

# Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2

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## Abstract

Previous work constructed a generalized truncated Brown-Peterson spectrum of chromatic height 2 at the prime 2 as an  $\mathcal{E}_\infty$ -ring spectrum, based on the study of elliptic curves with level-3 structure. We show that the natural map forgetting this level structure induces an  $\mathcal{E}_\infty$ -ring map from the spectrum of topological modular forms to this truncated Brown-Peterson spectrum, and that this orientation fits into a diagram of  $\mathcal{E}_\infty$ -ring spectra lifting a classical diagram of modules over the mod-2 Steenrod algebra. In an appendix we document how to organize Morava's forms of  $K$ -theory into a sheaf of  $\mathcal{E}_\infty$ -ring spectra.

## 1 Introduction

Some of the main applications of modern algebraic topology, including the development of structured ring spectra [EKMM97, HSS00], have been to the subject of algebraic  $K$ -theory. These new foundations introduce strictly associative and commutative ring objects in the category of spectra, together

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with their categories of modules. These provide a large library of new objects whose algebraic  $K$ -theory can be calculated and studied. These illuminate general phenomena that bear on old calculations of the algebraic  $K$ -theory of rings, of simplicial rings, and of spaces.

Based on computational studies in algebraic  $K$ -theory, Ausoni and Rognes [AR02, Introduction] now expect the existence of a redshift phenomenon (generalizing the Bott-periodic phenomena appearing in the algebraic  $K$ -theory of fields) and initiated a long-term program to study the relationship between algebraic  $K$ -theory and the chromatic filtration. These conjectures have been supported by their work showing that the algebraic  $K$ -theory of complex  $K$ -theory supports information at chromatic level 2.

The next computational steps in such a research program would involve study of the algebraic  $K$ -theory of objects at chromatic level 2. Ongoing work of Bruner and Rognes aims to compute the algebraic  $K$ -theory of the topological modular forms spectrum  $K_*(\mathrm{tmf}_{(2)})$  [BR] and of related spectra such as  $\mathrm{BP}\langle 2 \rangle$ . These computations take place using the machinery of topological cyclic homology.

This computational work is greatly assisted by the use of higher multiplicative structures. For an associative object  $R$ , the algebraic  $K$ -theory  $K(R)$  and topological cyclic homology  $TC(R)$  are spectra connected by a cyclotomic trace  $K(R) \rightarrow TC(R)$ . However, if the category of  $R$ -modules has a symmetric monoidal structure analogous to the tensor product of modules over a commutative ring, the algebraic  $K$ -theory and topological cyclic homology inherit the structure of ring objects themselves [EM06]. Computations in topological cyclic homology involve numerous spectral sequence calculations, and these are greatly assisted by the existence of ring structures (or the data of power operations [BR05]) and by naturality arguments.

In addition, many of these computations begin with the Bökstedt spectral sequence, which requires information about the mod- $p$  homology of the spectrum in question.

Previous work, based on a study of the moduli of elliptic curves with level  $\Gamma_1(3)$ -structures, showed the following result.

**Theorem 1.1** ([LN12, Theorem 1.1]). *There exists a 2-local complex oriented*

$\mathcal{E}_\infty$ -ring spectrum  $\mathrm{tmf}_1(3)_{(2)}$  such that the composite map of graded rings

$$\mathbb{Z}_{(2)}[v_1, v_2] \subset BP_* \rightarrow MU_{(2),*} \rightarrow \mathrm{tmf}_1(3)_{(2),*}$$

is an isomorphism.

(Here we say that a multiplicative cohomology theory is “complex oriented” if it is given compatible choices of orientation for all complex vector bundles; we employ the fact that this is equivalent to the choice of a map of ring spectra  $MU \rightarrow R$ .)

However, this result was obtained by means of obstruction theory, and only used the modular interpretation of  $\mathrm{tmf}_1(3)_{(2)}$  in a superficial way. The goal of the current paper is to gain a better understanding of the larger context inhabited by the spectrum  $\mathrm{tmf}_1(3)_{(2)}$ ; this is closely related to the study made by Mahowald and Rezk in [MR09]. With the above  $K$ -theoretic applications in mind, another goal is to exhibit the mod-2 cohomology of  $\mathrm{tmf}_1(3)_{(2)}$ . (Forthcoming work of Hill and the first author should recover a  $C_2$ -action and a connective spectrum  $\mathrm{tmf}_0(3)$ .)

There is a map of moduli stacks of generalized elliptic curves

$$\overline{\mathcal{M}}_1(3) \rightarrow \overline{\mathcal{M}}$$

([Con07, Theorem 4.1.1, (1) with  $N = 3$ ,  $n = 1$ ]). This is the unique map extending the map that takes a smooth elliptic curve with a 3-torsion point and forgets the point. This map ramified at exactly one of the two cusps of  $\overline{\mathcal{M}}_1(3)$  but is log-étale. The modular interpretation of  $\mathrm{tmf}$  suggests that this map should have a topological realization. In fact, we would like to construct a 2-local commutative diagram of  $\mathcal{E}_\infty$ -ring spectra corresponding to (the connective covers of) the global sections of sheaves of  $\mathcal{E}_\infty$ -ring spectra in the following diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}} & \xleftarrow{\text{cusp}} & [\mathrm{Spec}(\mathbb{Z}_{(2)})//\{\pm 1\}] \\ \uparrow & & \uparrow \\ \overline{\mathcal{M}}_1(3) & \xleftarrow{\text{ramified cusp}} & \mathrm{Spec}(\mathbb{Z}_{(2)}) \end{array}$$

A realization of this diagram is achieved by the following main result of this note.

To state it, we recall that, for each  $n$ , the mod-2 Steenrod algebra  $\mathcal{A}^*$  contains exterior subalgebras  $E(n)$  generated by the Milnor primitives  $Q^0, \dots, Q^n$ , and larger subalgebras  $\mathcal{A}(n)$  generated by  $Sq^1, \dots, Sq^{2^{n+1}}$ .

**Theorem 1.2.** *There is a commutative diagram of connective  $\mathcal{E}_\infty$ -ring spectra as follows:*

$$\begin{array}{ccc} \mathrm{tmf}_{(2)} & \xrightarrow{c} & ko_{(2)} \\ \downarrow o & & \downarrow \iota \\ \mathrm{tmf}_1(3)_{(2)} & \xrightarrow{\tilde{c}} & ku_{(2)} \end{array} \quad (1.1)$$

Here  $\iota$  is the complexification map,  $o$  is a  $\mathrm{tmf}_{(2)}$ -orientation of  $\mathrm{tmf}_1(3)$ ,  $c$  corresponds to the cusp on the moduli space of elliptic curves, and  $\tilde{c}$  corresponds to the unique ramified cusp on the moduli space of elliptic curves with level  $\Gamma_1(3)$ -structure.

In mod-2 cohomology, this induces the following canonical diagram of modules over the mod-2 Steenrod algebra  $\mathcal{A}^*$ :

$$\begin{array}{ccc} \mathcal{A}^* // \mathcal{A}(2) & \longleftarrow & \mathcal{A}^* // \mathcal{A}(1) \\ \uparrow & & \uparrow \\ \mathcal{A}^* // E(2) & \longleftarrow & \mathcal{A}^* // E(1). \end{array}$$

There exists a complex orientation of  $\mathrm{tmf}_1(3)_{(2)}$  such that in homotopy,  $\tilde{c}$  induces a map sending the Hazewinkel generators  $v_1$  to  $v_1$  and  $v_2$  to zero, and there is a cofiber sequence of  $\mathrm{tmf}_1(3)_{(2)}$ -modules

$$\Sigma^6 \mathrm{tmf}_1(3)_{(2)} \xrightarrow{v_2} \mathrm{tmf}_1(3)_{(2)} \xrightarrow{\tilde{c}} ku_{(2)}.$$

We note that the restriction to the ramified cusp of  $\overline{\mathcal{M}}_1(3)$  is not material in the above discussion. The unramified cusp also gives rise to a commutative diagram of the same form and with similar properties. However, the spectrum in the lower-right corner is no longer the connective  $K$ -theory spectrum  $ku_{(2)}$  if multiplicative structure is taken into account, but instead corresponds to a Galois twist of the multiplicative formal group law which is defined over  $\mathbb{Z}[1/3]$ . We discuss the modifications necessary to use this form of  $K$ -theory in an appendix. Its periodic version is most easily described as a homotopy fixed point spectrum:

$$KU^\tau = (KU \wedge \mathbb{S}_{(2)}[\omega])^{hC_2}$$

Here  $\mathbb{S}_{(2)}[\omega]$  is obtained by adjoining third roots of unity to the 2-local sphere spectrum. The generator of the cyclic group  $C_2$  acts by  $\psi^{-1} \wedge \sigma$ , where  $\psi^{-1}$  is the Adams operation associated to complex conjugation and  $\sigma$  is complex conjugation acting on the roots of unity.

We conclude with a short overview of the content. In Section 2 we collect basic facts about the moduli of generalized elliptic curves equipped with a  $\Gamma_1(3)$ -structure. In Section 3 we construct the  $\mathcal{E}_\infty$ -maps in Theorem 1.2 using Goerss-Hopkins obstruction theory and chromatic fracture squares. While familiar to the experts, we found it worthwhile to spell out the details of the  $K(1)$ -local obstruction theory. Section 4 shows  $H^*(\mathrm{tmf}_1(3)_{(2)}, \mathbb{F}_2) \cong \mathcal{A}^*//E(2)$  by showing more generally that every generalized  $\mathrm{BP}\langle n \rangle$  has the same cohomology as the standard  $\mathrm{BP}\langle n \rangle$  (Definition 4.1 and Theorem 4.3). Section 5 collects these results into a proof of Theorem 1.2. The appendix discusses forms of  $K$ -theory and the changes necessary to realize diagram (1.1) using the unramified cusp rather than the ramified one.

*Convention 1.3.* With the exception of the appendix, throughout this paper we will work in the category of 2-local spectra. In particular, the names  $\mathrm{tmf}$ ,  $\mathrm{tmf}_1(3)$ ,  $ko$ ,  $ku$ , and the like denote 2-localizations.

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## 2 The 2-localized moduli $\overline{\mathcal{M}}_1(3)_{(2)}$

This section collects basic facts about the moduli of generalized elliptic curves with a  $\Gamma_1(3)$ -structure. It provides background for the somewhat ad-hoc construction of [LN12, Section 8.2], where it was shown that a certain formal group law over  $\mathbb{Z}_{(2)}[A, B]$ , to be recalled presently, defines a  $\mathrm{BP}\langle 2 \rangle$ -realization problem at the prime 2 which can be solved. We will focus here on determining the ordinary and supersingular loci of the moduli stack, as those dictate

the chromatic properties of  $\mathrm{tmf}_1(3)$ .

Consider the curve  $\mathcal{E} \subseteq \mathbb{P}_{\mathbb{Z}_{(2)}[A,B]}^2$  defined by the affine Weierstrass equation

$$y^2 + Axy + By = x^3 \quad (2.1)$$

over the graded ring  $\mathbb{Z}_{(2)}[A, B]$ , where  $|A| = 1$ ,  $|B| = 3$ . There is a 3-torsion point at the origin  $(0, 0)$ , and conversely this Weierstrass form is forced by the requirements that  $(0, 0)$  is a 3-torsion point whose tangent line is horizontal; the elliptic curve  $\mathcal{E}$  is the universal generalized elliptic curve with a choice of 3-torsion point (namely  $[0 : 0 : 1]$ ) and a choice of invariant differential  $dy/3x^2$  by [MR09, Proposition 3.2]. The grading, which can be interpreted as acting on the invariant differential, is reflected in the action of the multiplicative group  $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[\lambda^{\pm 1}])$  determined by

$$A \mapsto \lambda A, B \mapsto \lambda^3 B.$$

This lifts to a  $\mathbb{G}_m$ -action on  $\mathcal{E}/\mathrm{Spec}(\mathbb{Z}_{(2)}[A, B])$  via

$$(x, y) \mapsto (\lambda^2 x, \lambda^3 y).$$

The curve  $\mathcal{E}$  has additive reduction only at  $(A, B) = (0, 0)$ .

**Proposition 2.1.** *The restriction of  $\mathcal{E}$  to  $\mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\}$  is a generalized elliptic curve with irreducible geometric fibers as follows:*

- i) A nodal curve of arithmetic genus one if  $A^3 = 27B$  or  $B = 0$ .*
- ii) A supersingular elliptic curve if  $A = 0$  and  $2 = 0$ .*
- iii) An ordinary elliptic curve otherwise.*

This, in particular, expresses  $A$  as a lift of the Hasse invariant  $v_1$ , which is well-defined mod 2; the element  $B$  is a lift of  $v_2$ , which is well-defined mod  $(2, v_1)$ .

