The product formula in unitary deformation K-theory

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Abstract. For finitely generated groups G and H, we prove that there is a weak equivalence $\mathcal{K}\mathrm{G} \wedge_{ku} \mathcal{K}\mathrm{H} \simeq \mathcal{K}(\mathrm{G} \times \mathrm{H})$ of ku-algebra spectra, where $\mathcal{K}$ denotes the "unitary deformation K-theory" functor. Additionally, we give spectral sequences for computing the homotopy groups of $\mathcal{K}\mathrm{G}$ and $\mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{K}\mathrm{G}$ in terms of connective K-theory and homology of spaces of G-representations.

1. Introduction

The underlying goal of many programs in algebraic K-theory is to understand the algebraic K-groups of a field F as being built from the K-groups of the algebraic closure of the field, together with the action of the absolute Galois group. Specifically, Carlsson's program (see [2]) is to construct a model for the algebraic K-theory *spectrum* using the Galois group and the K-theory spectrum of the algebraic closure $\overline{F}$.

In some specific instances, the absolute Galois group of the field F is explicitly the profinite completion $\hat{\mathrm{G}}$ of a discrete group G. (For example, the absolute Galois group of the field $k(z)$ of rational functions, where k is an algebraically closed of characteristic zero, is the profinite completion of a free group.) In the case where F contains an algebraically closed subfield, the profinite completion of a "deformation K-theory" spectrum $\mathcal{K}\mathrm{G}$ is conjecturally equivalent to the profinite completion of the algebraic K-theory spectrum $\mathbb{K}F$.

Additionally, it would be advantageous for this description to be compatible with the motivic spectral sequence. This deformation K-theory spectrum has an Atiyah-Hirzebruch spectral sequence arising from a spectrum level filtration. The filtration quotients are spectra built from isomorphism classes of representations of the group. It is hoped that this filtration is related to the motivic spectral sequence, and that this relation would give a greater understanding of the relationships between Milnor K-theory, Galois cohomology, and the representation theory of the Galois group.

We now outline the construction of deformation K-theory. To a finitely generated group G one associates the category $\mathcal{C}$ of finite dimensional unitary representations of G, with morphisms being equivariant isometric isomorphisms. Elementary methods of representation theory

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allow this category to be analyzed; explicitly, the category of unitary G-representations is naturally equivalent to a direct sum of copies of the (topological) category of unitary vector spaces.

However, there is more structure to $\mathcal{C}$. First, there is a bilinear tensor product pairing. Second, $\mathcal{C}$ can be given the structure of an internal category $\mathcal{C}^{top}$ in **Top**. This means that there are *spaces* $\mathrm{Ob}(\mathcal{C}^{top})$ and $\mathrm{Mor}(\mathcal{C}^{top})$, together with continuous domain, range, identity, and composition maps, satisfying appropriate associativity and unity diagrams. The topology on $\mathrm{Ob}(\mathcal{C}^{top})$ reflects the possibility that homomorphisms from G to $\mathrm{U}(n)$ can continuously vary from one isomorphism class of representations to another. The identity map on objects and morphisms is a continuous functor $\mathcal{C} \to \mathcal{C}^{top}$ that is bijective on objects.

Both of these categories have notions of direct sums and so are suitable for application of an appropriate infinite loop space machine, such as Segal's machine [19]. This yields a map of (ring) spectra as follows:

$$\mathbb{K}\mathcal{C} \simeq \bigvee ku \to \mathbb{K}\mathcal{C}^{top} \stackrel{\mathrm{def}}{=} \mathcal{K}\mathrm{G}. \tag{1}$$

Here $\mathbb{K}$ is an algebraic K-theory functor, ku is the connective K-theory spectrum, and the wedge is taken over the set of irreducible unitary representations of G. Note that $\pi_*(\mathbb{K}\mathcal{C}) \cong R[\mathrm{G}] \otimes \pi_*(ku)$ as a ring, where $R[\mathrm{G}]$ is the unitary representation ring of G.

The spectrum $\mathcal{K}\mathrm{G}$ is the *unitary deformation K-theory* of G. It differs from the C^*-algebra K-theory of G—for example, in section 8, we find that the unitary deformation K-theory of the discrete Heisenberg group has infinitely generated π_0.

When G is free on k generators, one can directly verify the formula

$$\mathcal{K}\mathrm{G} \simeq ku \vee \left(\bigvee^{k} \Sigma ku \right).$$

A more functorial description in this case is that $\mathcal{K}\mathrm{G}$ is the connective cover of the function spectrum $F(B\mathrm{G}_+, ku)$. This formula does not hold in general; even in simple cases $\mathcal{K}\mathrm{G}$ can be difficult to directly compute, such as when G is free abelian on multiple generators.

In this paper, we will prove the following product formula for unitary deformation K-theory.

THEOREM 1. *The tensor product map induces a map of commutative ku-algebra spectra $\mathcal{K}\mathrm{G} \wedge_{ku} \mathcal{K}\mathrm{H} \to \mathcal{K}(\mathrm{G} \times \mathrm{H})$, and this map is a weak equivalence.*

The reader should compare the formula

$$R[\mathrm{G}] \otimes R[\mathrm{H}] \cong R[\mathrm{G} \times \mathrm{H}]$$

for unitary representation rings.

The proof of Theorem 1 proceeds by making use of a natural filtration of $\mathcal{K}\mathrm{G}$ by subspectra $\mathcal{K}\mathrm{G}_n$. These subspectra correspond to representations of G whose irreducible components have dimension less than or equal to n. Specifically, we show in sections 5 and 6 that there is a homotopy fibration sequence of spectra

$$\mathcal{K}\mathrm{G}_{n-1} \to \mathcal{K}\mathrm{G}_n \to (\mathrm{Hom}(\mathrm{G},\mathrm{U}(n))/\mathrm{Sum}(\mathrm{G},n)) \underset{\mathrm{PU}(n)}{\wedge} ku^{\mathrm{PU}(n)}.$$

Here $\mathrm{Sum}(\mathrm{G},n)$ is the subspace of $\mathrm{Hom}(\mathrm{G},\mathrm{U}(n))$ consisting of those representations $\mathrm{G} \to \mathrm{U}(n)$ containing a nontrivial invariant subspace. The spectrum $ku^{\mathrm{PU}(n)}$ is a connective $\mathrm{PU}(n)$-equivariant K-homology spectrum discussed in section 3.

As side benefits of the existence of this filtration, Theorems 31 and 33 give spectral sequences for computing the homotopy groups of $\mathcal{K}\mathrm{G}$ and the homotopy groups of $\mathrm{H}\mathbb{Z} \wedge_{ku} \mathcal{K}\mathrm{G}$ respectively.

When G is free on k generators, Theorem 33 gives a spectral sequence converging to $\mathbb{Z}$ in dimension 0, $\mathbb{Z}^k$ in dimension 1, and 0 otherwise, but the terms in the spectral sequence are highly nontrivial—they are the homology groups of the spaces of k-tuples of elements of $\mathrm{U}(n)$, mod conjugation and relative to the subspace of k-tuples that contain a nontrivial invariant subspace. The method by which the terms in this spectral sequence are eliminated is a bit mysterious.

The layout of this paper is as follows. Section 2 gives the necessary background on G-equivariant Γ-spaces for a compact Lie group G, allowing identification of equivariant smash products. The model theory of such functors was considered when G is a finite group in [4], using simplicial spaces. Our approach to the proofs of the results we need follows the approach of [3]. In Section 3 we give explicit constructions of an equivariant version of connective K-theory. In Section 4 the unitary deformation K-theory of G is defined, and Sections 5 and 6 are devoted to constructing the localization sequences, in particular explicitly identifying the base as an equivariant smash product. In section 7 the algebra and module structures are made explicit by making use of results of Elmendorf and Mandell [6]. The proofs of the main theorems are completed in sections 8 and 9.

A proof of the product formula for representations in $\mathrm{GL}(n)$, rather than $\mathrm{U}(n)$, would also be desirable. This paper makes use of quite rigid constructions that make apparent the identification of the base in the localization sequence with a particular model for the equivariant smash product. In the case of $\mathrm{GL}(n)$, the definitions of both the cofiber in the localization sequence and the equivariant smash product need to be replaced by notions that are more well-behaved from the point of view of homotopy theory.

2. Preliminaries on G-equivariant Γ-spaces

In this section, G is a compact Lie group, and actions of G on based spaces are assumed to fix the basepoint; a *free* action will be one that is free away from the basepoint. We will now carry out constructions of Γ-spaces in a naïve equivariant context. When G is trivial these are definitions for topological Γ-spaces, as in the appendix of [17].

For any natural number k, denote the based space $\{*, 1, \ldots, k\}$ by k_+.

Let Γ_G^o be the category of *right* G-spaces that are isomorphic to ones the form $G_+ \wedge k_+$, with morphisms being G-equivariant. (Strictly speaking, we take a small skeleton for this category.) The set $\Gamma_G^o(X, Y)$ can be given the mapping space topology, giving this category an enrichment in spaces. Explicitly,

$$\Gamma_G^o(G_+ \wedge k_+, Z) \cong \prod^k Z$$

as a space. If G is trivial we drop it from the notation.

Definition 2. A Γ_G-space is a base-point preserving continuous functor $\Gamma_G^o \to \mathbf{Top}_*$.

Here $\mathbf{Top}_*$ is the category spaces, i.e. compactly generated weak Hausdorff pointed topological spaces with nondegenerate basepoint, which has internal function objects $F(-,-)$.

Any Γ_G-space M has an underlying Γ-space $M(G_+ \wedge -)$. This Γ-space inherits a continuous left G-action because the left action of G on the first factor of $G_+ \wedge Y$ is right G-equivariant. This can be expressed as a continuous map

$$\phi : G \to \Gamma_G^o(G_+ \wedge Y, G_+ \wedge Y),$$

given by $\phi(g)(h \wedge y) = gh \wedge y$. Then the composite map

$$M \circ \phi : G \to F(M(G_+ \wedge Y), M(G_+ \wedge Y)),$$

gives an action of G on $M(G_+ \wedge Y)$ that is natural in Y. This gives a G-action on the underlying Γ-space of M.

Remark 3. More generally, if $H \to G$ is a map of groups, the formula $M \mapsto M(- \wedge_H G_+)$ defines a restriction map from Γ_G-spaces to Γ_H-spaces with a left action of $C(H)$, the centralizer of H in G.

For technical reasons, we require the following definition.

Definition 4. A Γ_G-space M is semi-cofibrant if for any $Z \in \Gamma_G^o$, the inclusion

$$\tilde{M}(Z) = \bigcup_{Y \subsetneq Z} M(Y) \subset M(Z)$$

is a cofibration of spaces.

A cofibrant Γ-space as defined in [1] would require some freeness assumption about the action of $\mathrm{Aut}(Z)$ on $M(Z)/\tilde{M}(Z)$.

Remark 5. For any element $m \in M(Z)$, there is a unique minimal subobject Y of Z such that m is in the image of $M(Y)$. To see this, consider the inclusion $i_Y : Y \to Z$ and the map $\pi_Y : Z \to Y$ that is the identity on Y and sends the rest of Z to the basepoint. The map $i_Y \pi_Y$ acts as the identity on the image of $M(Y)$, and $(i_Y \pi_Y)(i_{Y'} \pi_{Y'}) = i_{Y \cap Y'} \pi_{Y \cap Y'}$.

Let $X, Z \in \Gamma_G^o$, Y a based set. The space $G_+ \wedge Y$ has commuting left and right G-actions. This gives rise to a continuous right action of G on $\Gamma_G^o(G_+ \wedge Y, Z)$. There is a map $\phi : X \to \Gamma_G^o(G_+ \wedge Y, X \wedge Y)$ given by $x \mapsto \phi_x$, where $\phi_x(g \wedge y) = xg \wedge y$. The map ϕ is clearly right G-equivariant. Composing the map ϕ with the functor M gives a continuous G-equivariant map from X to $F(M(G_+ \wedge Y), M(X \wedge Y))$, and the adjoint is a natural assembly map $X \wedge_G M(G_+ \wedge Y) \to M(X \wedge Y)$.

We can promote a Γ_G-space M to a functor on all free right G-spaces, as follows.

