this context, if R is a commutative ring, then the inverse map considered by Madsen is exactly the universally defined norm map

$$N_e^{C_p} \colon R \longrightarrow N_e^{C_p}(R)(C_p/C_p)$$

underlying the Tambara functor structure. While this map is not linear, it is so modulo the transfer [38]. In fact, just as in topology, this map is a right inverse to the "geometric fixed points" functor Φ^{C_p} on Mackey functors, the map which takes a Mackey functor $\underline{M}$ and returns the quotient group $\underline{M}(G/G)/\operatorname{im}(\mathrm{Tr})$, where $\operatorname{im}(\mathrm{Tr})$ denotes the image of the transfer: $\Phi^{C_p} \circ N_e^{C_p} = Id$.

We close by illustrating this all with an example which shows the failure of the "naive" tensor power approach and the strength (and relative computability) of the Tambara functor approach to the algebraic norm. Let $p = 2$, and let $R = \mathbb{Z}[x]$. Then the two-fold tensor power, C_2-equivariantly, is

$$\mathbb{Z}[C_2 \cdot x] = \mathbb{Z}[x, gx].$$

The transfer ideal is generated by 2 and $x + gx$, and modulo 2 and $x + gx$, the map $x \mapsto x \cdot gx$ induces the canonical surjection

$$\mathbb{Z}[x] \longrightarrow \mathbb{Z}/2[x \cdot gx].$$

In this example, the map from R to the quotient of the fixed points of $R^{\otimes 2}$ by the ideal given by the transfer is not an isomorphism; we can interpret the failure to be an isomorphism as a failure to correctly interpret the transfer of the element 1. In particular, restricting to the submodule generated by 1 we implicitly computed

$$N_e^{C_2}\mathbb{Z} = \mathbb{Z},$$

endowed with the trivial action. This is not what the algebraic norm computes for us!

For $G = C_2$ and for $R = \mathbb{Z}[x]$, the fixed points of $N_e^{C_2}(\mathbb{Z}[x])$ are the ring

$$\mathbb{Z}[t, y, x \cdot gx]/(t^2 - 2t, ty - 2y),$$

with the elements t and y the transfers of 1 and x respectively (the restriction map takes t to 2, y to $x + gx$ and $x \cdot gx$ to itself). In particular, we observe that the unit 1 generates not a copy of $\mathbb{Z}$ but rather a copy of the Burnside ring $\mathbb{Z}[t]/t^2 - 2t$. Thus, modulo the image of the transfer, this ring is simply $\mathbb{Z}[x \cdot gx]$, and the norm map $x \mapsto x \cdot gx$ is an isomorphism.

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Australian National University, Canberra, Australia
E-mail address: `vigleik.angeltveit@anu.edu.au`

University of Texas, Austin, TX 78712
E-mail address: `blumberg@math.utexas.edu`

Michigan State University, East Lansing, MI 48824
E-mail address: `teena@math.msu.edu`

University of California Los Angeles, Los Angeles, CA 90025
E-mail address: `mikehill@math.ucla.edu`

University of Minnesota, Minneapolis, MN 55455
E-mail address: `tlawson@math.umn.edu`

Indiana University, Bloomington, IN 47405
E-mail address: `mmandell@indiana.edu`