*Proof.* To say that  $\mathcal{E}/(\mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\})$  is a generalized elliptic curve with irreducible fibers in the sense of [DR73, Chapitre I, Définition 1.12] means

that the modular quantities  $c_4(\mathcal{E})$  and  $\Delta(\mathcal{E})$  have no common zero on  $\mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\}$ . This results from the following computations:

$$\begin{aligned} c_4(\mathcal{E}) &= A(A^3 - 24B) \\ \Delta(\mathcal{E}) &= B^3(A^3 - 27B) \\ j(\mathcal{E}[\Delta^{-1}]) &= \frac{A^3(A^3 - 24B)^3}{B^3(A^3 - 27B)} \end{aligned}$$

A geometric fiber is a nodal curve if and only if  $\Delta = 0$ , so i) is clear and ii) and iii) follow by recalling that the only supersingular  $j$ -invariant in characteristic 2 is  $j = 0$ .  $\square$

The  $\mathbb{G}_m$ -action on  $\mathcal{E}$  over  $\mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\}$  descends to determine a generalized elliptic curve over the quotient stack

$$\overline{\mathcal{M}}_1(3)_{(2)} \cong \left[ (\mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\}) // \mathbb{G}_m \right].$$

The stack  $\overline{\mathcal{M}}_1(3)_{(2)}$  is a stack with coarse moduli isomorphic to the weighted projective space  $\text{Proj}(\mathbb{Z}_{(2)}[A, B])$ . Since the zero section of a generalized elliptic curve lies in the smooth locus, we have associated with  $\mathcal{E}$  a 1-dimensional formal group  $\widehat{\mathcal{E}}/\overline{\mathcal{M}}_1(3)_{(2)}$ .

The function  $-x/y$  is a coordinate in a neighborhood of  $\infty$  on  $\mathcal{E}$ , and this gives an isomorphism between the pullback of the relative cotangent bundle  $\omega$  of  $\widehat{\mathcal{E}}/\overline{\mathcal{M}}_1(3)$  along the zero section and the tautological line bundle  $\mathcal{O}(1)$  on  $\overline{\mathcal{M}}_1(3)$ . Compatibility with the grading implies that the formal group  $\widehat{\mathcal{E}}$  comes from a graded formal group law, and is induced by a map of graded rings  $MU_* \rightarrow \mathbb{Z}_{(2)}[A, B]$ . Here we follow the standard convention that elements in algebraic grading  $k$  lie in topological grading  $2k$ . The elements  $A$  and  $B$  can then be interpreted as global sections:

$$\begin{aligned} A &\in H^0(\overline{\mathcal{M}}_1(3), \mathcal{O}(1)) \\ B &\in H^0(\overline{\mathcal{M}}_1(3), \mathcal{O}(3)) \end{aligned}$$

Let  $\mathbb{Z}_{(2)}[A, B] \rightarrow \mathbb{Z}_{(2)}[b]$  be the ungraded map given by  $A \mapsto 1$ ,  $B \mapsto b$ . The composite map

$$\text{Spec}(\mathbb{Z}_{(2)}[b]) \rightarrow \mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\} \rightarrow \overline{\mathcal{M}}_1(3)$$

is an open immersion and thus determines an affine coordinate chart  $V$  of  $\overline{\mathcal{M}}_1(3)$ . On this chart the elliptic curve  $y^2 + xy + by = x^3$  has no supersingular fibers by Proposition 2.1, ii).

**Proposition 2.2.** *The restriction of  $\widehat{\mathcal{E}}$  to  $V = \text{Spec}(\mathbb{Z}_{(2)}[b])$  is a formal 2-divisible group whose mod-2 reduction has constant height 1.*

*Proof.* By Proposition 2.1 the restriction  $\widehat{\mathcal{E}}|_V$  is a 1-dimensional formal group of constant height 1, so it is 2-divisible.  $\square$

Observe that  $V \subseteq \overline{\mathcal{M}}_1(3)$  is the maximal open substack over which  $\mathcal{E}$  is ordinary.

Let  $\mathbb{Z}_{(2)}[A, B] \rightarrow \mathbb{Z}_{(2)}[a]$  be the map given by  $A \mapsto a, B \mapsto 1$ . The composite map

$$\text{Spec}(\mathbb{Z}_{(2)}[a]) \rightarrow \mathbb{A}_{\mathbb{Z}_{(2)}}^2 \setminus \{(0, 0)\} \rightarrow \overline{\mathcal{M}}_1(3)$$

is étale. The stack-theoretic image is the quotient of  $\text{Spec}(\mathbb{Z}_{(2)}[a])$  by the action of the group  $\mu_3$  of third roots of unity, given by  $\omega \cdot a = \omega a$  for  $\omega$  any third root of unity. The induced map

$$W = [\text{Spec}(\mathbb{Z}_{(2)}[a]) // \mu_3] \rightarrow \overline{\mathcal{M}}_1(3)$$

is an open immersion, and so  $\text{Spec}(\mathbb{Z}_{(2)}[a])$  determines an étale coordinate chart for the stack near  $a = 0$ . On this chart the elliptic curve is defined by the Weierstrass equation  $y^2 + axy + y = x^3$ , with  $\mu_3$ -action given by  $\omega \cdot (x, y) = (\omega^2 x, y)$ .

Let  $U$  be the formal scheme  $\text{Spf}(\mathbb{Z}_2[[a]])$ , with formally étale map  $U \rightarrow \overline{\mathcal{M}}_1(3)$ . By Proposition 2.1, the pullback of  $\mathcal{E}$  to  $U$  has special fiber a supersingular elliptic curve. We denote by  $\mathbb{G}/U$  the 2-divisible group of  $\mathcal{E}|_U$ .

**Proposition 2.3.** *The 2-divisible group  $\mathbb{G}/U$  has height 2 and is a universal deformation of its special fiber.*

*Proof.* This is a restatement of [LN12, Proposition 8.2 and Remark 4.2].  $\square$

The common overlap of the coordinate charts  $V$  and  $W$  is determined by the identity  $a^3 b = 1$ . The Mayer-Vietoris sequence for this weighted projective space, using these affine coordinate charts, allows us to compute the cohomology of  $\overline{\mathcal{M}}_1(3)_{(2)}$  with coefficients in  $\mathcal{O}(\ast)$ .



**Corollary 2.4.** *The cohomology of  $\overline{\mathcal{M}}_1(3)$  with coefficients in  $\mathcal{O}(\ast)$  vanishes above degree 1.*

*The cohomology groups  $H^0(\overline{\mathcal{M}}_1(3), \mathcal{O}(\ast))$  form the graded ring  $\mathbb{Z}_{(2)}[A, B]$ .*

*The cohomology  $H^1(\overline{\mathcal{M}}_1(3), \mathcal{O}(\ast))$  is the module  $\mathbb{Z}_{(2)}[A, B]/(A^\infty, B^\infty)$  of elements  $A^{-n}B^{-m}D$ , where  $D$  is a duality class in  $H^1(\overline{\mathcal{M}}_1(3), \mathcal{O}(-4))$  annihilated by  $A$  and  $B$ .*

### 3 Constructing the maps

The goal of this section is to construct the  $\mathcal{E}_\infty$ -maps of Theorem 1.2, diagram (1.1), which we reproduce here:

$$\begin{array}{ccc} \mathrm{tmf} & \xrightarrow{c} & ko \\ \downarrow o & & \downarrow \iota \\ \mathrm{tmf}_1(3) & \xrightarrow{\tilde{c}} & ku \end{array}$$

All spectra appearing in this diagram are the connective covers of their  $K(0) \vee K(1) \vee K(2)$ -localizations. Accordingly, we will construct the required  $\mathcal{E}_\infty$ -maps using two chromatic fracture squares, followed by taking connective covers.

#### 3.1 The $K(2)$ -local maps

We identify the  $K(2)$ -localizations of the connective spectra in Theorem 1.2 as follows. We have that  $L_{K(2)}KU \simeq \ast$ , and as  $K(n)$ -localization does not distinguish between a spectrum and its connective cover we have  $L_{K(2)}ku \simeq \ast$  as well. From the familiar fibration

$$\Sigma ko \xrightarrow{\eta} ko \rightarrow ku, \tag{3.1}$$

it follows that the nilpotent map  $\eta$  induces an equivalence on  $L_{K(2)}ko$ , hence  $L_{K(2)}ko \cong \ast$  as well.

Let  $E$  denote the Lubin-Tate spectrum associated with the formal group of the supersingular elliptic curve

$$C: y^2 + y = x^3$$

over  $\mathbb{F}_4$  [GH04, Section 7]. The group  $G_{48} = \text{Aut}(C/\mathbb{F}_4) \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$  acts on  $E$  and  $L_{K(2)}\text{tmf} \simeq E^{hG_{48}}$  [HM]. The subgroup  $\langle \omega \rangle \subseteq \text{Aut}(C/\mathbb{F}_4)$  fixing the point at infinity on  $C$  (whose generator sends  $(x, y)$  to  $(\omega^2 x, y)$ ) is cyclic of order 3 and defines a subgroup

$$S_3 \cong \langle \omega \rangle \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \subseteq G_{48}. \quad (3.2)$$

We have  $L_{K(2)}\text{tmf}_1(3) \simeq E^{hS_3}$  by the construction of  $\text{tmf}_1(3)$  [LN12, proof of Theorem 4.4].

We define the  $K(2)$ -localizations of the maps from diagram (1.1) as follows:

$$\begin{array}{ccc} E^{hG_{48}} & \longrightarrow & * \\ \downarrow o_{K(2)} & & \downarrow \\ E^{hS_3} & \longrightarrow & * \end{array}$$

The map  $o_{K(2)}$  is defined to be the canonical map of homotopy fixed point spectra associated with the inclusion of equation (3.2).

## 3.2 The $K(1)$ -local maps

We refer the reader to [Lau04, Hop, AHS04], as well as [LN12, Sections 5 and 6] and references therein, for an account of basic results about  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectra which we will use freely.

To ease reading, in this subsection only we will abbreviate

$$\begin{aligned} \text{TMF} &= L_{K(1)}\text{tmf}, \\ \text{TMF}_1(3) &= L_{K(1)}\text{tmf}_1(3), \\ KO &= L_{K(1)}ko, \text{ and} \\ K &= L_{K(1)}ku. \end{aligned} \quad (3.3)$$

Furthermore, all smash products will implicitly be  $K(1)$ -localized and all abelian groups implicitly 2-completed. We use  $K_*^\vee(-)$  to denote ( $K(1)$ -localized)  $K$ -homology, so that

$$K_*^\vee(-) = \pi_* L_{K(1)}(K \wedge -).$$

In order to construct the required maps between the  $\mathcal{E}_\infty$ -ring spectra in equation (3.3), we first construct maps between the  $\psi$ - $\theta$ -algebras given by their  $K^\vee$ -homology.

**Proposition 3.1.** *All spectra in equation (3.3) have  $K_*^\vee$  concentrated in even degrees, and there are isomorphisms of  $\psi$ - $\theta$ -algebras as follows.*

$$K_0^\vee(\mathrm{TMF}) \cong V \tag{3.4}$$

$$K_0^\vee(KO) \cong \mathrm{Hom}_c(\mathbb{Z}_2^\times / \{\pm 1\}, K_0) \tag{3.5}$$

$$K_0^\vee(K) \cong \mathrm{Hom}_c(\mathbb{Z}_2^\times, K_0) \tag{3.6}$$

Here  $V$  is (the level 1-analogue of) Katz's ring of generalized modular functions [Kat75, (1.4.9.1)], where it is denoted  $V_{\infty, \infty}$ .

These fit into a commutative diagram of  $\psi$ - $\theta$ -algebras as follows:

$$\begin{array}{ccc} K_0^\vee(\mathrm{TMF}) \cong V & \xrightarrow{a} & \mathrm{Hom}_c(\mathbb{Z}_2^\times / \{\pm 1\}, K_0) \cong K_0^\vee(KO) \\ \downarrow d & & \downarrow b \\ K_0^\vee(\mathrm{TMF}_1(3)) & \xrightarrow{c} & \mathrm{Hom}_c(\mathbb{Z}_2^\times, K_0) \cong K_0^\vee(K) \end{array} \tag{3.7}$$

*Proof.* First, we review the structure of these  $\psi$ - $\theta$ -algebras. According to [Hop, Lemma 1] and [Lau03, Proposition 3.4], or [AHR, Proposition 9.2], we have isomorphisms of  $\psi$ - $\theta$ -algebras as follows:

$$K_*^\vee K \cong \mathrm{Hom}_c(\mathbb{Z}_2^\times, K_*) \tag{3.8}$$

$$K_*^\vee KO \cong \mathrm{Hom}_c(\mathbb{Z}_2^\times / \{\pm 1\}, K_*) \tag{3.9}$$

$$KO_*^\vee KO \cong \mathrm{Hom}_c(\mathbb{Z}_2^\times / \{\pm 1\}, KO_*) \tag{3.10}$$

Here we have obvious  $\mathbb{Z}_2^\times$ -actions and trivial  $\theta$ , in the sense that the ring homomorphism  $\psi^2$  is the identity.

