# In which I try to get the signs right for once

or: a public exercise in futility

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Not so long ago, a colleague of mine asked me the following question: What's the sign convention for commutativity of elements in the Adams spectral sequence?

Let's write $s$ for the Ext-degree and $t$ for the grading degree. I found in more than one source that for an element $x$ in bidegree $(s, t)$ and $y$ in bidegree $(s', t')$, the rule is:

$$xy = (-1)^{ss'+tt'} yx.$$

This is a reasonable-sounding convention that happens to be utterly false.

For example, there is a unique element in filtration 1, grading degree 1 in the Adams spectral sequence. No matter what your name for it is, it represents multiplication by $p$ and it quite simply does not anticommute with elements in odd total degree. When I tried to check what was going wrong, it quickly became apparent that my conventions for multiplication and the boundary operator were simply incompatible.

Unfortunately, my understanding of *why* certain things have the sign conventions they do was a house of cards, especially with respect to the cobar complex. So I tried to work things out by making as few arbitrary choices as possible, and working from reasoning that made more sense to me than trying to figure out exactly which symbols have which degrees and what counts as moving something across something else. I wrote it down here so that I hopefully never have to do it again.

This is not a particularly formal document but I'm posting it in case it helps someone else.

(In Sept. 2013 I added some material about the connecting homomorphism for exact triangles.)

## 1 Convention 0: Stars.

I'm not going to decorate chain complexes with stars. The chain complex is the object I'm interested in, not the modules that build it up, and so I'm just not going to do it.

## 2 Convention I: Homological algebra.

I'm going to try to hold the basic convention of homological algebra: if something of degree $p$ moves across something of degree $q$, it introduces the sign $(-1)^{pq}$. However, I'm going to decree it in as few places as possible.

As a result, graded abelian groups get a tensor product

$$(A \otimes B)_n = \oplus_{p+q=n} A_p \otimes B_q$$

where the symmetry isomorphism $\tau$ sends $a \otimes b$ to $(-1)^{|a||b|} b \otimes a$.

A ground ring is implicit. (If it becomes an issue I will come back and make conventions.)

## 3 Convention II: Homological indexing.

Cochain complexes will be chain complexes in negative degrees.

# 4 Convention III: Things operate on the left.

In general, unless specifically requested otherwise, functions and operators act on the left.

In particular, the boundary map $\partial$ of a chain complex then acts on the left.

# 5 Tensor products

Our demands for tensor products and boundary maps leads to the sign convention for the tensor product of two chain complexes:

$$\partial(a \otimes b) = (\partial a) \otimes b + (-1)^{|a|} a \otimes (\partial b)$$

This makes chain complexes into a symmetric monoidal category.

# 6 Shift operators

Here we start to get annoying. There are two standard conventions for the same shift operator on chain complexes. One is $\Sigma C$, and one is $C[1]$. These suggest different conventions for their boundary map.

The first notation suggests that the elements of $\Sigma C$ are of the form $\Sigma a$, and $\partial(\Sigma a) = -\Sigma(\partial a)$ by our sign conventions. The second convention suggests that the elements of $C[1]$ are of the form $a[1]$, and $\partial(a[1]) = (\partial a)[1]$. Note that this is NOT the convention that is often used in homological algebra for $C[1]$ - a sign is often introduced.

More bluntly, let $S$ be the chain complex which is $\mathbb{Z}$ in degree 1 and zero elsewhere. Then $\Sigma C$ under the above conventions is another name for $S \otimes C$, and $C[1]$ is another name for $C \otimes S$. They're isomorphic, but they're not the same. I'm going to try and keep things described in a way where I remember what convention I'm using.

# 7 Convention IV: Tensor up on the left.

In view of the previous section, I'm going to say that we're usually going to stick with the $\Sigma C$-type notation, where we tensor up with standardized objects on the left. This at least is consistent with the sign conventions in homological algebra, if not the right-side shift notation.

This will probably come back to bite me.

# 8 Simplices

I'm going to use another complex; the normalized chain complex $Z(\Delta[n])$ of the standard $n$-simplex. More specifically, $Z(\Delta[n])_0 = \mathbb{Z}^{n+1}$, with generators [0] through [1], etc, with $Z(\Delta[n])_n = \mathbb{Z}$ with a single generator $[0, 1, \ldots, n]$. Under our sign convention (and I use this in the loosest possible sense!), I'm going to define the boundary by

$$\partial[a_0, \cdots, a_p] = \sum(-1)^i [a_0, \ldots \hat{a}_i \ldots, a_p]$$

where the sign comes in because the edges are 1-dimensional.