Definition 6. For X a right G-space, the Γ-space $X \otimes_G M$ is defined by:

$$X \otimes_G M(Z) = \left(\coprod_{Y \in \Gamma_G^o} M(Y) \wedge F^G(Y, X \wedge Z) \right) / \sim .$$

Here the equivalence relation $\sim$ is generated by relations $(u \wedge f^* v) \sim (f_* u \wedge v)$ for $f : Y \to Y', u \in M(Y), v \in F^G(Y', X \wedge Z)$. More concisely, $X \otimes_G M(Z)$ can be expressed as the coend

$$\int^Y M(Y) \wedge F^G(Y, X \wedge Z).$$

Remark 7. If $X_.$ is a simplicial object in the category Γ_G^o, there is a natural homeomorphism $|X_.| \otimes_G M \to |M(X_.)|$. (A short proof can be given by expressing $|X_.|$ as a coend $\int^n X_n \wedge \Delta_+^n$ and applying "Fubini's theorem"—see [11], chapter IX.) The reason for allowing G-CW complexes rather than simply restricting to these simplicial objects is that some G-homotopy types cannot be realized by simplicial objects. For example, any such simplicial object of the form X_+ is a principal G-bundle over X/G_+, and is classified by an element in $H^1_{discrete}(X/G, G)$. A general X_+ is classified by an element in $H^1_{cont.}(X/G, G)$.

We will refer to the space $(X \otimes_G M)(1_+)$ as $M(X)$; this agrees with the notation already defined when $X \in \Gamma_G^o$. We will only apply this construction to cofibrant objects in a certain model category of based G-spaces; specifically, we will only apply this construction to based G-CW complexes with free action away from the basepoint. Such objects are formed by iterated cell attachment of $G_+ \wedge D_+^n$ along $G_+ \wedge S_+^{n-1}$.

It will be useful to have homotopy theoretic control on $X \otimes_G M$, for the purposes of which we introduce a less rigid tensor product.

Definition 8. For M a Γ_G-space, we can define a simplicial Γ_G-space $LM_.$ by setting $LM(Z)_p$ equal to

$$\bigvee_{Z_0,\ldots,Z_p \in \Gamma_G^o} \Gamma_G^o(Z_p, Z) \wedge \Gamma_G^o(Z_{p-1}, Z_p) \wedge \cdots \wedge \Gamma_G^o(Z_0, Z_1) \wedge M(Z_0).$$

The face maps are given by:

$$\begin{aligned} d_i(f_0 \wedge \ldots \wedge f_p \wedge m) &= f_0 \wedge \ldots \wedge f_i \circ f_{i+1} \wedge \ldots \wedge f_p \wedge m \quad \text{if } i < p, \\ d_p(f_0 \wedge \ldots \wedge f_p \wedge m) &= f_0 \wedge \ldots \wedge f_{p-1} \wedge (M f_p)(m). \end{aligned}$$

The degeneracy map s_i is an insertion of an identity map after f_i for $0 \leq i \leq p$.

Suppose M is a semi-cofibrant Γ_G-space. For any p and Z, the subspace of $LM(Z)_p$ consisting of degenerate objects is the union of the subspaces that contain an identity element in some component of the smash product. All the spaces in the smash product are nondegenerately based cofibrant objects, and so the inclusion of the degenerate subcomplex is a cofibration. As a result, this simplicial space is *good* in the sense of [19], Appendix A, and so the geometric realization of it is homotopically well-behaved.

The simplicial Γ_G-space $LM_.$ has a natural augmentation $LM_. \to M$. The augmented object $LM_. \to M$ has an extra degeneracy map s_{-1}, defined by

$$s_{-1}(f_0 \wedge \ldots \wedge f_p \wedge m) = id \wedge f_0 \wedge \ldots \wedge f_p \wedge m.$$

As a result, the map $|LM_{\cdot}(Z)| \to M(Z)$ is a homotopy equivalence for any $Z \in \Gamma^o_G$. (Note that $LM_{\cdot}$ is the bar construction $B(\Gamma^o_G, \Gamma^o_G, M)$.)

For X a right G-space, consider the simplicial Γ_G-space $X \otimes_G LM_{\cdot}$. We have

$$X \otimes_G LM_p = \bigvee_{Z_0,\ldots,Z_p} \Big[X \otimes_G \Gamma^o_G(Z_p, -)\Big] \wedge \Gamma^o_G(Z_{p-1}, Z_p) \wedge \cdots \wedge M(Z_0)$$

as a Γ-space, because the tensor construction distributes over wedge products and commutes with smashing with spaces. However, a straightforward calculation yields the formula

$$\Big[X \otimes_G \Gamma^o_G(Y, -)\Big](Z) \cong F^G(Y, X \wedge Z),$$

the space of G-equivariant based functions from Y to $X \wedge Z$.

PROPOSITION 9. *For M a semi-cofibrant Γ_G-space and X a G-CW complex with free action away from the basepoint, the augmentation map $X \otimes_G LM_{\cdot} \to X \otimes_G M$ is a levelwise weak equivalence of Γ-spaces after realization; i.e., the map $|(X \otimes_G LM_{\cdot})(Z)| \to (X \otimes_G M)(Z)$ is a weak equivalence for all $Z \in \Gamma^o$.*

Proof. There is a natural isomorphism $(X \otimes_G M)(Z) \cong M(X \wedge Z)$, so it suffices to prove that $|LM_{\cdot}(X)| \to M(X)$ is a weak equivalence for any G-CW complex X. We will prove this by showing that it is filtered by weak equivalences.

Suppose X is a G-CW complex. For any $n \in \mathbb{N}$, define

$$M(X)^{(n)} = \left(\coprod_{|Y/G| \le n, Y \in \Gamma^o_G} M(Y) \wedge F^G(Y, X) \right) / \sim .$$

Here the equivalence relation is the same as that defining $M(X)$.

We now prove that $M(X)^{(n)}$ is a subspace of $M(X)$. Suppose $Y \in \Gamma^o_G$, and $u \wedge v \in M(Y) \wedge F^G(Y, X)$. Call this element *minimal* if v is an embedding and u is not in the image of j_* for any proper inclusion j_*.

Given an element $u \wedge v$ as above, the map v uniquely factors through the surjection $p : Y \to \mathrm{Im}(v)$. By Remark 5, there is a unique minimal inclusion $j : Z \subset \mathrm{Im}(v)$ such that $p_* u = j_* m$ for some m. The element $m \wedge j$ is minimal and equivalent to $u \wedge v$. Note that if $|Y/G| \le n$, this equivalence holds in $M(X)^{(n)}$.

If $f : Y' \to Y$, $u \in M(Y')$, and $v : Y \to X$, then there is a natural factorization as follows:

$$\begin{tikzcd}
& Y' \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[d, two heads] \arrow[dr, "v"] & \\
Z \arrow[r, tail] & \mathrm{Im}(f^*v) \arrow[r, tail] & \mathrm{Im}(v) \arrow[r, tail] & X.
\end{tikzcd}$$

This shows that $f_*u \wedge v$ and $u \wedge f^*v$ are equivalent to the same minimal element.

We find that two elements are equivalent under the equivalence relation defining $M(X)$ or $M(X)^{(n)}$ if and only if they are equivalent to a common minimal element. As a result, an element in $M(X)^{(n)}$ is not in the image of $M(X)^{(n-1)}$ if and only if it is its own minimal factorization, i.e. it is of the form $u \wedge v$ where v is an embedding and u is not in the image of j_* for j any proper inclusion. Two such minimal elements $u \wedge v$ and $u' \wedge v'$ are equivalent if and only if there is an isomorphism f such that $f_*u = u'$ and $(f^{-1})^*v = v'$.

In particular, for any $n > 0$, there is a natural pushout square for constructing $M(X)^{(n)}$ from $M(X)^{(n-1)}$. Define $\Sigma_n \wr G$ to be the wreath product $G^n \rtimes \Sigma_n$, which is the automorphism group in Γ^o_G of $G_+ \wedge n_+$. Let $\tilde{F}(n_+, X)$ be the subset of $F(n_+, X) = F^G(G_+ \wedge n_+, X)$ consisting of those maps that are not embeddings, and let $\tilde{M}(G_+ \wedge n_+)$ denote the union of the images of $M(Y)$ over proper inclusions $Y \to G_+ \wedge n_+$.

There is a natural pushout diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & M(X)^{(n-1)} \arrow[d] \\
F(n_+, X) \wedge_{\Sigma_n \wr G} M(G_+ \wedge n_+) \arrow[r] & M(X)^{(n)},
\end{tikzcd}$$

where

$$A = \left(\tilde{F}(n_+, X) \underset{\Sigma_n \wr G}{\wedge} M(G_+ \wedge n_+) \right) \cup \left(F^G(Z, X) \underset{\Sigma_n \wr G}{\wedge} \tilde{M}(G_+ \wedge n_+) \right).$$

Similarly, there is a natural pushout diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & |LM_\cdot(X)|^{(n-1)} \arrow[d] \\
F(n_+, X) \wedge_{\Sigma_n \wr G} |LM_\cdot(G_+ \wedge n_+)| \arrow[r] & |LM_\cdot(X)|^{(n)},
\end{tikzcd}$$

where B is the corresponding union for $LM_\cdot$.

Because X is a free G-CW complex, the map $\tilde{F}(n_+, X) \to F(n_+, X)$ is a cofibration of $(\Sigma_n \wr G)$-spaces. Additionally, the map of $(\Sigma_n \wr G)$-

spaces $|LM_\cdot(G_+ \wedge n_+)| \to M(G_+ \wedge n_+)$ is a weak equivalence, so the square

$$\begin{tikzcd}
\tilde{F}(n_+, X) \wedge_{\Sigma_n \wr G} |LM_\cdot(G_+ \wedge n_+)| \arrow[r] \arrow[d] & \tilde{F}(n_+, X) \wedge_{\Sigma_n \wr G} M(G_+ \wedge n_+) \arrow[d] \\
F(n_+, X) \wedge_{\Sigma_n \wr G} |LM_\cdot(G_+ \wedge n_+)| \arrow[r] & F(n_+, X) \wedge_{\Sigma_n \wr G} M(G_+ \wedge n_+)
\end{tikzcd}$$

is homotopy cocartesian. Similarly, the weak equivalence $|\widetilde{LM}_\cdot| \to \tilde{M}$ shows that the square

$$\begin{tikzcd}
\tilde{F}(n_+, X) \wedge_{\Sigma_n \wr G} \left|\widetilde{LM}_\cdot(G_+ \wedge n_+)\right| \arrow[r] \arrow[d] & \tilde{F}(n_+, X) \wedge_{\Sigma_n \wr G} \tilde{M}(G_+ \wedge n_+) \arrow[d] \\
F(n_+, X) \wedge_{\Sigma_n \wr G} \left|\widetilde{LM}_\cdot(G_+ \wedge n_+)\right| \arrow[r] & F(n_+, X) \wedge_{\Sigma_n \wr G} \tilde{M}(G_+ \wedge n_+)
\end{tikzcd}$$

is homotopy cocartesian.

These two previous homotopy cocartesian squares imply that the square

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & F(n_+, X) \wedge_{\Sigma_n \wr G} |LM_\cdot(G_+ \wedge n_+)| \arrow[d] \\
A \arrow[r] & F(n_+, X) \wedge_{\Sigma_n \wr G} M(G_+ \wedge n_+)
\end{tikzcd}$$

is homotopy cocartesian. (This square would be honestly cocartesian if the previous squares were cocartesian.) The horizontal maps in this square are cofibrations.

Inductively assume that $|LM_\cdot(X)|^{(n-1)} \to M(X)^{(n-1)}$ is a weak equivalence. Weak equivalences are preserved by pushouts along cofibrations, so the map

$$|LM_\cdot(X)|^{(n-1)} \cup_A \left(F(n_+, X) \underset{\Sigma_n \wr G}{\wedge} M(G_+ \wedge n_+) \right) \to M(X)^{(n)}$$

is a weak equivalence. However, we know that there is a weak equivalence

$$F(n_+, X) \underset{\Sigma_n \wr G}{\wedge} M(G_+ \wedge n_+) \simeq A \cup_B \left(F(n_+, X) \underset{\Sigma_n \wr G}{\wedge} |LM_\cdot(G_+ \wedge n_+)| \right),$$

so we find that the map $|LM_\cdot(X)|^{(n)} \to M(X)^{(n)}$ is a weak equivalence, as desired.

COROLLARY 10. *If M is a semi-cofibrant Γ_G-space and a map $X \to Y$ of free G-CW complexes is k-connected, so is the map $M(X) \to M(Y)$.*

Proof. It suffices to show that the map $|LM_.(X)| \to |LM_.(Y)|$ is k-connected. If $k = 0$, this is clear.