Furthermore, the inclusion of the constant functions  $K_* \subseteq K_*^\vee K$  (resp.  $K_* \subseteq K_*^\vee KO$ ), as the ring of  $\mathbb{Z}_2^\times$ -invariants, is a split  $\mathbb{Z}_2^\times$ - (resp.  $\mathbb{Z}_2^\times / \{\pm 1\}$ -) Galois extension.

Next, we consider  $K_0^\vee(\mathrm{TMF}_1(3))$ . We have a generalized elliptic curve

$$E: y^2 + xy + by = x^3 \tag{3.11}$$

over  $\mathbb{Z}_2[b]_2^\wedge$ . By construction [LN12, Section 6.2], the Hurewicz map

$$\pi_0\mathrm{TMF}_1(3) \rightarrow K_0^\vee\mathrm{TMF}_1(3)$$

has domain  $\mathbb{Z}_2[b]_2^\wedge$ , and is the  $\mathbb{Z}_2^\times$ -Galois extension classifying isomorphisms  $\widehat{\mathbb{G}}_m \rightarrow \widehat{E}$ . We will refer to such an isomorphism as a trivialization of the ordinary elliptic curve  $E/\pi_0(\mathrm{TMF}_1(3))$ . The operation  $\theta: K_0^\vee(\mathrm{TMF}_1(3)) \rightarrow K_0^\vee(\mathrm{TMF}_1(3))$  is determined by the canonical subgroup of  $E$  [Gou88, page 35] and can be computed explicitly [LN12, proof of Proposition 8.5]. Still by construction, we also have  $K_1^\vee\mathrm{TMF}_1(3) = 0$ .

The  $\psi$ - $\theta$ -algebra structure on  $V$  is determined similarly. The ring  $V$  carries a universal isomorphism class of trivialization of its elliptic curve. It has a continuous action of  $\mathbb{Z}_2^\times$  (acting on the universal trivialization) and a canonical lift of Frobenius  $\psi^2$ , which induces the natural transformation on  $V$  determined by the quotient by the canonical subgroup. Since  $V$  is torsion free, there is a unique self-map  $\theta$  of  $V$  such that  $\psi^2(x) = x^2 + 2\theta(x)$  for all  $x \in V$ . This gives the structure of a  $\psi$ - $\theta$ -algebra to  $V$ .

We will now establish the isomorphism of equation (3.4). From [Lau04, Theorem 3 and Proposition 1], we know  $KO_*^\vee\mathrm{TMF} \cong KO_* \otimes KO_0^\vee\mathrm{TMF}$  and  $V \simeq KO_0^\vee\mathrm{TMF}$  as  $\psi$ - $\theta$ -algebras.

Since  $\mathrm{TMF}$  is equivalent to a  $(K(1)$ -local) wedge of copies of  $KO$  [Lau04, Corollary 3], we find that

$$K_*^\vee\mathrm{TMF} \cong \widehat{\bigoplus} K_*^\vee KO. \quad (3.12)$$

Therefore, by equation (3.9) we find that  $K_*^\vee\mathrm{TMF}$  is concentrated in even degrees. Moreover, by equation (3.10) we have that the map  $V \cong KO_0^\vee\mathrm{TMF} \rightarrow K_0^\vee\mathrm{TMF}$  is an isomorphism.

We next construct maps between these  $\psi$ - $\theta$ -algebras as required in diagram (3.7).

Construction of the map  $b$ . The map  $b$  is determined by equations (3.8), (3.9), and pull-back along the canonical projection  $\mathbb{Z}_2^\times \rightarrow \mathbb{Z}_2^\times/\{\pm 1\}$ , and is clearly a map of  $\psi$ - $\theta$ -algebras.

Construction of the map  $d$ . By construction there is a trivialization of the elliptic curve  $E$  over  $K_0^\vee\mathrm{TMF}_1(3)$ . As  $V$  carries the universal example of a trivialized elliptic curve over a 2-adically complete ring, this

determines a map  $d: V = K_0^\vee \text{TMF} \rightarrow K_0^\vee \text{TMF}_1(3)$ . A  $\Gamma_1(3)$  structure on an elliptic curve  $E$  determines a unique compatible structure on the quotient by the canonical subgroup, as 2 and 3 are relatively prime. This implies that the induced map of classifying rings is a map of  $\psi$ - $\theta$ -algebras.

Construction of the map  $a$ . The map  $a$  is constructed in the same manner as  $d$ : the ring  $K_0^\vee KO$  carries the Tate curve  $y^2 + xy = x^3$  as universal among isomorphism classes of nodal elliptic curve equipped with a choice of trivialization, and the map  $a: V \rightarrow K_0^\vee KO$  classifies it.

Construction of the map  $c$ . The map

$$\pi_0 \text{TMF}_1(3) \cong \mathbb{Z}_2[b]_2^\wedge \rightarrow K_0^\vee \cong \mathbb{Z}_2,$$

determined by sending  $b \mapsto 0$  specializes the elliptic curve  $E$  of equation (3.11) over  $\pi_0 \text{TMF}_1(3)$  to the Tate curve  $T: y^2 + xy = x^3$ . We fix an isomorphism of formal groups  $\widehat{T} \cong \widehat{\mathbb{G}}_m$ , so that the pullback of the elliptic curve  $E$  under the composite map  $\pi_0 \text{TMF}_1(3) \rightarrow K_0^\vee K$  has a trivialization. By the universal property of  $K_0^\vee \text{TMF}_1(3)$ , this trivialization determines a map  $c: K_0^\vee \text{TMF}_1(3) \rightarrow K_0^\vee K$ , and we need to show that it commutes with  $\psi^\alpha$  ( $\alpha \in \mathbb{Z}_2^\times$ ) and  $\psi^2$ .

Let  $(E, f: \widehat{E} \xrightarrow{\cong} \widehat{\mathbb{G}}_m)$  be the universal trivialization of  $E$  over  $K_0^\vee(\text{TMF}_1(3))$ . Then

$$(\psi^\alpha)_*(E, f) = (E, [\alpha] \circ f)$$

for any  $\alpha \in \mathbb{Z}_2^\times = \text{Aut}(\widehat{\mathbb{G}}_m)$ . Hence

$$(c \circ \psi^\alpha)_*(E, f) = (T, c_*([\alpha]) \circ c_*(f)) = (T, [\alpha] \circ c_*(f)).$$

On the other hand,

$$(\psi^\alpha \circ c)_*(E, f) = \psi_*^\alpha(T, c_*(f)) = (T, [\alpha] \circ c_*(f)),$$

so  $\psi^\alpha \circ c = c \circ \psi^\alpha$ , making  $c$  compatible with the  $\mathbb{Z}_2^\times$ -action.

Recall [Gou88, page 35] the canonical subgroup  $C \subseteq E$  is defined so that there is the following diagram of formal groups over  $K_0^\vee(\text{TMF}_1(3))$

with exact columns:

$$\begin{array}{ccc}
 C & \xrightarrow{\cong} & \mu_2 \\
 \downarrow & & \downarrow \\
 \widehat{E} & \xrightarrow[f \cong]{} & \widehat{\mathbb{G}}_m \\
 \downarrow & & \downarrow [2] \\
 \widehat{E/C} & \xrightarrow[\cong]{\bar{f}} & \widehat{\mathbb{G}}_m
 \end{array} \tag{3.13}$$

From this, we find that  $c_*(\bar{f}) = \overline{c_*(f)}$ . By construction of  $\psi^2$ , we have

$$(\psi^2)_*(E, f) = (E/C, \bar{f}),$$

By the functoriality of the canonical subgroup and (3.13), we therefore find that

$$(c \circ \psi^2)_*(E, f) = (c_*(E/C), c_*(\bar{f})) = (T/C, \overline{c_*(f)}),$$

On the other hand,

$$(\psi^2 \circ c)_*(E, f) = \psi_*^2(T, c_*(f)) = (T/C, \overline{c_*(f)}).$$

Hence  $\psi^2 \circ c = c \circ \psi^2$ , and  $c$  is indeed a map of  $\psi$ - $\theta$ -algebras.

To see that diagram (3.7) commutes, it suffices to remark that both composites  $b \circ a$  and  $c \circ d$  classify the same trivialized generalized elliptic curve over  $K_0^\vee K$ , and this is true by construction.  $\square$

We now start to realize diagram (3.7) as the  $K^\vee$ -homology of a commutative diagram of  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectra. The authors have not been able to locate a complete proof in the literature for the following result, though it is known to the experts and a proof sketch can be found in [DM10, Remark 2.2].

**Proposition 3.2.** *There is an  $\mathcal{E}_\infty$ -map  $c_{K(1)}: \mathrm{TMF} \rightarrow KO$  such that  $K_0^\vee(c_{K(1)}) = a$  as in diagram (3.7).*

*Proof.* We remind the reader of the presentation of  $KO$  and  $TMF$  as finite cell  $L_{K(1)}\mathbb{S}$ -algebras. We will write  $\mathbb{P}X$  for the free  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectrum on a  $K(1)$ -local spectrum  $X$ .

There is a generator  $\zeta \in \pi_{-1}(L_{K(1)}S^0)$ , and we define  $T_\zeta$  to be  $\mathbb{S} \cup_\zeta e^0$ : the pushout of the diagram

$$\mathbb{S} \xleftarrow{0} \mathbb{P}S^{-1} \xrightarrow{\zeta} \mathbb{S}$$

in the category of  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectra. We refer to this as the  $\mathcal{E}_\infty$ -cone over  $\zeta$ .

There are elements  $y$  and  $f$  in  $\pi_0 T_\zeta$ . We refer the reader to the discussion surrounding [Lau04, Proposition 5] for the definitions of these elements, and to [Lau04, end of appendix] for the existence of a factorization of the attaching maps

$$y: \mathbb{P}S^0 \xrightarrow{\theta(x)-h(x)} \mathbb{P}S^0 \xrightarrow{f} T_\zeta.$$

The spectrum  $KO$  is  $T_\zeta \cup_f e^1$ , the  $\mathcal{E}_\infty$ -cone on  $f$  [Hop, Proposition 13], and  $TMF \simeq T_\zeta \cup_y e^1$  [Lau04, Convention on page 390] as  $\mathcal{E}_\infty$ -ring spectra. Therefore, there is an  $\mathcal{E}_\infty$ -map  $c_{K(1)}: TMF \rightarrow KO$  factoring the given attaching maps.

It remains to see that  $K_0^\vee(c_{K(1)}) = a$  and to this end, we first consider the effect of  $c_{K(1)}$  in homotopy. Remembering that everything is implicitly 2-completed, we know that

$$\pi_0 TMF = \mathbb{Z}_2[f] = \mathbb{Z}_2[j^{-1}] \tag{3.14}$$

by [Lau04, Proposition 6 and Lemma 9].

By construction  $f$  maps to zero under  $c_{K(1)}$ , and this implies that  $j^{-1}$  also maps to zero by the following computation.

The element  $j^{-1}$  is a 2-adically convergent power series in  $f$ :

$$j^{-1} = \sum_{n=0}^{\infty} a_n f^n \tag{3.15}$$

Clearly  $c_{K(1)}$  sends  $j^{-1}$  to  $a_0$ .

We now pass to  $q$ -expansions. It is classical that  $j^{-1}(q) = q + O(q^2)$ . Since  $f$  is of the form  $f = \psi(b) - b$  for a suitable 2-adic modular function  $b$  and  $\psi$  the

Frobenius operator [Lau04, Equation (33)], we learn that the  $q$ -expansion

$$f(q) = \psi(b)(q) - b(q) = b(q^2) - b(q)$$

has constant term 0. Hence, taking  $q$ -expansions of equation (3.15) and setting  $q = 0$  yields  $0 = a_0$ , as desired.

Finally, knowing that  $c_{K(1)}$  sends  $j^{-1}$  to 0 implies that  $K_0^\vee(c_{K(1)}) = a$ , because  $K_0^\vee(c_{K(1)})$  is the map induced on the Igusa towers ([LN12, Definition 5.6]) by the map  $\pi_0(c_{K(1)})$ .  $\square$

All other maps of  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectra we require will be constructed by obstruction theory. (The reason the map  $c_{K(1)}$  cannot be thus constructed is that  $K_0^\vee(KO)$  is not an induced  $\mathbb{Z}_2^\times$ -module (Equation 3.5), which is a manifestation of the fact that  $KO$  is not complex orientable.)