Using 0 through $n$ isn't special, so I have a normalized chain complex associated to any sequence of symbols, and natural maps between them: e.g. from $[1, 2]$ to $[0, 1, 2]$.

Similarly, we'll have normalized chain complexes of arbitrary simplicial sets.

# 9 Mapping cylinders

If we can't even get the shift operator right without making choices, we're probably in trouble for most other definitions as well. Let's discuss mapping cylinders. There is more than one way to make a mapping cylinder. I'm going to think of the interval $[0, 1]$ as "directed" towards 1, so that 1 should be the range of a function.

If $f\colon A \to B$ is an arbitrary map, then I get a pushout defining a mapping cylinder:

$$\begin{array}{ccc} [1]\otimes A & \longrightarrow & Z(\Delta[1])\otimes A \\ \downarrow & & \downarrow \\ [1]\otimes B & \longrightarrow & M_f \end{array}$$

There's a composite inclusion map $[0] \otimes A \to Z(\Delta[1]) \otimes A \to M_f$. The map $[1] \otimes B \to M_f$ is a chain homotopy equivalence.

In degree $n$, this complex is

$$[0] \otimes A_n \oplus [1] \otimes B_n \oplus [0,1] \otimes A_{n-1},$$

which we can view as identified with $A_n \oplus B_n \oplus A_{n-1}$. Under this identification, the boundary map is

$$\partial(a, b, a') = (\partial a - a', \partial b + f(a'), -\partial a').$$

# 10 Mapping cones

The mapping cone $C_f$ of $f : A \to B$ is the quotient of the mapping cylinder $M_f$ by the inclusion of $[0] \otimes A$. In particular, in degree $n$ we can identify it as

$$[1] \otimes B_n \oplus [0,1] \otimes A_{n-1}$$

with

$$\partial(b, a') = (\partial b + f(a'), -\partial a').$$

This gives me at least one good reason to go with the tensoring on the left convention; this agrees with the definition of the mapping cone that I'm used to. If I'd tensored on the right, there would be some $(-1)^n$-terms, and if I'd done the pushout with $[0] \otimes B$, I'd have ended up with a different sign on $f(a')$.

# 11 Exact triangles

In the homotopy category of chain complexes, we're going to want exact triangles. We want these to be sequences $A \to B \to C \to \Sigma A$ which help us satisfy the axioms for a triangulated category, and $C$ is (nonuniquely) determined by the maps $A$ and $B$. Moreover, we want exact sequences $0 \to A \to B \to C \to 0$ to determine an exact triangle, via some map $C \to \Sigma A$ in the homotopy category.

This forces some things upon us. Let's write the following diagram, natural for injections $A \to B$:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 & & \\
 & & \| & & \uparrow{\scriptstyle\sim} & & \uparrow{\scriptstyle\sim} & & & & \\
0 & \longrightarrow & A & \longrightarrow & M_f & \longrightarrow & C_f & \longrightarrow & 0 & & \\
 & & & & \downarrow{\scriptstyle\sim} & & \| & & & & \\
 & & 0 & \longrightarrow & B & \longrightarrow & C_f & \longrightarrow & \Sigma A & \longrightarrow & 0
\end{array}$$

If we're going to expect the axioms for exact triangles to hold and to be compatible with equivalences, then the map $C \to \Sigma A$ which finishes our exact triangle is going to need to be represented in the homotopy category by the composite $C \xleftarrow{\sim} M_f \to \Sigma A$. The whole exact triangle is represented by $A \to M_f \to C_f \to \Sigma A$ (which is valid even if we do not require $f$ to be an injection).

Let's examine the connecting map in homology. We start with a homology element represented by a cycle $\alpha$ in $H_n C$, and lift it to a homology element in $H_n(C_f)$. Lifting $\alpha$ to $\tilde{\alpha}$ in $H_n(B)$ whose boundary is in $A$, we find that $\alpha$ lifts to the cycle $[0] \otimes \tilde{\alpha} - [0,1] \otimes \partial\tilde{\alpha}$ in $C_f$. The image of this under the morphism $C_f \to \Sigma A$ sends this cycle to $-[0,1] \otimes \partial\tilde{\alpha}$.