The simplicial space $LM_.(X)$ is good in the sense of Segal [19], Appendix A, so it suffices to show that the map of thick geometric realizations $||LM_.(X)|| \to ||LM_.(Y)||$ is k-connected. However, this map of simplicial spaces is levelwise of the form

$$\begin{array}{c} \bigvee_{Z_0,\ldots,Z_p} F^G(Z_p, X) \wedge \Gamma^o_G(Z_{p-1}, Z_p) \wedge \cdots \wedge \Gamma^o_G(Z_0, Z_1) \wedge M(Z_0) \\ \Big\downarrow \\ \bigvee_{Z_0,\ldots,Z_p} F^G(Z_p, Y) \wedge \Gamma^o_G(Z_{p-1}, Z_p) \wedge \cdots \wedge \Gamma^o_G(Z_0, Z_1) \wedge M(Z_0). \end{array}$$

This map is k-connected because the map $F^G(Z_p, X) \to F^G(Z_p, Y)$ is. The result follows because a levelwise k-connected map of simplicial spaces $A_. \to B_.$ induces a k-connected map of thick geometric realizations. We include a proof as follows.

Filtering the thick geometric realization by skeleta, we find that for any $n \geq 1$ there is a commutative square

$$\begin{tikzcd} \mathrm{sk}_{n-1}(||A_.||) \arrow[r] \arrow[d] & \mathrm{sk}_n(||A_.||) \arrow[d] \\ \mathrm{sk}_{n-1}(||A_.||) \arrow[r] & \mathrm{sk}_n(||B_.||). \end{tikzcd}$$

Assume inductively that the leftmost vertical map is k-connected.

We now recall the statement of the Blakers-Massey excision theorem, as in [7], Section 2. Suppose there is a commutative square of spaces

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_2"'] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

If the map from the homotopy pushout to D is ℓ-connected, and f_i is k_i connected for each i, then the map of of homotopy fibers

$$\mathrm{fib}(A \to C) \to \mathrm{fib}(B \to D)$$

is $\min(k_1 + k_2 - 1, \ell - 1)$-connected.

Applying this shows that the map from $sk_{n-1}(||A_.||)$ to $sk_n(||A_.||)$ is $(n-1)$-connected, and similarly for $||B_.||$. Consider the commutative square of skeleta. Because $k \geq 1$, the map from the homotopy pushout to $sk_n(||B_\cdot||)$ is 1-connected by the Seifert-Van Kampen theorem. By the relative Hurewicz theorem, the connectivity of this map is the same as the connectivity of its homotopy cofiber.

The total homotopy cofiber of the square of skeleta is the cofiber of the map $S^n \wedge LM_n(X)_+ \to S^n \wedge LM_n(Y)_+$, which is $(n+k)$-connected by assumption. The map of $(n-1)$-skeleta is k-connected, and the map from the $(n-1)$-skeleton to the n-skeleton is $(n-1)$-connected. Therefore, the Blakers-Massey theorem shows that the map of homotopy fibers is $(n+k-2)$-connected. As $n \geq 1$, it is in particular $(k-1)$-connected. The homotopy fiber of the map of $(n-1)$-skeleta is $(k-1)$-connected, so the homotopy fiber of the map of n-skeleta must be $(k-1)$-connected as well, as desired.

For any Γ_G-space M, we have an associated (naïve pre-)spectrum $\{M(G_+ \wedge S^n)\}$, which is the spectrum associated to the underlying Γ-space $M(G_+ \wedge -)$. A map of Γ_G-spaces $M \to M'$ is called a stable equivalence if the associated map of spectra is a weak equivalence.

PROPOSITION 11. *For any semi-cofibrant Γ_G-space M and free based G-CW complex X, the assembly map $X \wedge_G M(G_+ \wedge -) \to X \otimes_G M$ is a stable equivalence.*

Proof. It suffices to show that $X \wedge_G M(G_+ \wedge S^n) \to M(X \wedge S^n)$ is highly connected for large n. Using the levelwise weak equivalence $|X \otimes_G LM_\cdot| \to X \otimes_G M$, it suffices to show that this statement is true for Γ-spaces of the form $\Gamma^o_G(Y, -)$ for $Y \in \Gamma^o_G$.

In this case, we have the following diagram:

$$\begin{tikzcd}
X \wedge_G \bigvee_Y (G_+ \wedge S^n) \arrow[r] \arrow[d] & \bigvee_Y X \wedge_G (G_+ \wedge S^n) \arrow[d] \\
X \wedge_G \prod_Y (G_+ \wedge S^n) \arrow[r] \arrow[d, equal] & \prod_Y (X \wedge S^n) \arrow[d, equal] \\
X \wedge_G \Gamma^o_G(Y, G_+ \wedge S^n) \arrow[r] & \Gamma^o_G(Y, X \wedge S^n).
\end{tikzcd}$$

The top vertical arrows are isomorphisms on homotopy groups up to roughly dimension $2n$, as $G_+ \wedge S^n$ is $(n-1)$-connected. The uppermost horizontal arrow is an isomorphism. Therefore, the bottom map is an equivalence on homotopy groups up to roughly dimension $2n$, as desired.

3. Connective equivariant K-homology

In this section, we construct for each n a $\Gamma_{\mathrm{PU}(n)}$-space whose underlying spectrum is homotopy equivalent to ku, the connective K-theory spectrum.

Fix an integer n. For any $d \in \mathbb{N}$, we have the Stiefel manifold $V(nd)$ of isometric embeddings of $\mathbb{C}^n \otimes \mathbb{C}^d$ into $\mathbb{C}^\infty$, where these vector spaces have the standard inner products. The tensor product map $\mathrm{U}(n) \otimes \mathrm{U}(d) \to \mathrm{U}(nd)$ gives the space $V(nd)$ a free right action of $I \otimes \mathrm{U}(d)$ by precomposition, where I is the identity element of $\mathrm{U}(n)$. Denote the quotient space by $H(d)$, and write $H = \coprod_d H(d)$. (Note $H \simeq \coprod_d \mathrm{BU}(d)$.) The space H has a partially defined direct sum operation: if $\{V_i\}$ is a finite set of elements of H such that $V_i \perp V_j$ for $i \neq j$, there is a sum element $\oplus V_i$ in H.

There is also an action of $\mathrm{U}(n) \otimes I$ on $V(nd)$ that commutes with the action of $I \otimes \mathrm{U}(d)$, and hence passes to an action on the quotient $H(d)$. Because $\lambda I \otimes I = I \otimes \lambda I$, the scalars in $\mathrm{U}(n)$ act trivially on $H(d)$, so the action factors through $\mathrm{PU}(n)$. We therefore get a right action of $\mathrm{PU}(n)$ on H. The direct sum operation is $\mathrm{PU}(n)$-equivariant.

For any $Z \in \Gamma^o_{\mathrm{PU}(n)}$, define

$$ku^{\mathrm{PU}(n)}(Z) = \left\{ f \in F^{\mathrm{PU}(n)}(Z, H) \;\middle|\; f(z) \perp f(z') \text{ if } [z] \neq [z'] \right\}.$$

A point of $ku^{\mathrm{PU}(n)}(Z)$ consists of a subspace $\mathbb{C}^\infty$, isomorphic to $\mathbb{C}^{\dim f(z)}$, associated to each non-basepoint $[z]$ of $Z/\mathrm{PU}(n)$, such that the vector spaces associated to $[z]$ and $[z']$ are orthogonal if $[z] \neq [z']$.

Given a map $\alpha \in \Gamma^o_{\mathrm{PU}(n)}(Z, Z')$ and $f \in ku^{\mathrm{PU}(n)}(Z)$, we get an element $ku^{\mathrm{PU}(n)}(\alpha)(f) \in ku^{\mathrm{PU}(n)}(Z')$ as follows:

$$ku^{\mathrm{PU}(n)}(\alpha)(f)(z') = \bigoplus_{\alpha(z)=z'} f(z).$$

This direct sum is well-defined: if the preimage of z is the family $\{z_i\}$, then the z_i all lie in distinct orbits because the action of $\mathrm{PU}(n)$ is free away from the basepoint. Therefore, the subspaces associated to the z_i are orthogonal. The map $ku^{\mathrm{PU}(n)}(\alpha)(f)$ is also clearly $\mathrm{PU}(n)$-equivariant, and takes distinct orbits to orthogonal elements of H.

The underlying Γ-space is given as follows. For $Z \in \Gamma^o$,

$$ku^{\mathrm{PU}(n)}(\mathrm{PU}(n)_+ \wedge Z) = \{ f \in F(Z, H) \mid f(z) \perp f(z') \text{ if } z \neq z' \}.$$

The spectrum attached to the underlying Γ-space of $ku^{\mathrm{PU}(n)}$ is weakly equivalent to the connective K-theory spectrum ku—see [20].

The $\Gamma_{\mathrm{PU}(n)}$-space $ku^{\mathrm{PU}(n)}$ is semi-cofibrant: The image in $ku^{\mathrm{PU}(n)}(Z)$ of the spaces $ku^{\mathrm{PU}(n)}(Y)$ for $Y \subsetneq Z$ consists of those maps $Z \to H$ that map some nontrivial subset of Z to $H(0)$. This is a union of components of $ku^{\mathrm{PU}(n)}(Z)$.

We also define a second $\Gamma_{\mathrm{PU}(n)}$-space ku/β as follows. For any $z \in \Gamma^o_{PU(n)}$,

$$ku/\beta(Z) = \tilde{\mathbb{N}}[Z/\mathrm{PU}(n)],$$

where $\tilde{\mathbb{N}}$ is the reduced free abelian monoid functor. More explicitly, $ku/\beta(Z)$ is the quotient of the free abelian monoid on $Z/\mathrm{PU}(n)$ by the submonoid $\mathbb{N}[*]$. For $\alpha \in \Gamma^o_{\mathrm{PU}(n)}(Z, Z')$,

$$ku/\beta(\alpha)\left(\sum n_z[z]\right) = \sum n_z[\alpha(z)].$$

(The reason for the notation is that the underlying spectrum is the cofiber of the Bott map.) The $\Gamma_{\mathrm{PU}(n)}$-space ku/β is semi-cofibrant for the same reason as $ku^{\mathrm{PU}(n)}$.

For X a free right $\mathrm{PU}(n)$-space, $X \otimes_{\mathrm{PU}(n)} ku/\beta$ is the infinite symmetric product $\mathrm{Sym}^\infty(X/\mathrm{PU}(n))$.

There is a natural map $\epsilon : ku^{\mathrm{PU}(n)} \to ku/\beta$ of $\Gamma_{\mathrm{PU}(n)}$-spaces: if $f \in ku^{\mathrm{PU}(n)}(Z)$, define $\epsilon(f) = \sum_{[z]} \left(\frac{\dim f(z)}{n}\right)[z]$. The map ϵ represents the augmentation $ku \to \mathrm{H}\mathbb{Z}$ on the underlying spectra; it is the first stage of the Postnikov tower for ku.

The $\Gamma_{\mathrm{PU}(n)}$-space $ku^{\mathrm{PU}(n)}$ determines a homology theory for $\mathrm{PU}(n)$-spaces. Specifically, we can define

$$ku_*^{\mathrm{PU}(n)}(X) = \pi_*\left(X \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}\right).$$

Here π_* denotes the stable homotopy groups of the spectrum. In fact, because the underlying spectrum of $ku^{\mathrm{PU}(n)}$ is *special*, we can compute

$$ku_*^{\mathrm{PU}(n)}(X) = \pi_*\left(ku^{\mathrm{PU}(n)}(X)\right)$$

for X connected. (See [19], 1.4.)

4. Unitary deformation K-theory

In this section we will have a fixed finitely generated *discrete* group G. Carlsson, in [2], defined a notion of the "deformation K-theory" of G as a contravariant functor from groups to spectra, and in the introduction of this article an analogous notion of "unitary deformation K-theory" $\mathcal{K}$G was sketched. The following are weakly equivalent definitions of the corresponding notion of $\mathcal{K}$G:

– The spectrum associated to the the E_∞-H-space

$$\coprod_n \mathrm{EU}(n) \underset{\mathrm{U}(n)}{\times} \mathrm{Hom}(\mathrm{G}, \mathrm{U}(n)).$$

– The K-theory of a category of unitary representations of G. This category is an internal category in **Top**: i.e., the objects and morphism sets are both given topologies.

– The K-theory of the singular complex of the category above. (This is essentially the definition given in [2].)

We will now describe another model for the unitary deformation K-theory of G, equivalent to the first definition given above. The construction is based on the construction of connective topological K-homology of Segal in [20]. (Also see [22].)

Definition 12. Let $\mathcal{U} = \mathbb{C}^\infty$ be the infinite inner product space having orthonormal basis $\{e_i\}_{i=0}^\infty$, with action of the group $\mathrm{U} = \operatorname{colim} \mathrm{U}(n)$.

Definition 13. A G-plane V of dimension k is a pair (V, ρ), where V is a k-dimensional plane in $\mathcal{U}$ and $\rho : \mathrm{G} \to \mathrm{U}(V)$ is an action of G on V.