We recall, for a graded  $\psi$ - $\theta$ -algebra  $B_*$  over  $(K_p^\wedge)_*$ ,  $\Omega^t B_*$  is the kernel of the map of augmented  $\psi$ - $\theta$ -algebras

$$B_* \otimes_{(K_p^\wedge)_*} (K_p^\wedge)^* S^t \rightarrow B_*.$$

**Proposition 3.3.** *Let  $p$  be a prime and suppose  $X$  and  $Y$  are  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectra with the following properties.*

- i)  $K_0^\vee(X)$  and  $K_0^\vee(Y)$  are  $p$ -adically complete and  $K_1^\vee(X) = K_1^\vee(Y) = 0$ .
- ii) The inclusion  $(K_0^\vee(X))^{\mathbb{Z}_p^\times} \subseteq K_0^\vee(X)$  is the  $p$ -adic completion of an ind-étale extension.
- iii) The ring  $(K_0^\vee(X))^{\mathbb{Z}_p^\times}$  is the  $p$ -adic completion of a smooth  $\mathbb{Z}_p$ -algebra.
- iv) For all  $s > 0$  we have  $H_c^s(\mathbb{Z}_p^\times, K_*^\vee(Y)) = 0$ .

Then there is an isomorphism

$$f \mapsto K_0^\vee(f): \pi_0 \mathcal{E}_\infty(X, Y) \xrightarrow{\cong} \text{Hom}_{\psi-\theta}(K_0^\vee X, K_0^\vee Y),$$

given by the canonical map evaluating on  $K_0^\vee$ , from the set of connected components of the derived  $\mathcal{E}_\infty$ -mapping space to the set of  $\psi$ - $\theta$ -algebra maps.



*Proof.* The first assumption implies that  $A_* := K_*^\vee(X)$  and  $B_* := K_*^\vee(Y)$  are graded,  $p$ -adic, even-periodic  $\psi$ - $\theta$ -algebras. The remaining conditions are exactly those of [LN12, Lemma 5.14], application of which implies that for all  $s \geq 2$  or  $t \in \mathbb{Z}$  odd we have vanishing of the  $\psi$ - $\theta$ -algebra cohomology groups

$$H_{\psi-\theta}^s(A_*/(K_p^\wedge)_*, \Omega^t B_*).$$

The claim now follows from Goerss-Hopkins obstruction theory as in [LN12, Theorem 5.13, 3].  $\square$

We will make use of the following particular instances of this result.

**Proposition 3.4.** *For each dotted arrow between  $K(1)$ -local  $\mathcal{E}_\infty$ -ring spectra  $X$  and  $Y$  in the diagram*

$$\begin{array}{ccccc} & & \text{TMF} & & KO \\ & \swarrow \cdots & \downarrow \cdots & \searrow \cdots & \downarrow \cdots \\ L_{K(1)}L_{K(2)}\text{tmf}_1(3) & & \text{TMF}_1(3) & \cdots \rightarrow & K \end{array}$$

there is an isomorphism

$$f \mapsto K_0^\vee(f): \pi_0 \mathcal{E}_\infty(X, Y) \xrightarrow{\cong} \text{Hom}_{\psi-\theta}(K_0^\vee X, K_0^\vee Y)$$

given by evaluation on  $K_0^\vee$ .

*Proof.* In order to deduce this from Proposition 3.3, we need to know certain properties of the  $K^\vee$ -homology of the spectra involved, the local references for which we summarize in the following table.

	$KO$	$K$	$\text{TMF}_1(3)$	$L_{K(1)}L_{K(2)}\text{TMF}_1(3)$	$\text{TMF}$
$p$ -adic, even	(3.9)	(3.8)	[LN12, 5.4]	[LN12, 5.4]	(3.4)+(3.9)
ind-étale $K_0^\vee$	(3.9)	(3.8)	[LN12, 5.8]	[LN12, 5.8]	(3.4)
smooth subring	(3.9)	(3.8)	[LN12, 6.1]	[LN12, 3.5]	(3.14)
no cohomology	*	(3.8)	[LN12, 5.8,3]	[LN12, 5.8,3]	*

Here, the rows correspond to the itemized conditions in Proposition 3.3 and the columns to the spectra under consideration. Note an entry means that the given spectrum satisfies any assertions about either the domain spectrum  $X$  or the target  $Y$ . The statements labeled with an asterisk are actually

false:  $K_0^\vee(KO)$  and  $K_0^\vee(\mathrm{TMF})$  are not cohomologically trivial  $\mathbb{Z}_2^\times$ -modules. These statements are not needed, because this proposition does not make any assertions about maps *into*  $KO$  or  $\mathrm{TMF}$ .  $\square$

**Corollary 3.5.** *There is a diagram of  $\mathcal{E}_\infty$ -maps*

$$\begin{array}{ccc} \mathrm{TMF} = L_{K(1)}\mathrm{tmf} & \xrightarrow{c_{K(1)}} & KO = L_{K(1)}ko \\ \downarrow o_{K(1)} & & \downarrow \iota_{K(1)} \\ \mathrm{TMF}_1(3) = L_{K(1)}\mathrm{tmf}_1(3) & \xrightarrow{\tilde{c}_{K(1)}} & K = L_{K(1)}ku, \end{array} \quad (3.16)$$

with diagram (3.7) being realized by the  $K_0^\vee$ -homology of diagram (3.16) and diagram (3.16) commuting up to homotopy in the category of  $\mathcal{E}_\infty$ -ring spectra.

*Proof.* Applying Proposition 3.4, we obtain  $\mathcal{E}_\infty$ -maps  $o_{K(1)}$ ,  $\tilde{c}_{K(1)}$ , and  $\iota_{K(1)}$  that are characterized up to homotopy by satisfying  $K_0^\vee(o_{K(1)}) = d$ ,  $K_0^\vee(\tilde{c}_{K(1)}) = c$ , and  $K_0^\vee(\iota_{K(1)}) = b$ . From Proposition 3.2 we already have the  $\mathcal{E}_\infty$ -map  $c_{K(1)}$  satisfying  $K_0(c_{K(1)}) = a$ . Note that we do not need to know whether  $a$  is characterized by its effect in  $K^\vee$ -homology.

Proposition 3.4 then reduces the homotopy commutativity of diagram (3.16) to the previously established commutativity of diagram (3.7).  $\square$

### 3.3 Chromatic gluing of maps

We briefly remind the reader of chromatic pullbacks in stable homotopy, referring to the introduction of [GHMR05] for more details and references.

Fixing a prime  $p$ , every  $p$ -local spectrum  $X$  maps canonically to a tower of Bousfield localizations

$$X \longrightarrow (\cdots L_n X \longrightarrow L_{n-1} X \longrightarrow \cdots \longrightarrow L_0 X = X \otimes \mathbb{Q}),$$

and the various stages of this tower are determined by canonical homotopy pullbacks, called chromatic fracture squares [HS99] or [Lura, Lecture 23, Proposition 5]:

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1}(L_{K(n)} X) \end{array} \quad (3.17)$$

Here,  $K(n)$  denotes any Morava  $K$ -theory of height  $n$  at the prime  $p$  and the localization functor  $L_n$  is naturally equivalent to  $L_{K(0)\vee\dots\vee K(n)}$ . We will write  $L_{K(1)}L_{K(2)}X$  for the iterated localization  $L_{K(1)}L_{K(2)}X$ , and similarly for other iterates.

We will use other canonical homotopy pullbacks similar to (3.17), such as the following:

$$\begin{array}{ccc} L_{K(1)\vee K(2)}Y & \longrightarrow & L_{K(2)}Y \\ \downarrow & & \downarrow \\ L_{K(1)}Y & \longrightarrow & L_{K(1)}L_{K(2)}Y \end{array} \quad (3.18)$$

**Lemma 3.6.** *Assume  $f_{K(i)}: L_{K(i)}X \rightarrow L_{K(i)}Y$  ( $i = 1, 2$ ) are  $\mathcal{E}_\infty$ -maps such that the diagram*

$$\begin{array}{ccc} L_{K(1)}X & \longrightarrow & L_{K(1)}L_{K(2)}X \\ \downarrow f_{K(1)} & & \downarrow L_{K(1)}(f_{K(2)}) \\ L_{K(1)}Y & \longrightarrow & L_{K(1)}L_{K(2)}Y \end{array}$$

*commutes up to homotopy in the category of  $\mathcal{E}_\infty$ -ring spectra.*

*Then there is an  $\mathcal{E}_\infty$ -map  $f: L_{K(1)\vee K(2)}X \rightarrow L_{K(1)\vee K(2)}Y$ , not necessarily unique, such that the diagrams*

$$\begin{array}{ccc} L_{K(1)\vee K(2)}X & \xrightarrow{f} & L_{K(1)\vee K(2)}Y \\ \downarrow & & \downarrow \\ L_{K(1)}X & \xrightarrow{f_{K(1)}} & L_{K(1)}Y \end{array} \quad \begin{array}{ccc} L_{K(1)\vee K(2)}X & \xrightarrow{f} & L_{K(1)\vee K(2)}Y \\ \downarrow & & \downarrow \\ L_{K(2)}X & \xrightarrow{f_{K(2)}} & L_{K(2)}Y \end{array}$$

*commute up to homotopy.*

*Proof.* Apply the derived mapping-space functor  $\mathcal{E}_\infty(L_{K(1)\vee K(2)}X, -)$  to the chromatic fracture square (3.18).  $\square$

**Corollary 3.7.** *There exists an  $\mathcal{E}_\infty$ -map*

$$o_{K(1)\vee K(2)}: L_{K(1)\vee K(2)}\mathrm{tmf} \rightarrow L_{K(1)\vee K(2)}\mathrm{tmf}_1(3)$$

such that the diagrams of  $\mathcal{E}_\infty$ -maps

$$\begin{array}{ccc}
L_{K(1)\vee K(2)}\mathrm{tmf} & \xrightarrow{o_{K(1)\vee K(2)}} & L_{K(1)\vee K(2)}\mathrm{tmf}_1(3) \\
\downarrow & & \downarrow \\
L_{K(1)}\mathrm{tmf} & \xrightarrow{o_{K(1)}} & L_{K(1)}\mathrm{tmf}_1(3)
\end{array}
\quad
\begin{array}{ccc}
L_{K(1)\vee K(2)}\mathrm{tmf} & \xrightarrow{o_{K(1)\vee K(2)}} & L_{K(1)\vee K(2)}\mathrm{tmf}_1(3) \\
\downarrow & & \downarrow \\
L_{K(2)}\mathrm{tmf} & \xrightarrow{o_{K(2)}} & L_{K(2)}\mathrm{tmf}_1(3)
\end{array}$$

both commute up to homotopy.

*Proof.* This follows from Lemma 3.6, provided we can establish the commutativity up to homotopy of the following diagram of  $\mathcal{E}_\infty$ -ring spectra:

$$\begin{array}{ccc}
L_{K(1)}\mathrm{tmf} & \xrightarrow{o_{K(1)}} & L_{K(1)}\mathrm{tmf}_1(3) \\
\downarrow & & \downarrow \\
L_{K(1)}L_{K(2)}\mathrm{tmf} & \xrightarrow{L_{K(1)}(o_{K(2)})} & L_{K(1)}L_{K(2)}\mathrm{tmf}_1(3)
\end{array} \tag{3.19}$$

The initial and terminal objects in this diagram appear in Proposition 3.4, and so it suffices to see that the induced diagram in  $K^\vee$ -homology commutes:

$$\begin{array}{ccc}
V & \longrightarrow & K_0^\vee L_{K(1)}\mathrm{tmf}_1(3) \\
\downarrow & & \downarrow \\
K_0^\vee L_{K(1)}L_{K(2)}\mathrm{tmf} & \longrightarrow & K_0^\vee L_{K(1)}L_{K(2)}\mathrm{tmf}_1(3)
\end{array}$$

This holds true because both composites classify isomorphic trivializations of the elliptic curves  $y^2 + xy + a^{-3}y = x^3$  and  $y^2 + axy + y = x^3$  over  $\mathbb{Z}((a))_2^\wedge \cong \pi_0 L_{K(1)}L_{K(2)}\mathrm{tmf}_1(3)$  (see Section 2).  $\square$

**Proposition 3.8.** *There exists a diagram of  $\mathcal{E}_\infty$ -ring spectra which commutes up to homotopy as follows:*

$$\begin{array}{ccc}
L_{K(1)\vee K(2)}\mathrm{tmf} & \xrightarrow{e_{K(1)\vee K(2)}} & L_{K(1)\vee K(2)}ko \\
\downarrow o_{K(1)\vee K(2)} & & \downarrow \iota_{K(1)\vee K(2)} \\
L_{K(1)\vee K(2)}\mathrm{tmf}_1(3) & \xrightarrow{\tilde{e}_{K(1)\vee K(2)}} & L_{K(1)\vee K(2)}ku
\end{array}$$

*Proof.* We have the following diagram of  $\mathcal{E}_\infty$ -ring spectra which commutes up to homotopy:

$$\begin{array}{ccccc}
L_{K(1)\vee K(2)}\mathrm{tmf} & \longrightarrow & L_{K(1)}\mathrm{tmf} & \xrightarrow{c_{K(1)}} & L_{K(1)}ko & (3.20) \\
\downarrow o_{K(1)\vee K(2)} & & \downarrow o_{K(1)} & & \downarrow \iota_{K(1)\vee K(2)} \\
L_{K(1)\vee K(2)}\mathrm{tmf}_1(3) & \longrightarrow & L_{K(1)}\mathrm{tmf}_1(3) & \xrightarrow{\tilde{c}_{K(1)}} & L_{K(1)}ku
\end{array}$$

Here, the left square is from Corollary 3.7, and the right one is from Corollary 3.5. Since  $L_{K(1)\vee K(2)}ko \simeq L_{K(1)}ko$  and similarly for  $ku$ , we can define the upper and lower horizontal composites to be  $c_{K(1)\vee K(2)}$  and  $\tilde{c}_{K(1)\vee K(2)}$  respectively.  $\square$

### 3.4 The rational maps

We first note the following about rational  $\mathcal{E}_\infty$ -ring spectra.