Note that under the standard conventions for identifying the underlying abelian groups, this is *opposite* to the convention you would get by chasing the long exact sequence in homology. However, it appears to be forced by naturality and the particular identifications we are using.

## 12 Function complexes

Functions act on the left under our convention. This suggests that once we define a complex $F(C, D) = \mathrm{Hom}(C, D)$ of functions from one chain complex to another, there should be an "evaluation" pairing

$$F(C, D) \otimes C \to D.$$

The *only* sign convention that makes this into a chain map is

$$(\partial f) = \partial \circ f - (-1)^{|f|} f \circ \partial$$

for an element $f = (f_k : C_k \to D_{k+|f|})_{k \in \mathbb{Z}}$.

I'm using $F$ instead of Hom most of the time because it's shorter to write, and because it's already standard notation, and because I'm going to reserve Hom to denote the actual set of maps of chain complexes. In particular, I'm not going to underline it.

# 13 Dualizing

As a result of the previous section, if we have an abelian group $M$ we can convert a chain complex $C$ to a cochain complex $F(C, M)$ where we view $M$ as a complex concentrated in degree 0, and the boundary convention should be

$$\partial f = (-1)^{|f|+1}(f \circ \partial).$$

If you don't have this convention, and you are trying to do something like - for example - evaluate cap products, you end up with a bunch of diagrams that only commute up to sign. I have complained about this to more people than I should have.

# 14 Convention V: Adjunctions.

Ideally we would like for the tensor product and function complex to be adjoint. Even better, under our "functions act on the left" convention we'd like the evaluation pairing

$$ev\colon\ F(C, D) \otimes C \to D$$

to be the canonical instance of this adjunction, and more generally for our composition pairing to be written in the annoying but familiar order

$$F(D,E)\otimes F(C,D)\to F(C,E)$$

as another instance of said adjunction. In order for this to be true, I'm going to write the "standard" adjunction as

$$\mathrm{Hom}(A\otimes B,C)\cong \mathrm{Hom}(A,F(B,C)),$$

and its function-complex lift as

$$F(A\otimes B,C)\cong F(A,F(B,C)).$$

I could write it in the opposite order (which is often standard) and it's equivalent using the symmetry part of our symmetric monoidal structure, but I want to keep things simple.

Under this convention, $A$ — and more generally, $A\otimes X$ — is always a left $F(A,A)$-module.

## 15 Tensoring functions

We have a "tensor product" map

$$\otimes\colon\ F(A,B)\otimes F(C,D)\to F(A\otimes C,B\otimes D)$$

that has the meaning we hope it does when we're actually evaluating functions. Of course, this involves an interchange of parameters; it is adjoint to the map

$$F(A,B)\otimes F(C,D)\otimes (A\otimes C)\to B\otimes D$$

given by $(ev\otimes ev)\circ(1\otimes\tau\otimes 1)$.

In particular, suppose $f\in F(A,B)$ and $g\in F(C,D)$. Then $(f\otimes g)$ is defined by

$$(f\otimes g)(a\otimes b)=(-1)^{|g||a|}f(a)\otimes g(b)$$

as it should be.

## 16 Duality

If $R$ is the ground ring, we have a dualization functor $DX = F(X,R)$. This has a natural pairing $DX \otimes X \to R$ and hence the pairing $Y \otimes DX \otimes X \to Y$ is adjoint to a map $Y \otimes DX \to F(X,Y)$. If $X$ is a perfect complex, this is an isomorphism.

## 17 Dualization of coalgebras

Given a coassociative and possibly cocommutative coalgebra $C$, with comultiplication $C \to C \otimes C$, we can dualize it and get a differential graded algebra. By this, I will mean taking the Hom-complex $F(C,R)$ for a ring $R$ concentrated in degree zero. (A more complete discussion should include statements that hold when $R$ is a differential graded algebra itself.) Specifically, we have a composite map

$$\smile\colon\ F(C,R) \otimes F(C,R) \to F(C \otimes C, R \otimes R) \to F(C,R)$$

given by tensoring functions, then precomposing with the comultiplication and postcomposing with the multiplication. Specifically, suppose we have $f, g \in F(C,R)$ and $c \in C$ whose image under the comultiplication is $\sum c' \otimes c''$. Then

$$(f \smile g)(c) = \sum (-1)^{|g||c'|} f(c') g(c'').$$

If $R$ is the unit for the tensor product, then $C$ becomes a module over $F(C,R)$. If you've taught an algebraic topology course, you may have screwed this point up.