We now describe a (non-equivariant) Γ-space $\mathcal{K}\mathrm{G}$. Define

$$\mathcal{K}\mathrm{G}(X) = \left\{ (V_x, \rho_x)_{x \in X} \;\middle|\; V_x \text{ a G-plane, } V_x \perp V_{x'} \text{ if } x \neq x', V_* = 0 \right\}.$$

This is a special Γ-space. The underlying H-space is

$$\mathcal{K}\mathrm{G}(1_+) \cong \coprod_n V(n) \times_{\mathrm{U}(n)} \mathrm{Hom}(\mathrm{G}, \mathrm{U}(n)),$$

where $V(n)$ is the Stiefel manifold of n-frames in $\mathcal{U}$. We will now describe the simplicial space $X_. = \mathcal{K}\mathrm{G}(S^1)$. Because $\mathcal{K}\mathrm{G}$ is special, $\Omega|X_.| \simeq \Omega^\infty \mathcal{K}\mathrm{G}$.

For $p > 0$, X_p is the space

$$\left\{ (V_i, \rho_i)_{i=1}^p \;\middle|\; (V_i, \rho_i) \text{ a G-plane, } V_i \perp V_j \text{ if } i \neq j \right\}.$$

(X_0 is a point.) Face maps are given by taking sums of orthogonal G-planes or removing the first or last G-plane. Degeneracy maps are given by insertion of 0-dimensional G-planes.

The geometric realization of this simplicial space can be explicitly identified. Let Y be the space of pairs (A, ρ), where $A \in \mathrm{U}$ and ρ is a homomorphism $\mathrm{G} \to \mathrm{U}$ such that $\rho(g)A = A\rho(g)$ for all $g \in \mathrm{G}$. Call two such elements (A, ρ) and (A', ρ') equivalent if $A = A'$ and ρ, ρ' agree on all eigenspaces of A corresponding to eigenvalues $\lambda \neq 1$. Write the standard p-simplex Δ^p as the set of all $0 \leq t_1 \leq \ldots \leq t_p \leq 1$. Then there is a homeomorphism $|X_.| \to (Y \;/\sim)$ given by sending a point $((V_i, \rho_i)_{i=1}^p, 0 \leq t_1 \leq \ldots \leq t_p \leq 1)$ of $X_p \times \Delta^p$ to the pair (A, ρ), where A acts on V_i with eigenvalue $e^{2\pi i t_i}$ and by 1 on the orthogonal complement of ΣV_i, while ρ acts on V_i by ρ_i and acts by 1 on the orthogonal complement of ΣV_i. (Here ΣV_i is the span of the set of orthogonal subspaces V_i.) This map is a homeomorphism by the spectral theorem. (The essential details of this argument are from [9] and [13].)

We will refer to this space $|X_.| \cong (Y \;/\sim)$ as E. It is space 1 of the Ω-spectrum associated to $\mathcal{K}\mathrm{G}$, in the sense that $\Omega^\infty \mathcal{K}\mathrm{G} \simeq \Omega E$.

This method is applicable to various other categories of representations of G that we will now examine in detail.

For any $n \geq 0$, there is a sub-Γ-space $\mathcal{K}\mathrm{G}_n$ of $\mathcal{K}\mathrm{G}$. The space $\mathcal{K}\mathrm{G}_n(X)$ consists of those elements $\{(V_x, \rho_x)\}_{x \in X}$ of $\mathcal{K}\mathrm{G}(X)$ such that ρ_x breaks up into a direct sum of irreducible representations of dimension less than or equal to n. Each $\mathcal{K}\mathrm{G}_n$ is a special Γ-space.

PROPOSITION 14. *The map* $\operatorname{hocolim} \mathcal{K}\mathrm{G}_n \to \mathcal{K}\mathrm{G}$ *is a weak equivalence.*

Proof. Clearly $\mathcal{K}\mathrm{G}$ is the union of the sub-Γ-spaces $\mathcal{K}\mathrm{G}_n$. For any based set X, any element $\{(V_x, \rho_x)\}$ of $\mathcal{K}\mathrm{G}(X)$ has a well-defined total dimension $\sum \dim V_x$, and this dimension is locally constant. Therefore, $\mathcal{K}\mathrm{G}(X)$ breaks up as a disjoint union according to total dimension. The component consisting of elements of total dimension n is completely contained in the subspace $\mathcal{K}\mathrm{G}_N(X)$ for all $N \geq n$. In particular, if x is any point of $\mathcal{K}\mathrm{G}(X)$ whose total dimension is n, $\pi_*(\mathcal{K}\mathrm{G}(X), x) \cong \pi_*(\mathcal{K}\mathrm{G}_N(X), x)$ for all $N \geq n$.

Remark 15. As a result, for X a simplicial set finite in each dimension (such as a sphere), $\operatorname{hocolim} \mathcal{K}\mathrm{G}_n(X)$ can be formed levelwise, and is levelwise weakly equivalent to $\mathcal{K}\mathrm{G}(X)$. In particular, the natural map $\operatorname{hocolim} \mathcal{K}\mathrm{G}_n(S^k) \to \mathcal{K}\mathrm{G}(S^k)$ is a weak equivalence for all k, so the associated spectrum of $\mathcal{K}\mathrm{G}$ is weakly equivalent to the associated spectrum of $\operatorname{hocolim} \mathcal{K}\mathrm{G}_n$.

We have infinite loop spaces $E_n = |\mathcal{K}\mathrm{G}_n(S^1)|$. For any $n \in \mathbb{N}$, E_n is the subspace of E consisting of pairs (A, ρ) such that ρ is a direct sum of irreducible representations of G of dimension less than or equal to n.

This gives a sequence of inclusions

$$* = E_0 \subset E_1 \subset E_2 \subset \dots$$

of infinite loop spaces. Each of these inclusions is part of a quasifibration sequence $E_{n-1} \to E_n \to B_n$ where the base spaces will be explicitly identified. This gives rise to the following "unrolled exact couple" of infinite loop spaces:

$$\begin{tikzcd}
* \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[rr] \arrow[dl] & & E_3 \dots \arrow[dl] \\
& B_1 \arrow[ul, "\circ"] & & B_2 \arrow[ul, "\circ"] & & B_3. \arrow[ul, "\circ"] &
\end{tikzcd}$$

Additionally, the inclusions of infinite loop spaces E_n are induced by maps of E_∞-ku-modules, so the above is induced by an exact couple of ku-module spectra.

The intuition for the description of B_n is that the category of G-representations whose irreducible summands have dimension less than n forms a Serre subcategory of the category of G-representations whose irreducibles have dimension less than or equal to n, and the quotient category should be the category of sums of irreducible representations of dimension exactly n. The topology on the categories involved complicates the question of when a localization sequence of spectra exists in this situation, as the most obvious attempts to generalize of Quillen's Theorem B would not be applicable. We will construct the localization sequence explicitly.

There is a quotient Γ-space F_n of $\mathcal{K}\mathrm{G}_n$ by the equivalence relation $(V_x, \rho_x)_{x\in X} \sim (V'_x, \rho'_x)_{x\in X}$ if for all $x \in X$:

- The subspace W_x of V_x generated by irreducible subrepresentations of ρ of dimension n coincides with the corresponding subspace for ρ', and

- ρ and ρ' agree on W_x.

Again, F_n is a special Γ-space.

Define B_n to be the space $|F_n(S^1)|$. B_n is the quotient of E_n by the following equivalence relation. We say $(A, \rho) \sim (A', \rho')$ if:

- The subspace W of $\mathcal{U}$ generated by irreducible subrepresentations of ρ of dimension n is the same as the subspace of $\mathcal{U}$ generated by irreducible subrepresentations of ρ' of dimension n,

- ρ and ρ' have the same action on W, and

– A and A' have the same action on W.

Note that each equivalence class contains a unique pair (A, ρ) such that ρ acts trivially on the eigenspace of A for 1, and on the complementary subspace ρ is a direct sum of irreducible n-dimensional representations.

5. Proof of the existence of the localization sequence

The proof that $p_n : E_n \to B_n$ is a quasifibration (and hence induces a long exact sequence on homotopy groups) proceeds inductively using the following result.

THEOREM 16. (Hardie [8]). *Suppose that we have a diagram*

$$\begin{tikzcd} Q \arrow[d, "\lambda"] & f^*(E) \arrow[l, "h"'] \arrow[r] \arrow[d, "s"] & E \arrow[d, "p"] \\ Q' & A \arrow[l, "g"] \arrow[r, "f"'] & B. \end{tikzcd}$$

Here f is a cofibration, p is a fibration, $f^*(E)$ is the pullback fibration, and λ is a quasifibration. If $h : s^{-1}(a) \to \lambda^{-1}(ga)$ is a weak equivalence for all $a \in A$, then the induced map of pushouts $Q \coprod_{f^*(E)} E \to Q' \coprod_A B$ is a quasifibration.

PROPOSITION 17. *The map $p_n : E_n \to B_n$ is a quasifibration with fiber E_{n-1}.*

Remark 18. This is what we might expect, as the map $E_n \to B_n$ is precisely the map that forgets the irreducible subrepresentations of dimension less than n. The fact that E_{n-1} is the honest fiber over any point is clear, but we need to show that E_{n-1} is also the homotopy fiber.

Proof. We will proceed by making use of a rank filtration. These rank filtrations were introduced in [13] and [16]. In particular, Mitchell explicitly describes this rank filtration for the connective K-theory spectrum.

For any j, let $B_{n,j}$ be the subspace of B_n generated by those pairs (A, ρ) such that ρ contains at most a sum of j irreducible representations. There is a sequence of inclusions $B_{n,j-1} \subset B_{n,j}$. Write $E_{n,j}$ for the subset of E_n lying over $B_{n,j}$.

The map $E_{n,0} \to B_{n,0}$ is a quasifibration, because $B_{n,0}$ is a point. Now suppose inductively that $E_{n,j-1} \to B_{n,j-1}$ is a quasifibration.

Let Y_j be the space of triples (A, ρ, W), where W is an nj-dimensional subspace of $\mathcal{U}$, A is an element of $\mathrm{U}(W)$, and ρ is a representation of G on W commuting with A and containing irreducible summands of dimension n or less. Let X_j be the subset of Y_j of triples (A, ρ, W) such that (A, ρ) represents a pair in $B_{n,j-1}$; in other words, ρ contains less than j distinct n-dimensional irreducible summands on the orthogonal complement of the eigenspace for 1 of A.

Next, we define a space Y'_j of triples (A, ρ, W), where $(A, \rho) \in E_{i,j}$ and W is an A- and ρ-invariant nj-dimensional subspace of $\mathcal{U}$ containing all the n-dimensional irreducible summands of ρ. There is a map $Y'_j \to Y_j$ given by forgetting the actions of A and ρ off W. Let X'_j be the fiber product of X_j and Y'_j over Y_j; it consists of triples (A, ρ, W) where ρ contains less than j distinct n-dimensional summands.

There is a map $X_j \to B_{j-1}$ given by sending (A, ρ, W) to (A, ρ), and a similar map $X'_j \to E_{j-1}$. These maps all assemble into the diagram below.

$$\begin{tikzcd}
E_{n,j-1} \arrow[d, "p_n"] & X'_j \arrow[l] \arrow[r] \arrow[d] & Y'_j \arrow[d, "p"] \\
B_{n,j-1} & X_j \arrow[l] \arrow[r] & Y_j
\end{tikzcd}$$

There is an evident map from the pushout of the bottom row to $B_{n,j}$, and similarly a map from the pushout of the top row to $E_{n,j}$.

The map $X_j \to B_{n,j-1}$ is a quotient map; two points become identified by forgetting the "framing" subspace W, the non-n-dimensional summands of ρ, and the summands of ρ on the eigenspace for 1 of A. For points of Y_j not in X_j, the framing subspace W is determined by the image (A, ρ) in B_j because ρ must have j distinct n-dimensional irreducible summands covering all of W, and A can have no eigenspace for the eigenvalue 1. Therefore, the map from Y_j to the pushout of the bottom row is precisely the quotient map gotten by forgetting the framing W and any non-n-dimensional summands or summands lying on the eigenspace for 1 of A. This identifies the pushout with $B_{n,j}$. In exactly the same way, the pushout of the top row is $E_{n,j}$. The induced map of pushouts is the projection map $E_{n,j} \to B_{n,j}$.

The map $X_j \to Y_j$ is a cofibration because it is the colimit of geometric realizations of a closed inclusion of real points of algebraic varieties. (The subspace W is allowed to vary over the infinite Grassmannian. If we restrict its image to any finite subspace we get an inclusion of real algebraic varieties.)