**Lemma 3.9.** *Suppose  $X$  and  $Y$  are  $\mathcal{E}_\infty$ -ring spectra such that  $\pi_*X \otimes \mathbb{Q}$  is a free graded-commutative  $\mathbb{Q}$ -algebra on generators in even nonnegative degrees, and  $\pi_*Y$  is rational with homotopy in nonnegative odd degrees. Then the natural map*

$$\pi_0\mathcal{E}_\infty(X, Y) \rightarrow \mathrm{Hom}_{\mathrm{graded\ rings}}(X_*, Y_*)$$

*is bijective, and all path components of the derived mapping space  $\mathcal{E}_\infty(X, Y)$  are simply connected.*

*Proof.* Since  $Y$  is rational, the natural map

$$\mathcal{E}_\infty(X \otimes \mathbb{Q}, Y) \rightarrow \mathcal{E}_\infty(X, Y)$$

is a weak equivalence. As  $X \otimes \mathbb{Q}$  is equivalent to a free  $H\mathbb{Q}$ -algebra on some family of cells  $x_i: S^{2n_i} \rightarrow X$ , evaluation on the generators gives a weak equivalence, natural in  $Y$ , of the form

$$\mathcal{E}_\infty(X \otimes \mathbb{Q}, Y) \rightarrow \prod \Omega^{\infty+2n_i}Y.$$

The result follows by considering  $\pi_0$  and  $\pi_1$  of the right-hand side.  $\square$

**Theorem 3.10.** *There exists a strictly commutative diagram in the category of rational  $\mathcal{E}_\infty$ -ring spectra as follows:*

$$\begin{array}{ccc}
L_{K(0)}\mathrm{tmf} & \xrightarrow{\cdots} & L_{K(0)}ko \\
\downarrow \cdots & \searrow & \downarrow \cdots \\
L_{K(0)}\mathrm{tmf}_1(3) & \xrightarrow{\cdots} & L_{K(0)}ku \\
& \searrow & \searrow \\
& L_{K(0)}L_{K(1)\vee K(2)}\mathrm{tmf} & \xrightarrow{\quad} L_{K(0)}L_{K(1)\vee K(2)}ko \\
& \downarrow & \downarrow \\
& L_{K(0)}L_{K(1)\vee K(2)}\mathrm{tmf}_1(3) & \xrightarrow{\quad} L_{K(0)}L_{K(1)\vee K(2)}ku
\end{array}
\tag{3.21}$$

*In this diagram, the bottom square is the rationalization of the diagram displayed in Proposition 3.8 and the diagonal maps are arithmetic attaching maps.*

*Proof.* To construct diagram (3.21) we must first construct the maps in the top square of the cube as to render the entire diagram homotopy commutative. Recall:

$$\begin{aligned}
\pi_*\mathrm{tmf} \otimes \mathbb{Q} &\cong \mathbb{Q}[c_4, c_6] && \text{where } |c_i| = 2i \\
\pi_*\mathrm{tmf}_1(3) \otimes \mathbb{Q} &\cong \mathbb{Q}[A, B] && |A| = 2, |B| = 6 \text{ [LN12, proof of Theorem 1.1]} \\
\pi_*ku \otimes \mathbb{Q} &\cong \mathbb{Q}[\beta] && |\beta| = 2 \\
\pi_*ko \otimes \mathbb{Q} &\cong \mathbb{Q}[\beta^2]
\end{aligned}$$

In nonnegative degrees, the diagonal maps in diagram (3.21) are given on homotopy groups by extension of scalars from  $\mathbb{Q}$  to  $\mathbb{Q}_2$ .

Evaluating the modular forms  $c_4$  and  $c_6$  on

$$y^2 + Axy + By = x^3,$$

the universal elliptic curve with  $\Gamma_1(3)$ -structure used to construct  $\mathrm{tmf}_1(3)$ , we find that

$$c_4 \mapsto A^4 - 24AB, \quad c_6 \mapsto -A^6 + 36A^3B - 216B^2.$$

Similarly, evaluating at the Tate curve  $y^2 + \beta xy = x^3$ , we find that

$$A \mapsto \beta, B \mapsto 0, \quad (3.22)$$

and

$$c_4 \mapsto \beta^4, c_6 \mapsto -\beta^6.$$

These formulas make diagram (3.21) commutative on homotopy groups.

The homotopy groups of the spectra in the upper square of diagram (3.21) form polynomial algebras, and all spectra in the diagram have zero homotopy in positive odd degrees. Lemma 3.9 thus implies that constructing the maps in this diagram is equivalent to defining the maps on homotopy groups, and that the homotopy-commutativity of each square subdiagram is equivalent to the commutativity of the square on homotopy groups. This shows that the cubical diagram commutes in the homotopy category.

Finally, the obstruction to lifting a homotopy commutative cubical diagram to an honestly commutative, homotopy equivalent cubical diagram lies in  $\pi_1 \mathcal{E}_\infty(L_{K(0)} \text{tmf}, L_{K(0), K(1) \vee K(2)} ku)$ , which is the zero group (again by Lemma 3.9).  $\square$

**Corollary 3.11.** *There is a commutative square of  $\mathcal{E}_\infty$ -ring spectra as follows:*

$$\begin{array}{ccc} L_{K(0) \vee K(1) \vee K(2)} \text{tmf} & \xrightarrow{c_{K(0) \vee K(1) \vee K(2)}} & L_{K(0) \vee K(1) \vee K(2)} ko \\ \downarrow \circ_{K(0) \vee K(1) \vee K(2)} & & \downarrow \iota_{K(0) \vee K(1) \vee K(2)} \\ L_{K(0) \vee K(1) \vee K(2)} \text{tmf}_1(3) & \xrightarrow{\tilde{c}_{K(0) \vee K(1) \vee K(2)}} & L_{K(0) \vee K(1) \vee K(2)} ku \end{array} \quad (3.23)$$

*Proof.* For the  $K(0) \vee K(1) \vee K(2)$ -local spectra under consideration,  $L_{K(1) \vee K(2)}$  is  $p$ -adic completion. We have canonical arithmetic squares

$$\begin{array}{ccc} L_{K(0) \vee K(1) \vee K(2)} Y & \longrightarrow & L_{K(1) \vee K(2)} Y \\ \downarrow & & \downarrow \\ L_{K(0)} Y & \longrightarrow & L_{K(0)} L_{K(1) \vee K(2)} Y. \end{array}$$

We can then take levelwise homotopy pullbacks of the maps

$$L_{K(0)} Y \rightarrow L_{K(0)} L_{K(1) \vee K(2)} Y \leftarrow L_{K(1) \vee K(2)} Y$$

from the diagonals of diagram (3.21) and obtain the desired commutative square.  $\square$

## 4 The cohomology computation

The techniques used in this section are very similar to those employed in [Rez] to calculate  $H^*(\mathrm{tmf})$ .

Let  $p$  be a prime, abbreviate  $H := H\mathbb{F}_p$  and recall that the dual Steenrod algebra  $A_* = H_*H$  takes the form

$$A_* \cong \begin{cases} P(\bar{\xi}_1, \bar{\xi}_2, \dots) & \text{if } p = 2, \\ P(\bar{\xi}_1, \bar{\xi}_2, \dots) \otimes E(\bar{\tau}_0, \bar{\tau}_1, \dots) & \text{if } p \neq 2. \end{cases}$$

Suppose a  $p$ -local spectrum  $X$  is connective and of finite type, with a map  $X \rightarrow H$ , such that the mod- $p$  homology maps isomorphically to the sub-Hopf-algebra of  $A_*$  given by

$$H_*X \cong \begin{cases} P(\bar{\xi}_1^2, \dots, \bar{\xi}_{n+1}^2, \bar{\xi}_{n+2}, \dots) & \text{if } p = 2, \\ P(\bar{\xi}_1, \dots) \otimes E(\bar{\tau}_{n+1}, \dots) & \text{if } p \neq 2. \end{cases}$$

For example, this is true when  $X = \mathrm{BP}\langle n \rangle$ . In these circumstances the Adams spectral sequence degenerates, and we find that

$$\pi_*X \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

where  $|v_i| = 2(p^i - 1)$  [Rav86, Chapter 4, Section 2, page 111]. (This is only necessarily an isomorphism as graded abelian groups unless  $X$  is a homotopy commutative and associative ring spectrum.) We will establish a converse to this isomorphism under the assumption that  $X$  is a ring spectrum in Theorem 4.3.

**Definition 4.1.** A  $p$ -local ring spectrum  $R$  is a *generalized*  $\mathrm{BP}\langle n \rangle$  if it admits a complex orientation such that the resulting composite map

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subseteq \pi_*BP \rightarrow \pi_*MU_{(p)} \rightarrow \pi_*R$$

is an isomorphism.

We remark that as the element  $v_i$  is an invariant of the formal group modulo  $(p, v_1, \dots, v_{i-1})$ , the property of a  $p$ -local, homotopy commutative, complex orientable ring spectrum being a generalized  $\mathrm{BP}\langle n \rangle$  depends only on the



ring structure and is independent of the choice of complex orientation. In particular, it does not depend on the choice of Hazewinkel, Araki, or arbitrary other  $p$ -typical  $v_i$ -classes.

We also note that any generalized  $\mathrm{BP}\langle n \rangle$  is the connective cover of its  $L_n$ -localization, which is the same as its  $L_{K(0)\vee K(1)\vee \dots \vee K(n)}$ -localization. This fact served as the basis for the construction of  $\mathrm{tmf}_1(3)$  by chromatic fracture. There does not appear to be an easier method to prove this than direct calculation: the chromatic fracture squares give rise to an isomorphism

$$\pi_* L_n \mathrm{BP}\langle n \rangle \cong \mathrm{BP}\langle n \rangle_* \oplus \Sigma^{-k} \mathrm{BP}\langle n \rangle_* / (p^\infty, v_1^\infty, \dots, v_n^\infty),$$

where  $k = (\sum_{i=0}^n 2p^i - 2) - 1$ . There are many closely related spectra where the identification as a connective cover fails.

The authors are not aware of any results establishing uniqueness of  $\mathrm{BP}\langle n \rangle$  when  $n \geq 2$ . It is not clear when, after forgetting the ring spectrum structure, two generalized  $\mathrm{BP}\langle n \rangle$  with nonisomorphic formal groups might have the same underlying homotopy type. It is also not clear, for any particular formal group of the correct form, how many weak equivalence classes of generalized  $\mathrm{BP}\langle n \rangle$  might exist realizing this formal group. (There exist results if one assumes additional structure, such as that of an  $MU$ -module or  $MU$ -algebra; see, for example, [JW].)

The following is a consequence of Definition 4.1 and of the existence of the Quillen idempotent splitting  $MU_{(p)} \rightarrow BP$ .