I have to make a choice about conventions at this point. Because $F(C,R)$ is a ring, I might want it to act on the left on $C$. However, in Poincaré duality (the standard application) we typically pick a single fundamental class, and use it to define an operator taking cochains and producing chains. I am going to have the functions act on the left, especially in light of my convention for adjunctions later.

So convention that we want this to be a *left* module, the map you need to use is the following composite.

$$\frown\colon\ F(C,R)\otimes C \to F(C,R)\otimes C\otimes C \to C\otimes F(C,R)\otimes C \to C\otimes R \cong C$$

The point being that the dualization procedure essentially reverses the comultiplication order of $C$. In formulas, if $f \in F(C,R)$ and $c \in C$ has comultiplication $\sum c'\otimes c''$, then we have

$$f \frown c = \sum (-1)^{|f||c'|} c' f(c'').$$

If you try to evaluate on the left-hand factor of $C$ instead, it will quite simply not be a module - the multiplication order gets reversed. I'm also a little leery of trying to define

$$c \frown f = f(c')c'' \quad \text{(NOT A VALID FORMULA)}$$

because, even though it looks like it makes things into a right module, the sign convention makes me worry about trying to do any chain-level manipulations.

# 18 Poincaré duality

Let's say a complex $C$ with a comultiplication is a duality complex of dimension $r$ if there is a "dualizing cycle" $D \in C_r$ such that the cap product

$$f \mapsto f \frown D$$

induces a weak equivalence between $C$ and its dual. Namely, the element $D$ can be represented by a chain map $\Sigma^r R \to C$, and we get a composite chain map

$$F(C,R)\otimes \Sigma^r R \to F(C,R)\otimes C \overset{\frown}{\to} C.$$

and the statement that we have a duality complex is that this is a chain equivalence. Notice that the the $\Sigma^r R$ occurs on the right, whereas $\Sigma^r F(C,R)$ is defined as $\Sigma^r R \otimes F(C,R)$.

Using this, we can transport the multiplication on cochains to an "intersection pairing" on homology. Namely, we take the adjoint to the above duality map:

$$F(C,R) \to F(\Sigma^r R, C).$$

This is a natural equivalence of chain complexes. Duality takes the form of an isomorphism

$$F(\Sigma^r R, C) \cong C \otimes \Sigma^{-r} R.$$

It is very tempting to go back at this point and, instead of asking that $F(C, R)$ act on the left on $C$, ask that it act on the right. However, we should note that if we do this, we will instead be trying to construct the adjoint to a map

$$\Sigma^r R \otimes F(C, R) \to C,$$

and to apply our adjunction conventions we would then first need to apply the twist isomorphism and get a sign out of it.

# 19 Realizing simplicial objects

Suppose we have a simplicial chain complex $C_\bullet$. We'd like to get an internal realization of it as a chain complex, and we'd like to understand *why* we use the sign conventions that we do and not just make stuff up as we go along.

Here is the convention: We're going to go back to topology and build it the same way as we realize a simplicial complex. We'll built by skeleta, attaching along the boundary of a standard simplex.

The 0-skeleton is $sk_0(C_\bullet) = Z(\Delta[0]) \otimes C_0$, the normalized chain complex of the standard 0-simplex tensored with the 0'th chain complex $C_0$. This is naturally isomorphic to $C_0$, because nothing interesting has happened yes.