The right-hand square is a pullback by construction, and the map p_n is assumed to be a quasifibration.

The map p is a fiber bundle with fiber E_{n-1}: An equivalence class of points $(A, \rho, W) \in Y_j'$ consists of a choice of nj-dimensional subspace W of $\mathcal{U}$, a choice of element in $(\bar{A}, \bar{\rho}, W)$ in Y_j to determine the action of A and ρ on W, and a choice of (A', ρ') acting on the orthogonal complement of W such that ρ' is made up of summands of dimension less than n. In other words, there is a pullback square:

$$\begin{tikzcd} Y_j' \arrow[r] \arrow[d] & V \arrow[d] \\ Y_j \arrow[r] & \mathrm{Gr}(nj). \end{tikzcd}$$

Here $\mathrm{Gr}(nj)$ is the Grassmannian of nj-dimensional planes in $\mathcal{U}$, and V is the bundle over the Grassmannian consisting of nj-dimensional planes in $\mathcal{U}$ and elements of E_{n-1} acting on their orthogonal complements.

Given any point (A, ρ, W) of X_j, the fiber in X_j' is E_{n-1} acting on the orthogonal complement of W. Suppose that (A, ρ) in $B_{n,j-1}$ is in canonical form: ρ acts by a sum of irreducible dimension n representations on some subspace $W' \subset W$ and trivial representations on the orthogonal complement, and A has eigenvalue 1 on the orthogonal complement of W'. Then the fiber over (A, ρ) in $E_{n,j-1}$ consists of all possible actions of E_{n-1} on the orthogonal complement of W'. The map from the fiber over (A, ρ, W) to the fiber over (A, ρ) is the inclusion of E_{n-1} acting on $W^\perp$ to E_{n-1} acting on $(W')^\perp$. This inclusion is a homotopy equivalence.

Therefore, $E_{n,j} \to B_{n,j}$ is a quasifibration with fiber E_{n-1}. Taking colimits in j, $E_n \to B_n$ is a quasifibration with fiber E_{n-1}.

COROLLARY 19. *The maps $\mathcal{K}\mathrm{G}_{n-1} \to \mathcal{K}\mathrm{G}_n \to F_n$ realize to a fibration sequence in the homotopy category of spectra.*

Proof. This follows because the composite map is null, the map from $\mathcal{K}\mathrm{G}_{n-1}(S^1)$ to the homotopy fiber of $\mathcal{K}\mathrm{G}_n(S^1) \to F_n(S^1)$ is a weak equivalence, and all three of these Γ-spaces are special.

6. Identification of the Γ-space F_n

Using the results of section 2, we will now identify the Γ-spaces F_n as equivariant smash products.

Let $\mathrm{Sum}(\mathrm{G}, n)$ be the subspace of $\mathrm{Hom}(\mathrm{G}, \mathrm{U}(n))$ of reducible G-representations of dimension n. Define $R_n = \mathrm{Hom}(\mathrm{G}, \mathrm{U}(n))/\mathrm{Sum}(\mathrm{G}, n)$. There is a free right action of $\mathrm{PU}(n)$ on R_n by conjugation.

Remark 20. The action of $\mathrm{PU}(n)$ on R_n is free because the only endomorphisms of an irreducible complex representation are scalar multiplications. This fails for orthogonal representations.

According to a result of Park and Suh ([15], Theorem 3.7), the space $\mathrm{Hom}(\mathrm{G}, U(n))$, which is the set of real points of an algebraic variety, admits the structure of a $\mathrm{U}(n)$-CW complex. All isotropy groups contain the diagonal subgroup, so this structure is actually the structure of a $\mathrm{PU}(n)$-CW complex. The subspace $\mathrm{Sum}(\mathrm{G}, n)$ consists of those elements of $\mathrm{Hom}(\mathrm{G}, \mathrm{U}(n))$ that are not acted on freely by $\mathrm{PU}(n)$, and so it must be a CW-subcomplex. Therefore, R_n has an induced CW-structure.

PROPOSITION 21. *There is an isomorphism of* Γ*-spaces*

$$R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)} \to F_n.$$

Proof. By the universal property of the coend

$$R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(Z) = \int^Y ku^{\mathrm{PU}(n)}(Y) \wedge F^G(Y, R_n \wedge Z),$$

we can construct the map by exhibiting maps

$$ku^{\mathrm{PU}(n)}(Y) \wedge F^G(Y, R_n \wedge Z) \to F_n(Z),$$

natural in Z, that satisfy appropriate compatibility relations in Y.

Recall that a point of $ku^{\mathrm{PU}(n)}(Y)$ consists of an equivariant map $f : Y \to H = \coprod V(nd)/I \otimes \mathrm{U}(d)$ such that $f(y) \perp f(y')$ if $y \neq y'$.

Suppose $f \wedge h \in ku^{\mathrm{PU}(n)}(Y) \wedge F^G(Y, R_n \wedge Z)$. For every $y \in Y$ the element $h(y) = r(y) \wedge z(y)$ determines an irreducible action $r(y)$ of G on $\mathbb{C}^n$. The element $f(y) \in V(nd)/I \otimes \mathrm{U}(d)$ is the image of some element $\widetilde{f(y)} \in V(nd)$, which determines an isometric embedding $\mathbb{C}^n \otimes \mathbb{C}^d \to \mathcal{U}$. Combining these two gives an action of G on an nd-plane of $\mathcal{U}$, together with a marking $z(y)$ of the plane by an element of Z. The action of $I \otimes \mathrm{U}(d)$ commutes with the G-action on $\mathbb{C}^n \otimes \mathbb{C}^d$, so the choice of lift $\widetilde{f(y)}$ does not change the resulting G-plane. For $g \in \mathrm{G}$, $r(yg) = r(y) \cdot g = g^{-1}r(y)g$, and $f(yg) = (g^{-1} \otimes I)f(y)(g \otimes I)$, so the resulting plane only depends on the orbit yG. The resulting G-plane breaks up into irreducible summands of dimension precisely n. Assembling these G-planes over the distinct orbits gives a collection of orthogonal hyperplanes with G-actions, marked by points of Z, that break up into a direct sum of n-dimensional irreducible representations. As $r(y)$ approaches the basepoint of R_n, the representation becomes reducible, so the map determines a well-defined element of $F_n(Z)$. The compatibility of this map with maps in Y is due to the fact that it preserves direct sums.

This map is bijective; associated to any point of $F_n(Z)$ there is a unique equivalence class of points that map to it.

We now construct the inverse map $F_n \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$. As F_n is a quotient of $\mathcal{K}\mathrm{G}_n$, it suffices to construct a map

$$\mathcal{K}\mathrm{G}_n(Z) \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(Z),$$

natural in Z, that respects the equivalence relation.

For $d \in \mathbb{N}$, consider the space X_d of pairs $(\rho, \{e_i\})$ consisting of an action ρ of G on $\mathbb{C}^d$, together with a set of nonzero mutually orthogonal G-equivariant projection operators e_i whose image each have dimension n and that contain all of the n-dimension summands of $\mathbb{C}^d$. This space is compact Hausdorff. The forgetful map to $\mathrm{Hom}(\mathrm{G}, \mathrm{U}(d))$ identifies the image subspace X'_d, consisting of elements ρ that are a direct sum of representations of dimension n or less, with the quotient of X_d by the equivalence relation gotten by forgetting the e_i.

Recall that $V(d)$ is the space of embeddings of $\mathbb{C}^d$ in $\mathcal{U}$. We now define a natural map

$$\begin{aligned}\phi : V(d) \ \times \ & X_d \to \\ & \bigvee_m ku^{\mathrm{PU}(n)}(\mathrm{PU}(n)_+ \wedge m_+) \wedge F^{\mathrm{PU}(n)}(\mathrm{PU}(n)_+ \wedge m_+, R_n).\end{aligned}$$

Given an isometric embedding f, an action ρ of G on $\mathbb{C}^d$, and a collection of m G-planes of $\mathbb{C}^d$, we use the isometric embedding to obtain a collection $\{f(\mathrm{Im}\ e_i)\}$ of n-planes in $\mathcal{U}$, marked by points of R_n. Composing with the quotient map, we get a map

$$\phi' : V(d) \times X_d \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(1_+).$$

The image of a point depends only on the n-dimensional summands of the representation ρ.

The map ϕ' is invariant under the choice of direct sum decomposition $\{e_i\}$, so there is an induced map ϕ'' from the quotient space $V(d) \times$

X'_d. (This identification of the quotient space requires that we use the product in the category of compactly generated spaces.) The map ϕ'' is invariant under the diagonal action of $\mathrm{U}(d)$ on $V(d) \times X'_d$, so there is an induced map $V(d) \times_{\mathrm{U}(d)} X'_d \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(1_+)$. Assembling these together over d gives a map $\mathcal{K}\mathrm{G}_n(1_+) \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(1_+)$.

The map

$$R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(Z) \to \prod_Z R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(1_+)$$

is an inclusion of the subspace of mutually orthogonal elements. The composite

$$\mathcal{K}\mathrm{G}_n(Z) \to \prod_Z \mathcal{K}\mathrm{G}_n(1_+) \to \prod_Z R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}(1_+)$$

maps into this subspace. Therefore, this gives a lift to a map of Γ-spaces $\mathcal{K}\mathrm{G}_n \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$.

This map respects the equivalence relation defining F_n, as the image of a point of $\mathcal{K}\mathrm{G}_n$ only depended on its n-dimensional summands.

COROLLARY 22. *There is a stable equivalence of* Γ-*spaces*

$$R_n \underset{\mathrm{PU}(n)}{\wedge} ku^{\mathrm{PU}(n)} \to F_n.$$

Proof. This follows from Proposition 11.

7. E_∞-algebra and module structures

In this section we will make explicit the following. The tensor product of representations leads to the following multiplicative structures:

- ku is an E_∞-ring spectrum,
- $\mathcal{K}\mathrm{G}$ is an E_∞-algebra over ku,
- the sequence of maps $\mathcal{K}\mathrm{G}_1 \to \mathcal{K}\mathrm{G}_2 \to \cdots \to \mathcal{K}\mathrm{G}$ is a sequence of E_∞-ku-module maps,
- there are compatible E_∞-ku-linear pairings $\mathcal{K}\mathrm{G}_n \wedge \mathcal{K}\mathrm{G}_m \to \mathcal{K}\mathrm{G}_{nm}$ for all n, m, and
- the assembly map $R_n \wedge_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)} \to \mathcal{K}\mathrm{G}_n$ is a map of E_∞-ku-modules.

All of the above structures are natural in G.

To begin, we will first recall the definition of a *multicategory*. A multicategory is an "operad with several objects", as follows. See [6].

Definition 23. A multicategory $\mathbf{M}$ consists of the following data:

– a class of objects $\mathrm{Ob}(\mathbf{M})$,

– a set $\mathbf{M}_k(a_1, \ldots, a_k; b)$ for each $a_1, \ldots, a_k, b \in \mathrm{Ob}(\mathbf{M})$, $k \geq 0$ of "k-morphisms" from $(a_1, \ldots, a_k)$ to b,

– a right action of the symmetric group Σ_k on the class of all k-morphisms such that σ^* maps the set $\mathbf{M}_k(a_1, \ldots, a_k; b)$ to the set $\mathbf{M}_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}; b)$,

– an "identity" map $1_a \in M_1(a; a)$ for all $a \in \mathrm{Ob}(\mathbf{M})$, and

– a "composition" map

$$\begin{aligned}\mathbf{M}_n(b_1, \ldots, b_n; c) \times \mathbf{M}_{k_1}(a_{11}, \ldots, a_{1k_1}; b_1) \times \cdots \\ \to \mathbf{M}_{k_1+\cdots+k_n}(a_{11}, \ldots, a_{nk_n}; c)\end{aligned}$$

which is associative, unital, and respects the symmetric group action.

We will not make precise these last properties; they are essentially the same as the definitions for an operad. A map between multicategories that preserves the appropriate structure will be referred to as a multifunctor.

Example 24. Any symmetric monoidal category $(\mathbf{C}, \Box)$ is a multicategory, with

$$\mathbf{C}_k(a_1, \ldots, a_k; b) = \mathbf{C}(a_1 \Box \cdots \Box a_k, b).$$

For example, the categories of Γ-spaces or symmetric spectra under $\wedge$ are multicategories.

There is a (lax) symmetric monoidal functor $\mathbb{U}$ from Γ-spaces to symmetric spectra [12]. This is a multifunctor from the multicategory of Γ-spaces to the multicategory of symmetric spectra.