**Lemma 4.2.** *Suppose  $R$  is a generalized  $\mathrm{BP}\langle n \rangle$ . Then there are maps of ring spectra  $BP \rightarrow R \rightarrow H$ , with the former map  $(2p^{n+1} - 2)$ -connected and the latter unique up to homotopy. If  $R$  is an  $\mathcal{A}_\infty$ -ring spectrum or an  $\mathcal{E}_\infty$ -ring spectrum, then the map  $R \rightarrow H$  is a map of  $\mathcal{A}_\infty$ -ring spectra or  $\mathcal{E}_\infty$ -ring spectra accordingly.*

**Theorem 4.3.** *Suppose  $R$  is a generalized  $\mathrm{BP}\langle n \rangle$ . Then the map  $R \rightarrow H$  induces an isomorphism of  $H^*R$  with the left  $\mathcal{A}^*$ -module  $\mathcal{A}^*//E(n)$ , and of  $H_*R$  with the subalgebra  $B_*$  of the dual Steenrod algebra given as follows:*

$$B_* = \begin{cases} P(\bar{\xi}_1^2, \dots, \bar{\xi}_{n+1}^2, \bar{\xi}_{n+2}, \dots) & \text{if } p = 2, \\ P(\bar{\xi}_1^2, \dots) \otimes E(\bar{\tau}_{n+1}, \dots) & \text{if } p \neq 2. \end{cases}$$

*Proof.* We recall from [BJ02, Theorem 3.4] that the Brown-Peterson spectrum  $BP$  admits an  $\mathcal{A}_\infty$  ring structure (in fact, it admits many). In the following we choose one for definiteness. Smashing the maps from Lemma 4.2 on the right with  $BP$  gives a sequence of maps

$$BP \wedge BP \rightarrow R \wedge BP \rightarrow H \wedge BP.$$

Complex orientability of  $BP$ ,  $R$ , and  $H$  implies that on homotopy groups, this becomes the sequence of maps of polynomial algebras

$$BP_*[t_i] \rightarrow R_*[t_i] \rightarrow \mathbb{F}_p[t_i],$$

with the polynomial generators  $t_i$  mapped identically. We then apply the natural equivalence  $(- \wedge BP) \wedge_{BP} H \simeq (-) \wedge H$ , together with the Künneth spectral sequence [EKMM97, Theorem IV.4.1], to obtain a natural map of spectral sequences:

$$\mathrm{Tor}_{**}^{BP_*}(R_*[t_i], \mathbb{F}_p) \rightarrow \mathrm{Tor}_{**}^{BP_*}(\mathbb{F}_p[t_i], \mathbb{F}_p), \quad (4.1)$$

which strongly converges to the map  $R_*H \rightarrow H_*H$ . Here the action of the generators  $v_i \in BP_*$  is through their images under the right unit  $BP_* \xrightarrow{\eta_R} BP_*[t_i]$ . Modulo  $(p, v_1, \dots, v_{k-1})$ , the image of  $v_k$  under the right unit is equal to  $v_k$ .

Writing  $S_* = BP_*/(p, v_1, \dots, v_n)$ , we have an identification of derived tensor products

$$(- \otimes_{BP_*}^{\mathbb{L}} S_*) \otimes_{S_*}^{\mathbb{L}} \mathbb{F}_p \cong (-) \otimes_{BP_*}^{\mathbb{L}} \mathbb{F}_p.$$

This shows that the map of equation (4.1) is the abutment of a map of Cartan-Eilenberg spectral sequences:

$$\mathrm{Tor}_{**}^{S_*}(\mathrm{Tor}_{**}^{BP_*}(R_*[t_i], S_*), \mathbb{F}_p) \rightarrow \mathrm{Tor}_{**}^{S_*}(\mathrm{Tor}_{**}^{BP_*}(\mathbb{F}_p[t_i], S_*), \mathbb{F}_p) \quad (4.2)$$

The elements  $(p, v_1, \dots, v_n)$  form a regular sequence in  $R_*$ . The image of the regular sequence  $(p, v_1, \dots, v_n) \in BP_*$  under the map

$$BP_* \xrightarrow{\eta_R} BP_*BP \rightarrow R_*BP \simeq R_*[t_i]$$

is therefore a regular sequence, by induction, because every  $v_k$  is invariant modulo  $(p, \dots, v_{k-1})$  and because of the assumed properties of the map  $BP_* \rightarrow R_*$ .

This shows that the higher Tor-groups in

$$\mathrm{Tor}_{**}^{BP_*}(R_*[t_i], S_*)$$

are zero, with the zero'th term given by the tensor product  $R_*[t_i] \otimes_{BP_*} S_* \cong \mathbb{F}_p[t_i]$ .

By contrast, the image of  $v_k$  in  $\mathbb{F}_p[t_i]$  under the right unit is zero for all  $k$ , and hence the Tor-algebras are exterior algebras.

Therefore, the map of equation (4.2) degenerates to an edge inclusion:

$$\mathbb{F}_p[t_i] \otimes \Lambda[x_{n+1}, x_{n+2}, \dots] \rightarrow \mathbb{F}_p[t_i] \otimes \Lambda[x_1, \dots, x_n] \otimes \Lambda[x_{n+1}, x_{n+2}, \dots]$$

Here  $t_i$  is in total degree  $2p^i - 2$  and  $x_i$  is in total degree  $2p^i - 1$ . For the right-hand term, the associated graded vector space already has the same dimension in each total degree as the dual Steenrod algebra. Therefore, both the Cartan-Eilenberg and Künneth spectral sequences must degenerate, as the final target is the dual Steenrod algebra and any non-trivial differentials would result in a graded vector space with strictly smaller dimension in some degree.

We find that the map  $R_*H \rightarrow H_*H$  is an inclusion of right comodules over the dual Steenrod algebra, and the image below degree  $2p^{n+1} - 1$  consists only of terms in even degrees. On cohomology, this implies that the map  $\mathcal{A}^* \rightarrow H^*R$  is a surjection of left  $\mathcal{A}^*$ -modules, and the image of the generator  $1 \in \mathcal{A}^*$  is acted on trivially by the odd-degree Milnor primitives  $Q^0, \dots, Q^n$ . The induced map  $\mathcal{A}^*//E(n) \rightarrow H^*R$  is still a surjection and both sides are graded vector spaces of the same, levelwise finite, dimensions over  $\mathbb{F}_p$ .

This shows that  $H^*R$  has the desired form. The statement for homology follows by dualizing the cohomology description.  $\square$

## 5 Proof of Theorem 1.2

We have now assembled all the preliminaries needed to give the proof of Theorem 1.2, For ease of reference, we recall the statements we need to prove.

i) There is a commutative diagram of connective  $\mathcal{E}_\infty$ -ring spectra as follows:

$$\begin{array}{ccc} \mathrm{tmf}_{(2)} & \xrightarrow{c} & ko_{(2)} \\ \downarrow o & & \downarrow \iota \\ \mathrm{tmf}_1(3)_{(2)} & \xrightarrow{\tilde{c}} & ku_{(2)} \end{array}$$

ii) In mod-2 cohomology, this induces the following canonical diagram of modules over the mod 2 Steenrod algebra  $\mathcal{A}^*$ :

$$\begin{array}{ccc} \mathcal{A}^* // \mathcal{A}(2) & \longleftarrow & \mathcal{A}^* // \mathcal{A}(1) \\ \uparrow & & \uparrow \\ \mathcal{A}^* // E(2) & \longleftarrow & \mathcal{A}^* // E(1). \end{array}$$

iii) There exists a complex orientation of  $\mathrm{tmf}_1(3)_{(2)}$  such that in homotopy  $\tilde{c}$  sends the Hazewinkel generators  $v_1$  to  $v_1$  and  $v_2$  to zero.

iv) There is a cofiber sequence of  $\mathrm{tmf}_1(3)_{(2)}$ -modules

$$\Sigma^6 \mathrm{tmf}_1(3)_{(2)} \xrightarrow{v_2} \mathrm{tmf}_1(3)_{(2)} \xrightarrow{\tilde{c}} ku_{(2)}.$$

*Proof.* The existence of the desired commutative diagram of  $\mathcal{E}_\infty$ -ring spectra is established by taking connective covers of diagram (3.23).

It is well known that there are isomorphisms of  $\mathcal{A}^*$ -modules  $H^*(ko) \cong \mathcal{A}^* // \mathcal{A}(1)$  and  $H^*(ku) \cong \mathcal{A}^* // E(1)$ . Theorem 1.1, together with Theorem 4.3, implies  $H_* \mathrm{tmf}_1(3) \cong P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_3^2, \bar{\xi}_4, \dots)$  and  $H^* \mathrm{tmf}_1(3) \cong \mathcal{A}^* // E(2)$ . A well-known result, based on work on  $\mathrm{tmf}$  initiated by Hopkins, Mahowald and Miller, is that  $H^*(\mathrm{tmf}) \cong \mathcal{A} // \mathcal{A}(2)$  as a module over the Steenrod algebra [HM, Theorem 9.2]. To the best of the authors' knowledge, this result still awaits official documentation. A sketch based on the characterization of [Rez, Theorem 14.5] can be found in [Rez, Section 21].

The diagram of  $\mathcal{A}^*$ -modules in the statement of the theorem commutes because all appearing  $\mathcal{A}^*$ -modules are cyclic, generated by 1.

To address the remaining statements, note that the map  $\pi_*(\tilde{c}): \pi_* \mathrm{tmf}_1(3)_{(2)} \rightarrow \pi_* ku_{(2)}$  is determined by its rationalization

$$\mathbb{Q}[A, B] \rightarrow \mathbb{Q}[\beta],$$

where we have  $A \mapsto \beta$  and  $B \mapsto 0$  by construction (see Theorem 3.10). Now recall that there exists an orientation  $BP \rightarrow \mathrm{tmf}_1(3)_{(2)}$  which maps  $v_1$  to  $A$  and  $v_2$  to  $B$  by [LN12, Proposition 8.2].

The composite  $\mathrm{tmf}_1(3)_{(2)}$ -module map  $\tilde{c} \circ (\cdot v_2): \Sigma^6 \mathrm{tmf}_1(3)_{(2)} \rightarrow ku_{(2)}$  sends the generator to  $0 = \pi_*(\tilde{c})(v_2) \in \pi_6(ku_{(2)})$ , and hence there is a factorization in the category of  $\mathrm{tmf}_1(3)_{(2)}$ -modules

$$\begin{array}{ccccc}
 \Sigma^6 \mathrm{tmf}_1(3)_{(2)} & \xrightarrow{\cdot v_2} & \mathrm{tmf}_1(3)_{(2)} & \longrightarrow & \mathrm{cof}((\cdot v_2)) \\
 & \searrow 0 & \downarrow \tilde{c} & & \swarrow \text{dotted} \\
 & & ku_{(2)} & & 
 \end{array}$$

Examining homotopy groups, we get an induced equivalence between  $ku_{(2)}$  and the cofiber of  $v_2$  as spectra, and hence as  $\mathrm{tmf}_1(3)_{(2)}$ -modules.  $\square$

## A Appendix: Forms of $K$ -theory

In this section we describe how to functorially construct certain forms of  $K$ -theory [Mor89] as  $\mathcal{E}_\infty$ -ring spectra, and then give a discussion of a form of  $K$ -theory related to  $\mathrm{tmf}_1(3)_{(2)}$ . The core content is a restatement of the fact that complex conjugation acts on  $KU$  by  $\mathcal{E}_\infty$ -ring maps, that the group  $\mathbb{Z}_p^\times$  acts on  $KU_p^\wedge$  by  $\mathcal{E}_\infty$ -ring maps, and that the element  $-1$  acts compatibly with complex conjugation. (However, these forms receive less attention than they might, and the reader who has not read Morava's paper recently deserves a reminder to do so.)

Indeed, some of this section could be regarded as consequences of the Goerss-Hopkins-Miller theorem [Rez98, GH04]. After  $p$ -completion, any form of  $K$ -theory that we construct is a Lubin-Tate spectrum for the formal group law over its residue field, and the spaces of  $\mathcal{E}_\infty$ -maps are homotopically discrete and equivalent to certain sets of isomorphisms between the associated formal group laws.

**Definition A.1.** A *form of the multiplicative group scheme* over  $X$  is a group scheme over  $X$  which becomes isomorphic to  $\mathbb{G}_m$  after a faithfully flat extension. (This is the same as a one-dimensional torus in the sense of SGA3.)

If  $X$  is a formal scheme over  $\mathrm{Spf}(\mathbb{Z}_p)$ , a *form of the formal multiplicative group* over  $X$  is a 1-dimensional formal group over  $X$  whose reduction to  $X/p$  is of height 1.

*Remark A.2.* There are numerous examples of forms of  $\mathbb{G}_m$ . For  $b$  and  $c$  in  $R$ , there is a group scheme structure with unit 0 on the complement of the roots of  $1 + bx + cx^2$  in  $\mathbb{P}^1$ , given by

$$F(x, y) = \frac{x + y + bxy}{1 - cxy}.$$

The isomorphism class of such an object (by an isomorphism fixing the invariant differential) is determined by the isomorphism class of  $y^2 + by + c$  under translations  $y \mapsto y + r$ , and the object is a form of  $\mathbb{G}_m$  if and only if the discriminant  $b^2 - 4c$  is invertible. Morava was originally interested in the specific formal group laws of the form

$$F(x, y) = \frac{x + y + (1 - a)xy}{1 + axy}.$$

**Definition A.3.** We denote by  $\mathcal{M}_{\mathbb{G}_m}/\mathrm{Spec}(\mathbb{Z})$  the stack which is the moduli of forms of the multiplicative group scheme  $\mathbb{G}_m$ . For a fixed prime  $p$ , we denote by  $\mathcal{M}_{\widehat{\mathbb{G}}_m}/\mathrm{Spf}(\mathbb{Z}_p)$  the stack which is the moduli of forms of the formal multiplicative group  $\widehat{\mathbb{G}}_m$ .