The 1-skeleton is formed as a pushout:

$$\begin{array}{ccc} Z(\partial\Delta[1]) \otimes C_1 & \longrightarrow & Z(\Delta[1]) \otimes C_1 \\ \downarrow{\scriptstyle d_0, d_1} & & \downarrow \\ sk_0(C_\bullet) & \longrightarrow & sk_1(C_\bullet) \end{array}$$

Since the map $Z(\partial\Delta[1]) \to Z(\Delta[1])$ is an inclusion with cokernel $\Sigma\mathbb{Z}$, the above pushout tells us there is a natural distinguished triangle

$$sk_0(C_\bullet) \to sk_1(C_\bullet) \to \Sigma C_1.$$

If we actually want to get dirty with chain-level computations, we can do that. The degree $n$ elements in the 1-skeleton are of the form $[0] \otimes a \oplus [0,1] \otimes b$ for $a \in C_0$ of degree $n$ and $b \in C_1$ of degree $n-1$. If we simply write this as an ordered pair $(a, b)$, then the boundary formula is:

$$\partial(a, b) = (\partial a + d_0(b) - d_1(b), -\partial b)$$

And we keep going. For each $n$, we have a pushout:

$$\begin{array}{ccc} Z(\partial\Delta[n-1]) \otimes C_n & \longrightarrow & Z(\Delta[n]) \otimes C_n \\ \downarrow & & \downarrow \\ sk_{n-1}(C_\bullet) & \longrightarrow & sk_n(C_\bullet) \end{array}$$

(The left-hand vertical map is the annoying one to define; one uses that $Z(\partial\Delta[n])$ is a union of subcomplexes given by faces, and on each face you have a map given by the face maps in the simplicial object.) Taking the limit (which is equivalent to a union) gives us a chain complex $Tot(C_\bullet)$. We can identify the degree $n$ part with the direct sum over $k$ of the degree $(n-k)$-part of $C_k$, and under this convention the boundary of

$$(a_0, a_1, a_2, \ldots)$$

is

$$(\partial a_0 + d_0 a_1 - d_1 a_1, -\partial a_1 + d_0 a_2 - d_1 a_2 + d_2 a_2, \partial a_2 + d_0 a_3 - d_1 a_3 + d_2 a_3, \ldots)$$

In particular, the boundary map is the sum of the "vertical" boundary map $\partial$ applied levelwise — with signs! — and the "horizontal" boundary map $\sum(-1)^i d_i$.

# 20 Intermission on geometric realization

One way to describe the geometric realization from the previous section is in the same way as the realization of a semisimplicial object, namely as

$$\coprod_n Z(\Delta[n]) \otimes C_n / \sim$$

or more concisely as the coend

$$\int^{\Delta^{inj}} Z(\Delta[n]) \otimes C_n.$$

Here $\Delta^{inj}$ is the subcategory of the standard category of finite ordered sets, consisting of just the injections. This corresponds to the "thick" geometric realization of simplicial sets or simplicial spaces.

The construction outlined in the previous section isn't a "normalized" chain complex construction, meaning that it doesn't mod out by degeneracies. That would be described instead as the coend over the *whole* category of finite ordered sets:

$$\int^{\Delta} Z(\Delta[n]) \otimes C_n$$

This is the quotient of the previous notion by the subcomplex generated by the images of the degeneracy maps. It has the same homology.

In topology or stable homotopy theory, we have no choice but to use a definition of geometric realization like this because the ability to realize a simplicial object is dependent on having the entire simplicial structure. We don't have the ability to form alternating sums of maps in a coherent way that allows a "simpler" description of the total complex like the ones occurring in homological algebra. If we remember this extra bit to the geometric realization, the associated combinatorial structure gives us the signs we need.

# 21 The bar construction, take I

This brings us to a dear friend whose sign conventions have personally given me nightmares on more than one occasion. Namely, the bar construction — or specifically, in this case, the bar construction of a differential graded algebra with coefficients in a pair of differential graded modules. Suppose $A$ is a differential graded algebra with left module $N$ and right module $M$. Then the bar construction is the simplicial chain complex

$$B(M, A, N) = M \otimes N \leftleftarrows M \otimes A \otimes N \cdots$$

We will write the elementary tensors in the bar construction in simplicial degree $p$ as $m \otimes a_1 \otimes \cdots \otimes a_p \otimes n$ for now.

Using the convention of the previous section, we find that the convention for the boundaries in the totalization of the bar construction should be as follows:

$$\begin{aligned}
\partial(m \otimes n) &= (\partial m) \otimes n + (-1)^{|m|} m \otimes (\partial n) \\
\partial(m \otimes a \otimes n) &= -\partial m \otimes a \otimes n - (-1)^{|m|} m \otimes \partial a \otimes n - (-1)^{|m|+|a|} m \otimes a \otimes \partial n \\
&\quad +(ma) \otimes n - m \otimes (an) \\
&\vdots
\end{aligned}$$

Namely, you have a "vertical" differential applying to terms (that gets signs according to degree and according to moving the boundary map across terms) and a "horizontal" differential (that only gets a $(-1)^i$ sign) from multiplying adjacent terms.