Remark 25. The smash product of Γ-spaces of simplicial sets is defined using left Kan extension. As a result, we can equivalently define a multicategory structure on Γ-spaces without reference to the smash product by declaring the set of k-morphisms from $(M_1, \ldots, M_k)$ to N to be the set of collections of maps

$$M_1(Y_1) \wedge \cdots \wedge M_k(Y_k) \to N(Y_1 \wedge \cdots \wedge Y_k),$$

natural in $Y_1, \ldots, Y_k$.

We will now define multicategories **M**, **A**, and **P**, enriched over topological spaces, as parameter multicategories for E_∞-modules, algebras, and pairings. Let $\mathcal{E}(n)$ be the space of linear isometric embeddings of $\mathcal{U}^{\otimes n}$ in $\mathcal{U}$. Together the $\mathcal{E}(n)$ form an E_∞-operad.

Definition 26. The multicategory **M** has objects R and M, such that $\mathbf{M}_k(B_1, \ldots, B_k; C)$ is equal to $\mathcal{E}(k)$ in the following cases:

– $B_j = C = R$ for all j, or

– $B_i = C = M$ for some i, and $B_j = R$ for all $j \neq i$.

Otherwise, $\mathbf{M}_k(B_1, \ldots, B_k; C) = \emptyset$. Composition is given by the composition in $\mathcal{E}$.

Definition 27. The multicategory **A** has objects R and A, such that $\mathbf{A}_k(B_1, \ldots, B_k; C)$ is equal to $\mathcal{E}(k)$ in the following cases:

– $B_j = C = R$ for all j, or

– $C = A$.

Otherwise, $\mathbf{A}_k(B_1, \ldots, B_k; C) = \emptyset$. Composition is given by composition in $\mathcal{E}$.

Definition 28. The multicategory **P** has objects R, M, and N, and P, such that $\mathbf{P}_k(B_1, \ldots, B_k; C)$ is equal to $\mathcal{E}(k)$ in the following cases:

– $B_j = C = R$ for all j,

– $B_i = C = M$ for some i, and $B_j = R$ for all $i \neq j$,

– $B_i = C = N$ for some i, and $B_j = R$ for all $i \neq j$,

– $B_i = C = P$ for some i, and $B_j = R$ for all $i \neq j$, or

– $B_i = N$, $B_{i'} = M$, $C = P$, and $B_j = R$ for all $j \neq i, i'$.

Otherwise, $\mathbf{P}_k(B_1, \ldots, B_k; C) = \emptyset$. Composition is given by composition in $\mathcal{E}$.

There are multifunctors $j : \mathbf{M} \to \mathbf{A}$ and $k : \mathbf{P} \to \mathbf{A}$ with $j(R) = k(r) = R$, $j(M) = k(M) = k(N) = k(P) = A$. Similarly, there are multifunctors $i_1, i_2, h : \mathbf{M} \to \mathbf{P}$ that send R to R and such that $i_1(M) = M$, $i_2(M) = N$, and $h(M) = P$. These induce restriction maps on multifunctors out to other categories; for instance, j^* restricts E_∞-algebras to their underlying modules.

Additionally, all three of these multicategories have a common subcategory **R** with a single object R. Write η for the embedding of **R** in any of these multicategories.

PROPOSITION 29. *Associated to a group* G*, there is a collection of multifunctors as follows:*

- $r : \mathbf{R} \to \Gamma$*-spaces,*
- $m : \mathbf{M} \to \Gamma$*-spaces,*
- $m_n, K_n, T_n, S_n : \mathbf{M} \to \Gamma$*-spaces for* $n \in \mathbb{N}$*,*
- $K : \mathbf{A} \to \Gamma$*-spaces, and*
- $P_{n,m} : \mathbf{P} \to \Gamma$*-spaces for* $n, m \in \mathbb{N}$*.*

These multifunctors are continuous with respect to the enrichment in spaces, and satisfy the following properties:

- $r(R) = ku$*,*
- $\eta^*(F) = r$ *for any multifunctor* F *on the above list,*
- $m(M) = ku$*,* $m_n(M) = \mathcal{L}(n)_+ \wedge ku$*, where* $\mathcal{L}(n)$ *is the space of isometries* $\mathbb{C}^n \otimes \mathcal{U} \to \mathcal{U}$*,*
- $K(A) = \mathcal{K}\mathrm{G}$*,* $K_n(M) = \mathcal{K}\mathrm{G}_n$*,*
- $T_n(M) = R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$*,* $S_n(M)$ *is the underlying* Γ*-space of* $ku^{\mathrm{PU}(n)}$*, and*
- $i_1^*(P_{n,m}) = K_n$*,* $i_2^*(P_{n,m}) = K_m$*,* $h^*(P_{n,m}) = K_{nm}$*.*

There are weak equivalences $m \leftarrow m_n \to S_n$ *for each* n*, and natural transformations* $K_1 \to K_2 \to \ldots \to j^*K$ *that realize the inclusions of* $\mathcal{K}\mathrm{G}_n$ *into* $\mathcal{K}\mathrm{G}$*. Similarly, there are natural transformations* $P_{n,m} \to P_{n',m'}$ *for* $n \le n', m \le m'$ *and* $P_{n,m} \to k^*K$*, which all commute, and applying* i_1^**,* j_1^**, or* h^* *yields a family of natural transformations that realize the above inclusions.*

Additionally, there are natural transformations $K_n \to T_n$ *that realize the quotient map* $\mathcal{K}\mathrm{G}_n \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$*. The group* $\mathrm{PU}(n)$ *acts continuously on* S_n*, and there are* $\mathrm{PU}(n)$*-equivariant maps from* R_n *to the set of natural transformations* $\mathrm{Nat}(S_n, T_n)$*. The adjoints of these maps realize the assembly maps* $R_n \wedge_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)} \to R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$*.*

A map $\mathrm{G} \to \mathrm{G}'$ *induces a natural transformation of multifunctors in the opposite direction.*

Proof. Write $\mathcal{K}\mathrm{G}^{\mathcal{V}}$ for the Γ-space $\mathcal{K}\mathrm{G}$ indexed on the universe $\mathcal{V}$. For groups $\mathrm{G}_1, \ldots, \mathrm{G}_k$, there is a well-defined exterior tensor product of representations:

$$\mathcal{K}(\mathrm{G}_1)^{\mathcal{U}_1}(Z_1) \wedge \cdots \wedge \mathcal{K}(\mathrm{G}_k)^{\mathcal{U}_k}(Z_k) \to \mathcal{K}(\mathrm{G}_1 \times \cdots \times \mathrm{G}_k)^{\otimes_i \mathcal{U}_i}(Z_1 \wedge \cdots \wedge Z_k).$$

This tensor product is coherently commutative and associative with respect to the underlying coherently commutative and associative tensor product on inner product spaces. It is natural in the Z_i, and comes from a natural map of Γ-spaces

$$\mathcal{K}(\mathrm{G}_1)^{\mathcal{U}_1} \wedge \cdots \wedge \mathcal{K}(\mathrm{G}_k)^{\mathcal{U}_k} \to \mathcal{K}(\mathrm{G}_1 \times \cdots \times \mathrm{G}_k)^{\otimes_i \mathcal{U}_i}.$$

The tensor product induces coherent pairings

$$\mathcal{K}(\mathrm{G}_1)^{\mathcal{U}_1}_{n_1} \wedge \cdots \wedge \mathcal{K}(\mathrm{G}_k)^{\mathcal{U}_k}_{n_k} \to \mathcal{K}(\mathrm{G}_1 \times \cdots \times \mathrm{G}_k)^{\otimes_i \mathcal{U}_i}_{n_1 \cdots n_k}.$$

Now restrict to the case when $\mathcal{U}_i = \mathcal{U}$ for all i. Post-composition with linear isometric embeddings $\mathcal{U}^{\otimes k} \to \mathcal{U}$ then gives maps of Γ-spaces

$$\mathcal{E}(k)_+ \wedge \mathcal{K}(\mathrm{G}_1)_{n_1} \wedge \cdots \wedge \mathcal{K}(\mathrm{G}_k)_{n_k} \to \mathcal{K}(\mathrm{G}_1 \times \cdots \times \mathrm{G}_k)_{n_1 \cdots n_k}.$$

If all G_i are equal to G or the trivial group, we can pull back along the diagonal map to get a map

$$\mathcal{E}(k)_+ \wedge B_1 \wedge \cdots \wedge B_k \to C,$$

where the B_i and C are all of the form $\mathcal{K}\mathrm{G}$, $\mathcal{K}\mathrm{G}_n$, or ku. These maps have continuous adjoints that define the multifunctors r, m, K, K_n, and $P_{n,m}$. The multifunctors m_n are formed by smashing m with the spaces $\mathcal{L}(n)_+$; projection from $\mathcal{L}(n)_+ \to S^0$ gives the weak equivalence $m_n \to m$.

Similarly, the underlying Γ-space of $ku^{\mathrm{PU}(n)}$ admits an exterior tensor product. A point of $ku^{\mathrm{PU}(n)}(\mathrm{PU}(n)_+ \wedge Z)$ is a map

$$f : Z \to \coprod V^{\mathcal{U}}(nd)/I \otimes U(d),$$

where $V^{\mathcal{U}}(nd)$ is the Stiefel manifold of nd-frames in $\mathcal{U}$, such that $f(z) \perp f(z')$ if $z \neq z'$. The tensor product of frames induces a map

$$\left[V^{\mathcal{U}}(nd)/I \otimes U(d)\right] \wedge \left[V^{\mathcal{V}}(k)/U(k)\right] \to V^{\mathcal{U} \otimes \mathcal{V}}(ndk)/I \otimes U(dk).$$

This product is coherently commutative and associative with respect to the tensor product on universes. Just as with $\mathcal{K}\mathrm{G}$, post-composition with linear isometries gives maps

$$\mathcal{E}(k)_+ \wedge ku \wedge \cdots \wedge ku^{\mathrm{PU}(n)} \wedge ku \wedge \cdots \wedge ku \to ku^{\mathrm{PU}(n)},$$

giving the desired multifunctor S_n. These pairings are clearly $\mathrm{PU}(n)$-equivariant, hence $\mathrm{PU}(n)$ acts on the functor S_n.

Tensoring with $\mathbb{C}^n$ and precomposing with an isometry $\mathbb{C}^n \otimes \mathcal{U} \to \mathcal{U}$ gives a weak equivalence

$$\mathcal{L}(n)_+ \wedge V^{\mathcal{U}}(d)/\mathrm{U}(d) \to V^{\mathcal{U}}(nd)/\mathrm{U}(nd).$$

This commutes with the exterior tensor product, and so defines a natural weak equivalence of multifunctors $m_n \to S_n$.

To define the multifunctor T_n, one can either go through the same argument with $R_n \otimes_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)}$ or recall that it is isomorphic to the Γ-space F_n. F_n is a quotient of $\mathcal{K}\mathrm{G}_n$ by an equivalence relation that is respected by tensor product with trivial representations. The natural map $\mathcal{K}\mathrm{G}_n \to F_n$ yields a natural transformation of multifunctors.

The only remaining issue is to check that the assembly map is a map of E_∞-ku-modules. For this, it suffices to note that the assembly map commutes with the exterior tensor pairing

$$(ku^{\mathrm{PU}(n)})^{\mathcal{U}}(\mathrm{PU}(n)_+ \wedge Y) \wedge ku^{\mathcal{V}}(Z) \to ku^{\mathcal{U}\otimes\mathcal{V}}(\mathrm{PU}(n)_+ \wedge Y \wedge Z)$$

for $Y, Z \in \Gamma^o$, $\mathcal{U}$ and $\mathcal{V}$ universes.

We now prove the following rigidification result. Recall from Example 24 that $\mathbb{U}$ is the functor which takes a Γ-space to the associated (topological) symmetric spectrum.