**Proposition A.4.** *The stack  $\mathcal{M}_{\mathbb{G}_m}$  is equivalent to the stack  $BC_2$  classifying principal  $C_2$ -torsors, and the stack  $\mathcal{M}_{\widehat{\mathbb{G}}_m}$  is equivalent to the stack  $B\mathbb{Z}_p^\times = \lim B(\mathbb{Z}/p^k)^\times$  classifying compatible systems of principal  $(\mathbb{Z}/p^k)^\times$ -torsors.*

*Proof.* The group scheme  $\mathbb{G}_m$  is defined over  $\mathbb{Z}$  and its sheaf of automorphisms is the constant group scheme  $C_2$  of order two; as a result, for any  $X$  equipped with  $\mathbb{G} \rightarrow X$  a form of  $\mathbb{G}_m$ , there is a principal  $C_2$ -torsor  $Y = \mathrm{Iso}_X(\mathbb{G}_m, \mathbb{G})$  classifying isomorphisms between  $\mathbb{G}$  and  $\mathbb{G}_m$ ; on  $Y$  there is a chosen isomorphism  $\mathbb{G}_m \rightarrow \mathbb{G}$ . Conversely, given such a  $C_2$ -torsor  $Y \rightarrow X$  we can recover  $\mathbb{G}$  as the quotient  $\mathbb{G}_m \times_{C_2} Y \rightarrow \mathrm{Spec}(Z) \times_{C_2} Y = X$ .

In other language, the moduli stack  $\mathcal{M}_{\mathbb{G}_m}$  is equivalent to the stack  $BC_2$  classifying principal  $C_2$ -torsors.

Similarly, for a formal group law  $\mathbb{G}$  let  $\mathbb{G}[p^k]$  be the group scheme of  $p^k$ 'th roots of unity. The sheaf of automorphisms of  $\mathbb{G}_m[p^k]$  is the constant group

scheme  $(\mathbb{Z}/p^k)^\times$ . Any formal group  $\mathbb{G}$  of height 1 on a formal scheme  $X$  over  $\mathbb{Z}_p$  carries a sequence of covers  $Y_k = \text{Iso}(\mathbb{G}_m[p^k], \mathbb{G}[p^k])$ ; any height 1 formal group étale-locally has its torsion isomorphic to that of  $\mathbb{G}_m$ , and so the  $Y_k$  form a tower of principal  $(\mathbb{Z}/p^k)^\times$ -torsors on  $X$ . Conversely, the inductive system of schemes  $\mathbb{G}[p^k]$  can be recovered as  $\mathbb{G}_m[p^k] \times_{(\mathbb{Z}/p^k)^\times} Y_k$ .

Because  $X$  is a formal scheme over  $\text{Spf}(\mathbb{Z}_p)$ ,  $X$  is a colimit of schemes where  $p$  is nilpotent; we then wish to know that the formal group  $\mathbb{G}$  is equivalent data to the directed system  $\mathbb{G}[p^k]$ . One can recover this from an equivalence between formal groups of height 1 and  $p$ -divisible groups of height 1 and dimension 1.

At its core, however, this is locally based on the observation that for a formal group *law* of height 1 on a  $p$ -adically complete ring  $R$ , the Weierstrass preparation theorem implies that the Hopf algebra  $R[[x]]/[p^k](x)$  representing  $\mathbb{G}[p^k]$  is free on the basis  $\{1, x, \dots, x^{p^k-1}\}$ . Both  $R[[x]]$  and its multiplication are recovered uniquely by the inverse limit of these finite stages. The map  $\mathbb{G}[p] \rightarrow \mathbb{G}[p^k]$  induces an isomorphism on cotangent spaces at the identity, and so any coordinate on  $\mathbb{G}[p]$  automatically produces coordinates on  $\mathbb{G}[p^k]$  and then  $\mathbb{G}$  itself.  $\square$

Both of these moduli stacks naturally carry 1-dimensional formal groups, as follows. The stack  $\mathcal{M}_{\widehat{\mathbb{G}}_m}$  carries a universal formal group by definition, and the stack  $\mathcal{M}_{\mathbb{G}_m}$  by taking the completion of its universal group scheme (which is affine and one-dimensional since these are flat-local properties). Therefore, it makes sense to ask if these formal groups can be realized by spectra; see [Goe10, Section 4.1] for details on such realization problems. The formal groups give rise to a commutative diagram of maps to the moduli stack  $\mathcal{M}_{fg}$  of formal groups:

$$\begin{array}{ccc} (\mathcal{M}_{\mathbb{G}_m})_p^\wedge & \longrightarrow & \mathcal{M}_{\widehat{\mathbb{G}}_m} \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathbb{G}_m} & \longrightarrow & \mathcal{M}_{fg} \end{array}$$

Here  $(\mathcal{M}_{\mathbb{G}_m})_p^\wedge$  is the base change of  $\mathcal{M}_{\mathbb{G}_m}$  to  $\text{Spf}(\mathbb{Z}_p)$ . The realization problem for most of the above diagram has a solution.

**Theorem A.5.** *There exist lifts  $\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{\text{top}}$  (resp.  $\mathcal{O}_{\mathcal{M}_{\widehat{\mathbb{G}}_m}}^{\text{top}}$ ) of the structure sheaves of  $\mathcal{M}_{\mathbb{G}_m}$  (resp.  $\mathcal{M}_{\widehat{\mathbb{G}}_m}$ ) to sheaves of weakly even-periodic  $E_\infty$ -ring spec-*

tra, along with an isomorphism between the formal group from the complex-orientable spectrum structure and the formal group pulled back from  $\mathcal{M}_{fg}$ . The homotopy groups of both  $\mathcal{O}^{top}$  and  $\mathcal{O}$  in degree  $2k$  are, on affine charts, the tensor powers  $\omega^{\otimes k}$  of the sheaf of invariant differentials.

Given a diagram of solutions to the realization problem on the above stacks, one would expect to obtain the following upon taking global sections.

$$\begin{array}{ccc} KO_p^\wedge & \longleftarrow & \mathbb{S}_{K(1)} \\ \uparrow & & \uparrow \\ KO & \longleftarrow & \mathbb{S} \end{array}$$

The maps in this diagram are all unit maps for the associated ring spectra.

*Proof.* We first construct pre-sheaves of  $\mathcal{E}_\infty$ -ring spectra defined on affines.

Given an étale map  $\mathrm{Spec}(R) \rightarrow \mathcal{M}_{\mathbb{G}_m}$ , we form the pullback:

$$\begin{array}{ccc} \mathrm{Spec}(T) & \longrightarrow & \mathrm{Spec}(R) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Z}) & \longrightarrow & \mathcal{M}_{\mathbb{G}_m} \end{array}$$

(Since the constant group scheme  $\mathbb{Z}/2\mathbb{Z}$  is affine, so is the canonical map  $\mathrm{Spec}(\mathbb{Z}) \rightarrow \mathcal{M}_{\mathbb{G}_m}$ , showing the above pull-back is indeed affine.)

Then  $\mathrm{Spec}(T) \rightarrow \mathrm{Spec}(R)$  is a Galois cover with Galois group  $C_2$ . The cohomology groups  $H^i(C_2; T)$  are therefore trivial for  $i > 0$ , and the fixed subring  $H^0(C_2, T)$  is equal to  $R$ . Furthermore, if  $\epsilon$  denotes the  $C_2$ -module  $\mathbb{Z}$  with the sign action, the cohomology groups  $H^i(C_2; \epsilon \otimes T)$  vanish for  $i > 0$ . (This will be relevant because  $\pi_2 KU \cong \epsilon$  as a  $C_2$ -module.)

The map  $\mathbb{Z} \rightarrow T$  is étale, and so there is a homotopically unique, and  $C_2$ -equivariant, realization  $\mathbb{S} \rightarrow \mathbb{S}(T)$  such that  $\pi_* \mathbb{S}(T) = \pi_* \mathbb{S} \otimes T$  by the results of [BR07, Section 2] or [Lurb, Theorem 7.5.0.6]. Write  $\sigma$  for the generator of  $C_2$ . The  $C_2$ -action on  $T$  is compatible with the negation action on the multiplicative group scheme over  $T$ .

Since  $\pi_* \mathbb{S}(T) = \pi_* \mathbb{S} \otimes_{\mathbb{Z}} T$  is flat over  $\pi_* \mathbb{S}$ , we have an isomorphism

$$\pi_*(KU \wedge \mathbb{S}(T)) \cong \pi_* KU \otimes_{\pi_* \mathbb{S}} \pi_* \mathbb{S}(T) \cong \pi_* KU \otimes_{\mathbb{Z}} T.$$



The group  $C_2$  acts via the diagonal  $\psi^{-1} \wedge \sigma$  on  $KU \wedge \mathbb{S}(T)$ ; this action lifts the  $C_2$ -action on the formal group law of  $KU \wedge \mathbb{S}(T)$ . We therefore obtain a homotopy fixed-point spectrum  $\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{pre}(R) := (KU \wedge \mathbb{S}(T))^{hC_2}$ . The vanishing of the higher group cohomology implies that the homotopy fixed-point spectral sequence degenerates into an isomorphism

$$\pi_* \mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{pre}(R) \xrightarrow{\cong} H^0(C_2; KU_* \otimes_{\mathbb{Z}} T).$$

We have

$$\pi_2 \mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{pre}(R) \cong H^0(C_2, \epsilon \otimes T) \cong T^{\sigma=-1} \subseteq T. \quad (\text{A.1})$$

This is a projective  $R$ -module of rank 1 and hence invertible, and one concludes that this ring of invariants is weakly even-periodic. An application of [LN12, Lemma 3.8] shows the formal group of  $\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{pre}(R)$  is the one classified by the given map  $R \rightarrow \mathcal{M}_{\mathbb{G}_m}$ .

The construction of  $\mathbb{S}(T)$  can be made functorial in the  $C_2$ -equivariant algebra  $T$ , and so this gives rise to the desired presheaf  $\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{pre}$  on affine objects over  $\mathcal{M}_{\mathbb{G}_m}$ .

We now discuss a similar setup for  $\mathcal{M}_{\widehat{\mathbb{G}}_m}$  and forms of the multiplicative formal group, based on work of Behrens and Davis.

Following [BD10, Section 8], let  $F_1$  be the homotopy colimit of the Devinatz-Hopkins homotopy fixed-point spectra:

$$F_1 := \operatorname{colim}_n (KU_p^\wedge)^{dh(1+p^n \mathbb{Z}_p)}$$

This is a discrete  $\mathcal{E}_\infty \mathbb{Z}_p^\times$ -spectrum with  $K(1)$ -localization  $KU_p^\wedge$ .

Given a  $p$ -adic  $\mathbb{Z}_p$ -algebra  $R$  with an étale map

$$\operatorname{Spf}(R) \rightarrow \mathcal{M}_{\widehat{\mathbb{G}}_m} = \lim[\operatorname{Spf}(\mathbb{Z}_p) // (\mathbb{Z}/p^n)^\times]$$

classifying a formal group  $\mathbb{G}$  over  $R$ , we form the system of pullbacks

$$\begin{array}{ccc} \operatorname{Spf}(T_n) & \longrightarrow & \operatorname{Spf}(\mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \operatorname{Spf}(R) & \longrightarrow & \mathcal{M}_{\widehat{\mathbb{G}}_m} \longrightarrow [\operatorname{Spf}(\mathbb{Z}_p) // (\mathbb{Z}/p^n)^\times] \end{array}$$

The maps  $\mathrm{Spf}(T_n) \rightarrow \mathrm{Spf}(R)$  are the Galois covers with Galois group  $(\mathbb{Z}/p^n)^\times$  trivializing  $\mathbb{G}[p^n]$ . We consider the  $\mathbb{Z}_p$ -algebra  $T = \mathrm{colim} T_n$ , which is an ind-Galois extension of  $R$  with Galois group  $\mathbb{Z}_p^\times$  and discrete  $\mathbb{Z}_p^\times$ -action. The maps  $\mathbb{Z}_p \rightarrow T_n$  are étale because the classifying map of  $\mathbb{G}$  is.