Even verifying that $\partial^2 = 0$ on $m \otimes a \otimes n$ is annoying using an explicit formula like this, and you might get the impression that nobody's checked it. When you check it, like me you will probably the signs wrong the first time, too, but if you use the formalism you know that the signs are correct and you are not. If nothing else, I'd like to say that you should build your conventions around a foundation, and not the other way around.

# 22 Why this bar construction is annoying

The conventions of the previous section are absolutely terrible in some ways. For one thing, we often use the bar construction $B(A, A, N)$ as a resolution of $N$ as an $A$-module. But since we've tensored up with $Z(\Delta[n])$ on the left, the structure of a left $A$-module is given on elementary tensors by

$$a \cdot a_0 \otimes \cdots \otimes a_p \otimes n = (-1)^{p|a_0|}(aa_0) \otimes \cdots \otimes a_p \otimes n.$$

This is annoying. Moving the simplex over to the right-hand side fixes resolutions of left modules but not of right modules. Also, this construction does not play nicely with the standard sign conventions in homological algebra.

## 23 Simplices, take II

There is another way to describe the normalized chain complex associated to the standard simplex. Namely, we can say that it is the chain complex freely generated by elements of the form

$$\sigma^{\epsilon_0} \otimes \sigma^{\epsilon_1} \otimes \cdots \otimes \sigma^{\epsilon_n} \otimes \alpha$$

where $\epsilon_i \in 0, 1$, $\sum \epsilon_i > 0$, $\sigma$ is viewed as having degree 1, $\alpha$ has degree $-1$, $\partial\sigma = 1$, and $\partial\alpha = 0$. For example, $Z(\Delta[2])$ has generators

$$\sigma \otimes 1 \otimes \alpha, 1 \otimes \sigma \otimes \alpha, \sigma \otimes \sigma \otimes \alpha$$

which correspond to the elements $[1]$, $[0]$, and $[0, 1]$ in the standard normalized chain complex respectively.

You could write $C$ for a chain complex generated freely by generators $\sigma$ in degree 1, 1 in degree 0, such that $\partial\sigma = 1$. This is chain contractible, and the normalized chain complex of $\Delta[n]$ is the quotient of $C^{\otimes(n+1)} \otimes \Sigma^{-1}\mathbb{Z}$ by $\mathbb{Z}^{\otimes(n+1)} \otimes \Sigma^{-1}\mathbb{Z}$. This kind of description seems to be analogous to describing an $n$-simplex as the convex hull of those centers of the faces of $[0, 1]^{n+1}$ that don't touch $(0, \cdots, 0)$.

## 24 The bar construction, take II

Using our new description of the normalized chain complex of the standard $n$-simplex as a quotient of $C^{\otimes(n+1)}$, we can then redescribe the bar construction. Namely,

$$Z(\Delta[p]) \otimes M \otimes A^{\otimes p} \otimes N$$

is equivalent to a quotient of

$$C^{\otimes(p+1)} \otimes \Sigma^{-1}\mathbb{Z} \otimes M \otimes A^{\otimes p} \otimes N,$$

and this is isomorphic to a quotient of

$$M \otimes C \otimes A \otimes C \otimes \cdots \otimes A \otimes C \otimes N \otimes \Sigma^{-1}\mathbb{Z}.$$

This uses the symmetric monoidal structure to intersperse the tensor factors in, in a method compatible with the simplicial realization maps. Namely, the generators in the total complex of the form

$$[0, 1, \ldots, p] \otimes m \otimes a_1 \otimes \cdots \otimes a_p \otimes n$$

which come from simplicial degree $p$ can be rewritten as

$$\sigma^{\otimes(p+1)} \otimes \alpha \otimes m \otimes a_1 \otimes \cdots \otimes a_p \otimes n.$$

After permuting the appropriate tensor factors, this becomes

$$m \otimes \sigma \otimes a_1 \otimes \sigma \otimes \cdots \otimes \sigma \otimes a_p \otimes \sigma \otimes n \otimes \alpha.$$

It is convenient to drop most of the tensor signs at this point and simply write it in standard bar-construction notation,

$$m[a_1|\cdots|a_p]n$$

and remember that the bars (and brackets) are actually of degree 1, and there's a right-hand term globally shifting everything down by one degree. (When $p = 0$ the notation is less good, because there should really be only one symbol in between $m$ and $n$; it is occasionally better, but less readable, to write it as $m|a_1|\cdots|a_p|n$.)