PROPOSITION 30. *There is a commutative ring object in symmetric spectra ku^r with module spectra $ku^{\mathrm{PU}(n),r}$, and contravariant functors $\mathcal{K}(-)^r_n$, $\mathcal{K}(-)^r$ from finitely generated discrete groups to connective ku^r-module symmetric spectra with the following properties:*

- *there are isomorphisms in the stable homotopy category of symmetric spectra $F^r \cong \mathrm{Sing}\,\mathbb{U}(F)$, where F is one of ku, $ku^{\mathrm{PU}(n)}$, $\mathcal{K}(-)$, or $\mathcal{K}(-)_n$,*
- *as a ku^r-module, $ku^{\mathrm{PU}(n),r}$ is weakly equivalent to ku^r,*
- *there are ku^r-module maps $\mathcal{K}\mathrm{G}^r_1 \to \mathcal{K}\mathrm{G}^r_2 \to \cdots \to \mathcal{K}\mathrm{G}^r_\infty$, and $\mathcal{K}\mathrm{G}^r$ is weakly equivalent to the homotopy colimit,*
- *there are strictly commutative and associative ku^r-module pairings $\mathcal{K}\mathrm{G}^r_n \wedge_{ku} \mathcal{K}\mathrm{G}^r_m \to \mathcal{K}\mathrm{G}^r_{nm}$ that commute with the above maps,*
- *for any n, $ku^{\mathrm{PU}(n),r}$ is acted on by $\mathrm{Sing}\,\mathrm{PU}(n)$, and the homotopy cofiber of the map $\mathcal{K}\mathrm{G}^r_{n-1} \to \mathcal{K}\mathrm{G}^r_n$ is a ku^r-module equivalent to the derived smash product $\mathrm{Sing}\,R_n \wedge_{\mathrm{Sing}\,\mathrm{PU}(n)} ku^{\mathrm{PU}(n),r}$.*

All of the above are natural in G.

Proof. The multifunctors constructed in Proposition 29 are multifunctors of categories enriched in topological spaces. Given such a multifunctor $g : \mathbf{C} \to \Gamma$-spaces, the composite functor $\mathbb{U} \circ g$ takes values in topological symmetric spectra. Similarly, the singular complex functor Sing is a Quillen equivalence that is lax symmetric monoidal with respect to the smash product $\wedge$, so there is a simplicial multifunctor $\mathrm{Sing}(\mathbb{U} \circ g)$ from $\mathrm{Sing}\,\mathbf{C}$ to symmetric spectra. Additionally, there is a simplicial multifunctor π from $\mathrm{Sing}\,\mathbf{C}$ to the constant simplicial multicategory $\pi_0\mathbf{C}$. π is a weak equivalence.

In [6], a simplicial closed model structure is constructed on the category of simplicial multifunctors from a simplicial multicategory $\mathbf{D}$ to $\mathcal{S}$, the category of symmetric spectra. In particular, Theorem 1.4 of [6] proves that if $f : \mathbf{D} \to \mathbf{D}'$ is a simplicial multifunctor, the restriction map $f^* : \mathcal{S}^{\mathbf{D}'} \to \mathcal{S}^{\mathbf{D}}$ has a left adjoint f_*, and if f is a weak equivalence then this adjoint pair is a Quillen equivalence.

Let $\mathbb{L}\pi_*$ denote the total left derived functor of π_*, which consists of cofibrant resolution followed by π_*. We obtain a rigidified symmetric spectrum $\mathrm{Rig}(g) = \mathbb{L}\pi_*\,\mathrm{Sing}(\mathbb{U} \circ g)$ such that $\pi^*\mathrm{Rig}(g)$ is isomorphic to $\mathrm{Sing}(\mathbb{U} \circ g)$ in the homotopy category of multifunctors $\pi_0(\mathbf{C}) \to \mathcal{S}$.

Additionally, in our case the multicategory $\mathbf{C}$ accepts a "unit" map η from the multicategory $\mathbf{R}$ such that $\eta^* g = r$. Let $\tilde{g} \to \mathrm{Sing}(\mathbb{U} \circ g)$ be a cofibrant replacement (an acyclic fibration where $\tilde{g}$ is a cofibrant object) and similarly $\tilde{r} \to \mathrm{Sing}(\mathbb{U} \circ r)$. The map η^* preserves weak equivalences and fibrations, so the map $\eta^*\tilde{g} \to \eta^*\,\mathrm{Sing}(\mathbb{U} \circ g)$ is an acyclic fibration. The isomorphism $r \to \eta^* g$ therefore lifts to a weak equivalence $\tilde{r} \to \eta^*\tilde{g}$.

Composing the weak equivalences

$$\tilde{r} \to \eta^*\tilde{g} \to \eta^*\pi^*\pi_*\tilde{g} \to \pi^*\eta^*\pi_*\tilde{g},$$

we get an adjoint weak equivalence $\pi_*\tilde{r} \to \eta^*\pi_*\tilde{g}$. Both objects are strictly commutative ring symmetric spectra weakly equivalent to the object $\mathrm{Sing}(\mathbb{U} \circ r)$. In other words, there is a natural weak equivalence $\eta' : \mathrm{Rig}(r) \to \eta^*\mathrm{Rig}(g)$.

Define $ku^r = \mathrm{Rig}(r)(R)$, $\mathcal{K}\mathrm{G}^r_n = \mathrm{Rig}(K_n)(M)$, $\mathcal{K}\mathrm{G}^r = \mathrm{Rig}(K)(A)$, and $ku^{\mathrm{PU}(n),r} = \mathrm{Rig}(S_n)(M)$. The unit maps η' make all of these objects and maps between them maps of ku^r-modules. The existence of algebra structures and pairings of modules are restatements of the multicategory structures on these modules.

The weak equivalence of ku^r with $ku^{\mathrm{PU}(n),r}$ follows from the weak equivalences of multifunctors $m \leftarrow m_n \to S_n$.

The weak equivalence between $\mathcal{K}\mathrm{G}^r$ and the homotopy colimit of the sequence $\mathcal{K}\mathrm{G}^r_n$ follows from Proposition 14, as we have $\mathcal{K}\mathrm{G}^r_n \simeq \mathrm{Sing}\,\mathbb{U}\mathcal{K}\mathrm{G}_n$ and $\mathcal{K}\mathrm{G} \simeq \mathrm{Sing}\,\mathbb{U}\mathcal{K}\mathrm{G} \simeq \mathrm{Sing}\,\mathbb{U}(\mathrm{hocolim}\,\mathcal{K}\mathrm{G}_n)$.

The homotopy cofiber sequence

$$\mathrm{Sing}\,\mathbb{U}\mathcal{K}\mathrm{G}_{n-1} \to \mathrm{Sing}\,\mathbb{U}\mathcal{K}\mathrm{G}_n \to \mathrm{Sing}\,\mathbb{U}F_n$$

is weakly equivalent to the sequence of maps

$$\mathcal{K}\mathrm{G}^r_{n-1} \to \mathcal{K}\mathrm{G}^r_n \to \mathrm{Rig}(T_n)(M)$$

in the category of ku^r-module symmetric spectra; we now prove that this last space is weakly equivalent to an equivariant smash product.

The ku^r-module $R_n \wedge_{\mathrm{PU}(n)} S_n(M)$ is weakly equivalent to the geometric realization of the bar construction $B(R_n, \mathrm{PU}(n)_+, S_n(M))$ because R_n is a free $\mathrm{PU}(n)$-CW complex. The structure map $X \wedge \mathbb{U}N \to \mathbb{U}(X \wedge N)$ for a space X and a Γ-space N is an isomorphism, and similarly $\mathrm{Sing}(X \wedge E) \to \mathrm{Sing}(X) \wedge \mathrm{Sing}(E)$ is a weak equivalence for a CW-complex X and a topological symmetric spectrum E. Therefore, both of these functors preserve this bar construction up to weak equivalence. The result is that there is a weak equivalence of symmetric spectra

$$\mathrm{Sing}\,\mathbb{U}\left(R_n \underset{\mathrm{PU}(n)}{\wedge} S_n(M)\right) \to \mathrm{Sing}(R_n) \underset{\mathrm{Sing}\,\mathrm{PU}(n)}{\wedge} \mathrm{Sing}\,\mathbb{U}S_n(M), \tag{2}$$

where this latter smash product is taken in the derived sense.

The continuous $\mathrm{PU}(n)$-equivariant map $R_n \to \mathrm{Nat}(S_n, T_n)$ induces a $\mathrm{Sing}\,\mathrm{PU}(n)$-equivariant map $\mathrm{Sing}\,R_n \to F(\mathrm{Rig}(S_n)(M), \mathrm{Rig}(T_n)(M))$. There is an adjoint map

$$\mathrm{Sing}\,R_n \underset{\mathrm{Sing}\,\mathrm{PU}(n)}{\wedge} \mathrm{Rig}(S_n)(M) \to \mathrm{Rig}(T_n)(M),$$

where the smash product is taken in the derived category. This is weakly equivalent to the map

$$\mathrm{Sing}\,R_n \underset{\mathrm{Sing}\,\mathrm{PU}(n)}{\wedge} \mathrm{Sing}\,\mathbb{U}S_n(M) \to \mathrm{Sing}\,\mathbb{U}T_n(M).$$

Using the weak equivalence of Equation 2, this is weakly equivalent to the map

$$\mathrm{Sing}\,\mathbb{U}\left(R_n \underset{\mathrm{PU}(n)}{\wedge} S_n(M)\right) \to \mathrm{Sing}\,\mathbb{U}T_n(M).$$

The map $R_n \wedge_{\mathrm{PU}(n)} S_n(M) \to T_n(M)$ is a weak equivalence by Corollary 22. Therefore, the map $\mathrm{Sing}\,R_n \wedge_{\mathrm{Sing}\,\mathrm{PU}(n)} ku^{\mathrm{PU}(n),r} \to \mathrm{Rig}(T_n)(M)$ is a weak equivalence of ku^r-modules.

8. The exact couple for $\mathcal{K}\mathrm{G}$

There is the following chain of weak equivalences of symmetric spectra. Here the smash products are taken in the derived category to assure associativity.

$$\begin{aligned} \mathrm{H}\mathbb{Z} \underset{ku^r}{\wedge} \left(\operatorname{Sing} R_n \underset{\operatorname{Sing}\mathrm{PU}(n)}{\wedge} ku^{\mathrm{PU}(n),r} \right) &\simeq \operatorname{Sing} R_n \underset{\operatorname{Sing}\mathrm{PU}(n)}{\wedge} \mathrm{H}\mathbb{Z} \\ &\simeq \mathrm{H}\mathbb{Z} \wedge (R_n/\mathrm{PU}(n)). \end{aligned}$$

Define $\mathrm{QIrr}(\mathrm{G}, n) = R_n/\mathrm{PU}(n)$. $\mathrm{QIrr}(\mathrm{G}, n)$ is the quotient space of isomorphism classes of representations G of dimension n modulo decomposable representations. (The notation is to avoid confusion with the standard notation for the *subspace* of isomorphism classes of irreducible representations.)

Proposition 30 identifies the following homotopy cofiber sequences. The homotopy colimit of the top row is weakly equivalent to $\mathcal{K}\mathrm{G}^r$.

$$\begin{tikzcd} * \arrow[r] & \mathcal{K}\mathrm{G}_1^r \arrow[r] \arrow[d] & \mathcal{K}\mathrm{G}_2^r \arrow[r] \arrow[d] & \cdots \\ & \operatorname{Sing} R_1 \wedge ku^r & \operatorname{Sing} R_2 \wedge_{\operatorname{Sing}\mathrm{PU}(2)} ku^{\mathrm{PU}(2),r} & \end{tikzcd}$$

Using the weak equivalence $\mathcal{K}\mathrm{G}^r \simeq \operatorname{Sing}(\mathbb{U}\mathcal{K}\mathrm{G})$, the following spectral sequence results.

THEOREM 31. *There exists a convergent right-half-plane spectral sequence of the form*

$$E_1^{p,q} = ku_{q-p+1}^{\mathrm{PU}(n)}(R_{p-1}) \Rightarrow \pi_{p+q}(\mathcal{K}\mathrm{G}).$$

Remark 32. This uses a Serre indexing convention, so that d_r maps $E_r^{p,q}$ to $E_r^{p-r,q+r-1}$. We should remark that by the homotopy group π_n of a symmetric spectrum X, we mean the set $[S^n, X]$ in the stable homotopy category, or equivalently the stable homotopy groups of an Ω-spectrum weakly equivalent to X. The symmetric spectra $\mathcal{K}\mathrm{G}$ and $\mathcal{K}\mathrm{G}_n$ are almost Ω-spectra, and so in this case the notion coincides with the classical notion of stable homotopy groups.

Smashing the previous diagram over ku^r with $\mathrm{H}\mathbb{Z}$ (with smash product taken in the derived category of ku^r-modules) yields the following. The homotopy colimit of the top row is weakly equivalent to $\mathrm{H}\mathbb{Z} \wedge_{ku^r} \mathcal{K}\mathrm{G}^r$.

$$\begin{tikzcd}
{*} \arrow[r] & \mathrm{H}\mathbb{Z} \wedge_{ku^r} \mathcal{K}\mathrm{G}_1^r \arrow[r] \arrow[d] & \mathrm{H}\mathbb{Z} \wedge_{ku^r} \mathcal{K}\mathrm{G}_2^r \arrow[r] \arrow[d] & \mathrm{H}\mathbb{Z} \wedge_{ku^r} \mathcal{K}\mathrm{G}_3^r \ldots \arrow[d] \\
& \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathrm{G}, 1) & \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathrm{G}, 2) & \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathrm{G}, 3)
\end{tikzcd}$$

Again, the diagram results in a spectral sequence.