As in the case of  $\mathcal{M}_{\mathbb{G}_m}$  above, we functorially realize this to obtain a  $\mathbb{Z}_p^\times$ -equivariant directed system  $\mathbb{S}_p^\wedge \rightarrow \mathbb{S}_p^\wedge(T_n)$  of  $\mathcal{E}_\infty$ -ring spectra, with the action on  $\mathbb{S}_p^\wedge(T_n)$  factoring through  $(\mathbb{Z}/p^n)^\times$ , such that  $\pi_* \mathbb{S}_p^\wedge(T_n) = \pi_* \mathbb{S}_p^\wedge \otimes_{\mathbb{Z}_p} T_n$ . We finally introduce the discrete  $\mathcal{E}_\infty$   $\mathbb{Z}_p^\times$ -spectrum

$$\mathbb{S}_p^\wedge(T) = \mathrm{hocolim}_n \mathbb{S}_p^\wedge(T_n).$$

The homotopy of  $\mathbb{S}_p^\wedge(T)$  is isomorphic to  $\pi_* \mathbb{S}_p^\wedge \otimes_{\mathbb{Z}_p} T$ . The  $\mathcal{E}_\infty$ -ring spectrum  $F_1 \wedge \mathbb{S}_p^\wedge(T)$  is a discrete  $\mathbb{Z}_p^\times$ -spectrum with the diagonal action, acting as the Morava stabilizer on  $F_1$  and via the Galois action on  $\mathbb{S}_p^\wedge(T)$ . As this spectrum is  $E(1)$ -local, the  $K(1)$ -localization of  $F_1 \wedge \mathbb{S}_p^\wedge(T)$  is equivalent to the  $p$ -completion  $(KU \wedge \mathbb{S}_p^\wedge(T))_p^\wedge$ , with homotopy groups isomorphic to  $\pi_* KU_p^\wedge \hat{\otimes}_{\mathbb{Z}_p} T_p^\wedge$ . We define our presheaf to take  $R$  to be a continuous homotopy fixed-point object:

$$\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{\mathrm{pre}}(R) = L_{K(1)} \left( (F_1 \wedge \mathbb{S}_p^\wedge(T))^{h\mathbb{Z}_p^\times} \right)$$

The homotopy fixed-point spectrum is  $E(1)$ -local, and so  $K(1)$ -localization is still simply  $p$ -adic completion.

We will now show that the resulting spectrum is even-periodic using the homotopy fixed-point spectral sequence (see [LN12, Theorem 5.1])

$$H_c^s(\mathbb{Z}_p^\times, (KU_p^\wedge)_t \otimes_{\mathbb{Z}_p} T_p^\wedge) \Rightarrow \pi_{t-s}(\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{\mathrm{pre}}(R)).$$

Fix  $n \in \mathbb{Z}$ . We will show that the continuous cohomology of  $\mathbb{Z}_p^\times$  with coefficients in  $W := KU_{p,2n}^\wedge \hat{\otimes}_{\mathbb{Z}_p} T_p^\wedge$  vanishes for  $s > 0$ , and the zero'th cohomology group is free of rank one over  $R$ . Multiplication by the Bott element shows that the  $\mathbb{Z}_p^\times$ -module  $W$  is isomorphic to  $T_p^\wedge$  twisted by the  $p$ -adic character  $\alpha \mapsto \alpha^n$ .

Since  $\mathrm{Spf}(R) = \mathrm{Spf}(T_1) \rightarrow \mathrm{Spf}(\mathbb{Z}_p)$  is étale,  $T_1 = R$  must be isomorphic to a finite product of copies of  $W(\mathbb{F}_q)$  with  $q$  a power of  $p$ . Therefore, without loss of generality we may assume  $R = W(\mathbb{F}_q)$ .

For any  $m \geq 1$ , there exists a sufficiently large  $k$  so that the subgroup  $U := (1 + p^k \mathbb{Z}_p)^\times < \mathbb{Z}_p^\times$  acts trivially on  $\mathbb{Z}/p^m$  by the character  $\alpha \mapsto \alpha^n$ . The continuous cohomology of  $U$  with coefficients in  $W/p^m$  then coincides with the continuous cohomology of  $U$  with coefficients in  $T/p^m$ . However, as  $T/p^m$  is a Galois extension of  $T_k/p^m$  with Galois group  $U$ , the higher cohomology vanishes and the zero'th cohomology is isomorphic to  $T_k/p^m$ . The Lyndon-Hochschild-Serre spectral sequence associated to  $1 \rightarrow U \rightarrow \mathbb{Z}_p^\times \rightarrow (\mathbb{Z}/p^k)^\times \rightarrow 1$  then degenerates to an isomorphism

$$H_c^s(\mathbb{Z}_p^\times; W/p^m) \cong H^s((\mathbb{Z}/p^k)^\times; \pi_{2n} KU/p^m \otimes T_k).$$

However, the map  $R/p^m \rightarrow T_k/p^m$  is a Galois extension with Galois group  $(\mathbb{Z}/p^k)^\times$ ; it is in particular faithfully flat, and the fixed-point functor is part of an equivalence between  $R/p^m$ -modules and  $T_k/p^m$ -modules equipped with a semilinear Galois action. In particular, the higher cohomology groups vanish, and by faithfully flat descent the  $R/p^m$ -module  $(\pi_{2n} KU/p^m \otimes T_k)^{(\mathbb{Z}/p^k)^\times}$  is projective of rank one. The base ring  $R/p^m$  is isomorphic to  $W(\mathbb{F}_q)/p^m$ , which is local, so the module is actually free of rank one. Moreover, the isomorphism  $\pi_{2n} \otimes_R \pi_{2m} \rightarrow \pi_{2(n+m)}$  induced by multiplication descends to an isomorphism on invariants.

Taking limits in  $m$ , we find that the higher continuous cohomology groups with coefficients in  $W$  vanish, and that the module of invariants of  $W$  is free of rank one.

As a result, we find  $\pi_* \mathcal{O}_{\mathcal{M}_{\widehat{\mathbb{G}}_m}}^{pre}(R) = R[u^{\pm 1}]$  for some unit  $u$  in degree 2. In particular, then,  $\mathcal{O}_{\mathcal{M}_{\widehat{\mathbb{G}}_m}}^{pre}(R)$  is complex orientable.

The formal group  $\mathbb{G}$  and the formal group associated with a complex orientation of  $\mathcal{O}_{\mathcal{M}_{\widehat{\mathbb{G}}_m}}^{pre}(R)$  both arise from the same descent data on  $T_p^\wedge$ , and so the associated formal group is isomorphic to  $\mathbb{G}$ .

We now take these spectra defined on affine étale charts and extend them to sheaves. Specifically, functoriality in  $R$  allows us to sheafify, associated fibrant objects in the Jardine model structure are functors  $\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{top}$  and  $\mathcal{O}_{\mathcal{M}_{\widehat{\mathbb{G}}_m}}^{top}$  of  $\mathcal{E}_\infty$ -ring spectra on general schemes over  $\mathcal{M}_{\mathbb{G}_m}$  and  $\mathcal{M}_{\widehat{\mathbb{G}}_m}$  respectively. On any affine  $R$ , the map  $\mathcal{O}^{pre} \rightarrow \mathcal{O}^{top}$  is a weak equivalence on stalks, and the presheaves of homotopy groups of  $\mathcal{O}^{pre}$  are the quasicohherent sheaves  $\omega^t$  of invariant differentials on any affine  $R$ . The sheaves associated to the homotopy groups of  $\mathcal{O}^{top}$  are therefore the same, and the map  $\mathcal{O}^{pre}(R) \rightarrow$

$\mathcal{O}^{top}(R)$  is an equivalence for affine  $R$ , and for a general  $R$  it is recovered as a homotopy limit. For a more detailed account of this argument in a very similar situation see [Beh, Sections 2.3-2.5].  $\square$

*Remark A.6.* i) The above construction is motivated by the fact that for the  $K(1)$ -local ( $\mathcal{E}_\infty$ -ring) spectrum  $X = \mathcal{O}_{\mathcal{M}_{\widehat{\mathbb{G}}_m}}^{top}(R)$ , one has a canonical equivalence

$$X \xrightarrow{\simeq} (L_{K(1)}(KU_p^\wedge \wedge X))^{h\mathbb{Z}_p^\times},$$

[DT, Theorem 1.3], where on the right-hand side the group  $\mathbb{Z}_p^\times$  acts through the factor  $KU_p^\wedge$  alone. Briefly,  $X$  can be recovered from its  $K$ -theory. Our construction is such that there is an equivalence of continuous  $\mathbb{Z}_p^\times$ -spectra

$$L_{K(1)}(KU_p^\wedge \wedge X) \simeq L_{K(1)}(KU_p^\wedge \wedge \mathbb{S}_p^\wedge(T)),$$

where  $\mathbb{S}_p(T)$  is built from the principal torsor on  $\pi_0 X = R$  and  $\mathbb{Z}_p^\times$  on the right-hand side acts diagonally on both factors.

- ii) During the above proof we saw that the étale site of  $\mathcal{M}_{\widehat{\mathbb{G}}_m}$  is not very big. Specifically, all its affine objects are disjoint unions of maps from some  $\mathrm{Spf}(W(\mathbb{F}_q))$  classifying a formal group of height one. These formal groups over  $W(\mathbb{F}_q)$  in turn are classified by  $p$ -adic units  $\alpha \in \mathbb{Z}_p^\times$ . All morphisms of the site are generated by change-of-base and by automorphisms inducing  $p$ -adic Adams operations. Existence and uniqueness of these realizations on affine coordinate charts is the content of the Goerss-Hopkins-Miller theorem, which lifts the local version of Morava's construction to  $\mathcal{E}_\infty$ -rings.
- iii) Morava considers many more forms of  $K$ -theory than are given by the values of these sheaves, Tate  $K$ -theory being a prominent example [Mor89, Theorem 2]. We take this as an opportunity to document the well-known fact that  $\mathcal{E}_\infty$ -realizations put very restrictive conditions on ramification. The authors learned the following argument from Mike Hopkins. Though these types of arguments have been known (e.g. by Ando, Strickland, and others) for several decades, the authors were unable to track down a published reference.

Specifically, assume  $p \neq 2$  and consider the extension  $KU_p^\wedge(-) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]$  which adjoins a  $p$ 'th root of unity (for  $p = 2$  a similar argument considering the extension  $\mathbb{Z}_2 \subseteq \mathbb{Z}_2[i]$  works). Since the extension  $\mathbb{Z}_p \rightarrow \mathbb{Z}_p[\zeta_p]$

is flat, this is a multiplicative cohomology theory which is a form of  $K$ -theory in Morava's sense. The corresponding homotopy-commutative ring spectrum does not admit an  $\mathcal{E}_\infty$ -refinement. The  $K(1)$ -local power operations would provide a lift of Frobenius  $\varphi: \mathbb{Z}_p[\zeta_p] \rightarrow \mathbb{Z}_p[\zeta_p]$  reducing to the  $p$ 'th power map mod  $(p)$ . However, the only endomorphisms of this ring are automorphisms, and the  $p$ 'th power map is not injective mod  $p$ . That a fourth root of unity cannot be adjoined to the sphere at  $p = 2$  is shown in [SVW99].

To conclude this appendix, we explain an application of Theorem A.5 relevant to the main concern of the present paper. Consider the generalized elliptic curve

$$y^2 + 3xy + y = x^3$$

over  $\mathbb{Z}[1/3]$ . It lies over the unramified cusp of  $\overline{\mathcal{M}}_1(3)$ . To identify the resulting formal group we observe that the curve has a nodal singularity at  $(-1, 1)$ , and that the coordinate  $t = (x + 1)/(y - 1)$  gives an isomorphism between the smooth locus of this curve and  $\mathbb{P}^1 \setminus \{t \mid t^2 + 3t + 3 = 0\}$ , with multiplication

$$G(t, t') = \frac{tt' - 3}{t + t' + 3}$$

and unit at  $t = \infty$ . The coordinate  $t^{-1}$  gives an associated formal group law

$$F(x, y) = \frac{x + y + 3xy}{1 - 3xy}.$$

This is not isomorphic to the multiplicative formal group over  $\mathbb{Z}[1/3]$ , but becomes so after adjoining a third root of unity  $\omega$ . It is classified by an étale map  $f: \mathbb{Z}[1/3] \rightarrow \mathcal{M}_{\mathbb{G}_m}$  and we can consider the form of  $K$ -theory  $KU^\tau$  given by  $\mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}^{\text{top}}(f)$ . Denoting  $\beta \in \pi_2 KU$  the Bott element one can check

$$KU_*^\tau = \mathbb{Z}[1/3][(\sqrt{-3}\beta)^{\pm 1}] \subseteq KU_* \otimes_{\mathbb{Z}} \mathbb{Z}[1/3][\omega].$$

For any  $p \neq 3$ , the  $p$ -adic completion of  $KU^\tau$  is a Lubin-Tate spectrum for its formal group law. In the obstruction theory of Section 3, one can equally well substitute the ramified cusp  $KU^\tau$  for  $KU$  into the construction. In place of the maps of equation (3.22), we would then have

$$A \mapsto (\sqrt{-3}\beta), \quad B \mapsto -\frac{1}{27} (\sqrt{-3}\beta)^3.$$

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