This is still isomorphic to $M \otimes A^{\otimes p} \otimes N$, but the isomorphism induced by all this shifting has a fairly large sign. Specifically, the transformation from the conventions of "bar construction I" to "bar construction II" are:

$$m \otimes a_1 \otimes \cdots \otimes a_p \otimes n \mapsto (-1)^{|n|+|a_{p-1}|+|a_{p-3}|+\cdots}$$

One thing to note is that the extra "shift" operator $\alpha$ at the right-hand edge makes it more obvious that, when you have an $A$-algebra $A'$, any bar construction $B(-, A, A')$ has a right $A'$-module structure that requires an extra sign not obvious in the standard notation.

(Roughly, this is because the total degree of $m[a_1|\cdots|a_p]n$ in the bar complex is one off from the sum of the degrees plus the number of bars.)

## 25 Coproducts

Now suppose $A$ is an augmented DGA over $R$. There is a coproduct on the geometric realization of the bar construction $B(R, A, R)$, which in degree $p$ is $A^{\otimes p}$.

Namely, one combines the "Alexander-Whitney" diagonal $Z(\Delta[p]) \to \otimes Z(\Delta[p'])\otimes Z(\Delta[p''])$ with the subdivision in the bar complex. This can be pretty concisely expressed as a map of coends. One point to make is that if the signs are not interspersed as in the "second" bar construction, then one needs to use a transposition in $\Delta[p'] \otimes \Delta[p''] \otimes R^{\otimes p'+p''}$ to get the associated map of bar complexes.

At any rate, this comultiplication is the standard Alexander-Whitney formula

$$[a_1|\cdots|a_p] \mapsto \sum_{p'+p''=p} [a_1|\cdots|a_{p'}] \otimes [a_{p'+1}|\cdots|a_p].$$

## 26 Cosimplicial objects

Cosimplicial objects are defined in the same way as we defined in the same way as we defined simplicial objects. Namely, we go back to the topological realizations and apply them in the context of homological algebra.

So say $D^\bullet$ is a cosimplicial chain complex. In this case, we construct a Tot-tower of chain complexes. The zero'th part of the Tot-tower is $Tot^0(D^\bullet) = F(Z(\Delta[0]), D^0) \cong D^0$. We then inductively build a sequence of pullback diagrams as follows:

$$\begin{array}{ccc} Tot^n(D^\bullet) & \longrightarrow & Tot^{n-1}(D^\bullet) \\ \downarrow & & \downarrow \\ F(Z(\Delta[n]), D^n) & \longrightarrow & F(Z(\partial\Delta[n]), D^n) \end{array}$$

(Again, the existence of the right-hand map requires decomposing the normalized chains and defining the map one face at a time, checking that they agree on intersections.)

The inverse limit is the totalization of the cosimplicial object. In degree $n$, it is the product over $k$ of the degree $(n+k)$-part of $D^k$. The boundary map is the confluence of numerous sign conventions now, and on the element

$$(a^0, a^1, a^2, \cdots)$$

it is given by

$$(\partial a^0, \partial a^1 - d^0 a^0 + d^1 a^0, \partial a^2 + d^0 a^1 - d^2 a^1 + d^2 a^2, \ldots)$$

(Namely, on $a^k$ it is the ordinary boundary $\partial a^k$ plus $\sum_{i=0}^{k+1} (-1)^{k+i+1} d^i(a^k)$.)

There are similar descriptions of this or an equivalent realization in terms of an end

$$\int F(Z(\Delta[p]), D^p).$$

We observed that $Z(\Delta[p])$ is a quotient of a certain tensor power complex, and this makes the Hom-complex $F(Z(\Delta[p]), D^p)$ into a subcomplex of $D^p \otimes \Sigma Z \otimes (DC)^{\otimes(p+1)}$.