THEOREM 33. *There exists a convergent right-half-plane spectral sequence of the form*

$$E_1^{p,q} = H_{q-p+1}(\mathrm{QIrr}(\mathrm{G}, p-1)) \Rightarrow \pi_{p+q}(\mathrm{H}\mathbb{Z} \underset{ku^r}{\wedge} \mathcal{K}\mathrm{G}).$$

Example 34. When G is finite or nilpotent, the cofiber sequences are all split. When G is finite, this is clear. When G is nilpotent, results of [10] show that the space of irreducible representations of dimension n is closed in $\mathrm{Hom}(\mathrm{G}, \mathrm{U}(n))$, which provides the desired splitting $R_n \wedge_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n)} \to \mathcal{K}\mathrm{G}_n$.

As a result, for these groups we have a weak equivalence

$$\mathcal{K}\mathrm{G} \simeq \bigvee \left(R_n \underset{\mathrm{PU}(n)}{\wedge} ku^{\mathrm{PU}(n)} \right).$$

In this case, the spectral sequence of Theorem 33 degenerates at the E_1 page. For example, consider the integer Heisenberg group of 3×3 strict upper triangular matrices with integer entries. The structure of the space of all irreducible representations appears in [14].

The $E_1 = E_\infty$ page of the spectral sequence of Theorem 33 looks, in part, as follows:

$$\begin{array}{|ccccccc}
 & & & \vdots & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z} & 0 & 0 & 0 & 0 & 0 & \\
\mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & 0 & 0 & \\
\mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & 0 & \cdots \\
\hline
 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & 0 & \\
 & & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 & \\
 & & & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & \\
 & & & \vdots & & &
\end{array}$$

Example 35. Suppose G is free on k generators. As mentioned in the introduction, there is a weak equivalence $\mathcal{K}\mathrm{G} \simeq ku \vee (\vee^k \Sigma ku)$.

When G is free on two generators, explicit computations with the spectral sequence of Theorem 33 give the following picture of the E_1 page.

$$
\begin{array}{|ccccc}
 & & & \vdots & \\
0 & 0 & ? & ? & \\
0 & \mathbb{Z} & ? & ? & \\
0 & \mathbb{Z}^2 & ? & ? & \\
\mathbb{Z} \leftarrow & \mathbb{Z} & ? & ? & \\
\mathbb{Z}^2 & 0 & ? & ? & \\
\mathbb{Z} & 0 & ? & ? & \cdots \\
\hline
 & 0 & 0 & ? & \\
 & & 0 & 0 & \\
 & & & \vdots &
\end{array}
$$

The differential $d_1 : E_1^{1,2} \to E_1^{0,2}$ is an isomorphism. The terms $E_1^{p,q}$ are zero on the set $\{p > 0, p+q < 2\}$, and also on the set $\{q > p^2+p+2\}$. The terms where $q = p^2 + p + 2$ are all isomorphic to $\mathbb{Z}$.

This spectral sequence converges to $\mathbb{Z}$ in dimension 0, $\mathbb{Z}^2$ in dimension 1, and 0 in all other dimensions. The classes in $E_1^{0,0}$ and $E_1^{0,1}$ are precisely those classes that survive to the E_∞ term.

9. Proof of the product formula

In this section we will prove Theorem 1, the product formula for deformation K-theory spectra.

The proof requires the following lemmas.

LEMMA 36. *A map $M' \to M$ of connective ku^r-module spectra is a weak equivalence if and only if the map $\mathrm{H}\mathbb{Z} \wedge_{ku^r} M' \to \mathrm{H}\mathbb{Z} \wedge_{ku^r} M$ is a weak equivalence. (Smash products are taken in the derived category.)*

Proof. By taking cofibers, it suffices to prove the equivalent statement that a connective ku^r-module spectrum M'' is weakly contractible if and only if $\mathrm{H}\mathbb{Z} \wedge_{ku^r} M'' \simeq *$.

However, smashing M'' with the homotopy cofiber sequence $\Sigma^2 ku^r \to ku^r \to \mathrm{H}\mathbb{Z}$ of ku^r-module spectra shows that $\mathrm{H}\mathbb{Z} \wedge_{ku^r} M'' \simeq *$ if and only if the Bott map $\beta : \Sigma^2 M'' \to M''$ is a weak equivalence. This would imply that the homotopy groups of M'' are periodic; M'' is connective, so the result follows.

LEMMA 37. *Irreducible unitary representations of* $\mathrm{G}\times\mathrm{H}$ *are precisely of the form* $V\otimes W$ *for* V, W *irreducible unitary representations of* G *and* H *respectively.*

Proof. That the tensor product of irreducible group representations is irreducible, and conversely irreducible representations of $\mathrm{G}\times\mathrm{H}$ are tensor products, is well known. (See, for example, [21] Section 4.4, Theorem 6 and the comment afterwards.)

Suppose V and W are nontrivial irreducible representations of G and H respectively. Clearly invariant inner products on V and W induce one on $V\otimes W$. Conversely, if W is nonzero V appears as a sub-G-representation of $V\otimes W$, and hence an invariant inner product on $V\otimes W$ induces one on the subspace V. Therefore, $V\otimes W$ admits a unitary structure if and only if both V and W do.

Remark 38. Lemma 37 fails when we consider representations of the group $\mathrm{G}\times\mathrm{H}$ in other groups such as orthogonal groups and symmetric groups.

Proof. [of Theorem 1] The proof consists of constructing a filtration of the spectrum $\mathcal{K}\mathrm{G}^r\wedge_{ku^r}\mathcal{K}\mathrm{H}^r$ that agrees with the existing filtration on $\mathcal{K}(\mathrm{G}\times\mathrm{H})^r$.

We apply the results of Proposition 30 to get a map of ku^r-algebras

$$\mathcal{K}\mathrm{G}^r\underset{ku^r}{\wedge}\mathcal{K}\mathrm{H}^r\to\mathcal{K}(\mathrm{G}\times\mathrm{H})^r\underset{ku^r}{\wedge}\mathcal{K}(\mathrm{G}\times\mathrm{H})^r\to\mathcal{K}(\mathrm{G}\times\mathrm{H})^r.$$

Similarly, whenever $p\cdot q\leq n$ there is a corresponding map of ku^r-modules

$$\mathcal{K}\mathrm{G}^r_p\underset{ku^r}{\wedge}\mathcal{K}\mathrm{H}^r_q\to\mathcal{K}(\mathrm{G}\times\mathrm{H})^r_n.$$

This diagram is natural in p, q, and n.

Let Γ denote a cofibrant replacement functor for ku^r-modules. If we define new ku^r-module spectra $M_n=\mathrm{hocolim}_{p\cdot q\leq n}\Gamma\mathcal{K}\mathrm{G}^r_p\wedge_{ku^r}\Gamma\mathcal{K}\mathrm{H}^r_q$, then there are induced ku-module maps

$$f_n: M_n\to\mathcal{K}(\mathrm{G}\times\mathrm{H})^r_n.$$

The maps $(\mathrm{hocolim}\,\mathcal{K}\mathrm{G}^r_p)\to\mathcal{K}\mathrm{G}^r$ and $(\mathrm{hocolim}\,\mathcal{K}\mathrm{H}^r_q)\to\mathcal{K}\mathrm{H}^r$ are weak equivalences, so there is a weak equivalence

$$\mathrm{hocolim}\,M_n\simeq\underset{p,q}{\mathrm{hocolim}}\,\Gamma\mathcal{K}\mathrm{G}^r_p\underset{ku^r}{\wedge}\Gamma\mathcal{K}\mathrm{H}^r_q\simeq\Gamma\mathcal{K}\mathrm{G}^r\underset{ku}{\wedge}\Gamma\mathcal{K}\mathrm{H}^r.$$

Therefore, it suffices to show $M_n\to\mathcal{K}(\mathrm{G}\times\mathrm{H})^r_n$ is a weak equivalence for all n. We have an induced map of homotopy cofiber sequences:

$$\begin{tikzcd}
M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & M_n \arrow[r] \arrow[d, "f_n"'] & M_n/M_{n-1} \arrow[d, "g_n"'] \\
\mathcal{K}(\mathrm{G}\times\mathrm{H})^r_{n-1} \arrow[r] & \mathcal{K}(\mathrm{G}\times\mathrm{H})^r_n \arrow[r] & R_n(\mathrm{G}\times\mathrm{H}) \wedge_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n),r}.
\end{tikzcd}$$

To prove the theorem it suffices to show that the map g_n is a weak equivalence for all n.

The spectra M_n/M_{n-1} and $R_n(\mathrm{G}\times\mathrm{H}) \wedge_{\mathrm{PU}(n)} ku^{\mathrm{PU}(n),r}$ are connective ku^r-module spectra. Applying Lemma 36, it suffices to prove that the map $\mathrm{H}\mathbb{Z} \wedge_{ku^r} g_n$ is a weak equivalence.

Because the map $M_{n-1} \to M_n$ is a map from the homotopy colimit of a subdiagram into the full diagram, we can explicitly compute the homotopy cofiber of this map. The homotopy cofiber is weakly equivalent to the wedge

$$\bigvee_{p\cdot q=n} \left(\Gamma\mathcal{K}\mathrm{G}^r_p/\Gamma\mathcal{K}\mathrm{G}^r_{p-1}\right) \underset{ku^r}{\wedge} \left(\Gamma\mathcal{K}\mathrm{H}^r_q/\Gamma\mathcal{K}\mathrm{H}^r_{q-1}\right).$$

To see this, one notes that we can take the homotopy colimit by replacing the diagram by a homotopy equivalent diagram made up of cofibrations, and then take the take the ordinary colimit. For any such diagram $\{F_{p,q}\}$, we clearly have

$$\left(\bigcup_{p\cdot q\leq n} F_{p,q}\right) \Big/ \left(\bigcup_{p\cdot q<n} F_{p,q}\right) = \bigvee_{p\cdot q=n} F_{p,q}/\left(F_{p-1,q} \cup_{F_{p-1,q-1}} F_{p,q-1}\right).$$

The "pushout product axiom" [18] shows that in the case where $F_{p,q} \simeq A_p \wedge B_q$ and the maps $A_{p-1} \to A_p$ and $B_{p-1} \to B_p$ are cofibrations, we have

$$F_{p,q}/\left(F_{p-1,q} \cup_{F_{p-1,q-1}} F_{p,q-1}\right) \simeq A_p/A_{p-1} \underset{ku^r}{\wedge} B_q/B_{q-1}.$$

The spectra $\mathcal{K}\mathrm{G}^r_p/\mathcal{K}\mathrm{G}^r_{p-1}$, and the corresponding spectra for H, are weakly equivalent to those that were identified as equivariant smash product spectra in Corollary 19 and Proposition 21. Smashing over ku^r with $\mathrm{H}\mathbb{Z}$ gives us the following identity.

$$\begin{aligned}
\mathrm{H}\mathbb{Z} \underset{ku^r}{\wedge} M_n/M_{n-1} &\simeq \bigvee_{p\cdot q=n} \Big(\mathrm{H}\mathbb{Z}\wedge \mathrm{QIrr}(\mathrm{G},p)\Big) \underset{\mathrm{H}\mathbb{Z}}{\wedge} \Big(\mathrm{H}\mathbb{Z}\wedge\mathrm{QIrr}(\mathrm{H},q)\Big) \\
&\simeq \mathrm{H}\mathbb{Z}\wedge\left(\bigvee_{p\cdot q=n} \mathrm{QIrr}(\mathrm{G},p)\wedge\mathrm{QIrr}(\mathrm{H},q)\right).
\end{aligned}$$

The map $\mathrm{H}\mathbb{Z} \wedge_{ku} g_n$ can be identified with the map

$$\mathrm{H}\mathbb{Z} \wedge \left(\vee_{p \cdot q = n} \mathrm{QIrr}(\mathrm{G}, p) \wedge \mathrm{QIrr}(\mathrm{H}, q) \right) \to \mathrm{H}\mathbb{Z} \wedge \mathrm{QIrr}(\mathrm{G} \times \mathrm{H}, n)$$

that is induced by the tensor product of representations. The tensor product map $\otimes : \vee_{p \cdot q = n} \mathrm{QIrr}(\mathrm{G}, p) \wedge \mathrm{QIrr}(\mathrm{H}, q) \to \mathrm{QIrr}(\mathrm{G} \times \mathrm{H}, n)$ is a continuous map between compact Hausdorff spaces. It is bijective by Lemma 37. Therefore, it is a homeomorphism.

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