# 27 The cobar construction

As with the bar construction, the cobar construction is worth making explicit, and it's worth interspersing terms as well.

Suppose $A$ is a coassociative and counital coalgebra, with left comodule $N$ and right comodule $M$. We have a cobar construction $C(M, A, N)$ which, in (co?)degree $p$, is $M \otimes A^{\otimes p} \otimes N$.

By carrying out the previous program of "interspersing" we can write the cobar complex as generated by elementary elements. If $[0, 1, \ldots, p] = \sigma^{\otimes(p+1)} \otimes \alpha$, then its dual is $\alpha^* \otimes (\sigma^*)^{\otimes(p+1)}$. We can combine the duality isomorphism $F(Z(\Delta[p]), -) \cong (-) \otimes DZ(\Delta[p])$ with this identification, and then intersperse elements in the following form.

$$m \otimes \sigma^* \otimes a^1 \otimes \cdots \otimes a^p \otimes \sigma^* \otimes n \otimes \alpha^*$$

Here $\alpha^*$ has degree 1 and $\sigma^*$ has degree $-1$. I'm going to write this in the notation

$$m[a^1|\cdots|a^p]n.$$

In more direct terms, there is a correspondence between functions on the normalized chain complex of $\Delta[p]$ and these cobar elements, given by

$$([0,1,\ldots,p] \mapsto m \otimes a^1 \otimes \cdots \otimes a^p \otimes n) \leftrightarrow (-1)^{p+1+|a^1|+|a^3|+\cdots} m[a^1|\cdots|a^p]n.$$

I should point out that I left $\alpha^*$ at the end rather than moving it to the beginning. One motivation is that this makes it so that it doesn't interfere with my formula for the boundary operator. Here's another, better motivation. Under this sign convention, the formula for the boundary on an element $[a] = 1[a]1$, when $|a| = p$, is

$$-[\partial a] + \sum_{p'+p''=p} (-1)^{p'} [\Delta_{p,p''} a]$$

where $\Delta_{p',p''} a$ is the part of the comultiplication on $a$ that lands in $A_{p'} \otimes A_{p''}$. This is precisely the sign convention introduced in Adams' paper on the cobar construction.

Another motivation for this sign has to do with the multiplication, coming up.

# 28 Products in the cobar construction

Now let's suppose $A$ is a coalgebra which is coassociative over $R$. Then there's an Alexander-Whitney comultiplication in the realization of the cobar complex $C(R, A, R)$ for $A$, coming from the diagonal in $Z(\Delta[n])$ and dividing the tensor product $A^{\otimes p+q}$ into $A^{\otimes p} \otimes A^{\otimes q}$.

Namely, suppose $f$ and $g$ are basic elements in the realization of the cobar complex, which are zero on all simplices except one:

$$f\colon\ [0,1,\ldots,p] \mapsto a^1 \otimes \cdots \otimes a^p g\colon\ [0,1,\ldots,q] \mapsto b^1 \otimes \cdots \otimes b^q$$

Then $fg$ is the function given by precomposition with the Alexander-Whitney diagonal. It is only nonzero on the standard $p+q$-simplex, and is given there by

$$(fg)[0,1,\ldots,p+q] = (-1)^{p|g|} f([0,\ldots,p]) \otimes g([0,\ldots,q]) = (-1)^{p|g|} a^1 \otimes \cdots \otimes b^q.$$

Note the presence of a sign because we had to move some simplex coordinates across $g$ in order to apply $f$.

If we change to using the "interspersed" sign conventions from the previous section, this simplifies to the formula

$$[a^1|\cdots|a^p]\cdot[b^1|\cdots|b^q] = [a^1|\cdots|a^p|b^1|\cdots|b^q].$$

And thus all is right with the universe.

One convention (which I learned from Baues' paper) is that you should perhaps decorate the elements in the cobar complex with a symbol, such as (say) $\Omega$ just to be difficult. Then an element in the cobar complex is a formal product $\Omega a_1 \cdots \Omega a_p$, with the differential being a derivation, satisfying

$$\partial(\Omega a) = (\Omega \otimes \Omega)\Delta a - \Omega(\partial a).$$

So the signs come about from moving the boundary across $\Omega$ and from moving $\Omega$ across terms in the comultiplication.

# 29 Underlying lesson

Go back and read Adams first if you're confused about the signs.