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E_∞-orientations for K-theory

Maps between K-theory spectra

Thm (Adams ~~[illegible]~~ Harris-Switzer):

$$K_0 K \cong \{ h(x) \in \mathbb{Q}[x,x^{-1}] \mid h(n) \in \mathbb{Z}[\tfrac{1}{n}] \text{ for all } n \neq 0, n \in \mathbb{Z} \}$$

Corollary: $(K_0K)^\wedge_p = \pi_0 (K \wedge K)^\wedge_p$

$$\cong \mathrm{map}_{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$$

So $[K^\wedge_p, K^\wedge_p] \cong \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ where $\mathbb{Z}_p^\times$ corresponds to Adams-Ops

$\cong \hom(\mathrm{map}_{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p)$ "functionals"

$\pi_* K \cong \mathbb{Z}[u,u^{-1}]$ with $|u| = 2$, so any $f \in [K^\wedge_p, K^\wedge_p]$ gives a sequence of elements (c_n) in $\mathbb{Z}_p$ such that

$$f_*(u^n) = c_n \cdot u^n.$$

Question: Can you construct f from the sequence (c_n)?

Proposition. $f \in [K^\wedge_p, K^\wedge_p] \mapsto (c_n)_{n \geq n_0} \in \prod \mathbb{Z}_p$ is injective; the image consists of those (c_n) such that:

for any polynomial $h(x) \in \mathbb{Q}_p[x] \cdot x^{n_0}$ such that $h(\mathbb{Z}_p^\times) \subseteq \mathbb{Z}_p$,

then $\sum a_i c_i \in \mathbb{Z}_p$, where $h(x) = \sum a_i x^i$.

Congruences determine sequences (c_n) which come from maps.

Example: $h(x) = \frac{1}{p^k}(x^m - x^n)$ for certain k, m, n.

For odd p this gives:

$$c_m \equiv c_n \mod p^k \text{ if } m \equiv n \mod (p-1)p^{k-1}$$

For $p=2$: $\quad C_m \equiv C_n \mod 2$ and

$$C_m = C_n \mod 2^k \text{ if } m \equiv n \mod 2^{k-2} \text{ for } k \geq 3.$$

These are "Kummer congruences"; the more general ones (involving $h(x) \in \mathbb{Q}_p[x]$) are "extended Kummer congruences".

One should be able to recover the beginning of an admissible sequence (c_n) from the tail of the sequence. The Kummer congruences say how:

$$c_n = \lim_{k \to +\infty} c_{n+(p-1)p^{k-1}} \in \mathbb{Z}_p .$$

Remark :

$$\begin{array}{ccc} \pi_* K & \xrightarrow{\cong} & \mathbb{Z}[u,u^{-1}] \\ \uparrow & & \uparrow \\ \pi_* KO & \longrightarrow & \mathbb{Z}[u^2,u^{-2}] \end{array}$$

isomorphism away from 2.

$$K_*^\wedge KO = \mathrm{map}(\mathbb{Z}_p^\times/_{(\pm 1)}, \mathbb{Z}_p) = \text{even functions on } \mathbb{Z}_p^\times$$

$$[KO_p^\wedge, KO_p^\wedge] = \hom(\ \downarrow\ , \mathbb{Z}_p) = \text{functionals on functions.}$$

$[K_p^\wedge, KO_p^\wedge] =$ functionals on $\mathrm{map}(\mathbb{Z}_{p,1}^\times, \mathbb{Z}_p)$ which vanish on odd functions.

Example :

$\mu(h(x)) = h(1) + h(-1)$ is a functional which vanishes on odd functions

$$\mu(x^n) = \begin{cases} 0 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even} \end{cases}$$

This functional corresponds to complexification $K \to KO$.

Adams operations $\Longleftrightarrow$ "Dirac measure" functional

$$\mu_{\psi^\lambda}(h(x)) = h(\lambda).$$

Atiyah-Bott-Shapiro orientation

M^n Spin manifold $\rightsquigarrow \varphi(M) \in \pi_n KO$

corresponds to Hirzebruch characteristic series

$$K(x) = \frac{x/2}{\sinh(x/2)} = \frac{x}{e^{x/2}-e^{-x/2}} = \exp\left(\sum_{n\geq 2} g_n \cdot \frac{x^n}{n!}\right)$$

where $g_n = \frac{-B_n}{n}$ for even n, and $g_n = 0$ for odd n.

We want an E_∞ ring spectra map

$MSpin \longrightarrow KO$, where $MSpin = \mathrm{Thom}(BSpin) = \mathrm{Th}(BO\langle 4\rangle)$

$$\Sigma^{-1} bo\langle 4\rangle \xrightarrow{j\text{-homom.}} gl_1(S) \xrightarrow{gl_1(i)} gl_1(KO)$$

Ando's talk: if the composite is zero, then there exists an orientation.
Orientations are classified by null-homotopies.

Proposition There is a spectrum map $gl_1(KO_p^\wedge) \longrightarrow KO_p^\wedge$
which is an iso on homotopy groups π_* for $* \geq 2$
on π_{2n} for even n, it is multiplication by $(1-p^{n-1})$
Also iso on π_{8n+1}, π_{8n+2}, except for π_1.

So $gl_1(KO_p^\wedge)\langle 2\rangle \simeq KO_p^\wedge\langle 2\rangle$

(Sullivan, Adams-Priddy: $BSO_\otimes \underset{p}{\simeq} BSO_\oplus$)

The proof will come later.

Moreover, the above map induces an equivalence

$$L_{K(1)}\, gl_1(KO) \xrightarrow{\sim} KO_p^\wedge$$

and $L_{K(1)}\, gl_1(S) \simeq L_{K(1)} S$

<u>Theorem:</u> There exists an E_∞ ring map $MSpin \longrightarrow KO^\wedge_p$.

<u>Proof</u>:

$$\Sigma^{-1} bo\langle 4\rangle \xrightarrow{j} gl_1(S) \longrightarrow gl_1(KO^\wedge_p) \underset{\simeq \text{ after } \langle 2\rangle}{\longrightarrow} KO^\wedge_p$$

Fact: $\pi_n \, \underline{map}(KO^\wedge_p, KO^\wedge_p) \cong \pi_0\, map(KO^\wedge_p, KO^\wedge_p) \otimes \pi_n KO$

$$= 0 \text{ if } n = -1, -2, -3.$$

So there are no essential maps, thus no obstructions.
But the orientation is (p-adically) not unique! $\square$

<u>Hasse - Sullivan square</u>

For spectrum X, there is a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod X^\wedge_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod X^\wedge_p\right)_{\mathbb{Q}} \end{tikzcd}$$

We'll use this for $X = gl_1 R$.

<u>Stable Adams Conjecture (Friedlander)</u> p odd,

The composite $bo\langle 4\rangle \xrightarrow{1-\psi^c} bo\langle 4\rangle^\wedge_p \longrightarrow \Sigma gl_1(S^\wedge_p)$ for $c \in \mathbb{Z}_p^\times$

is null as a map of spectra. Equivalently, the map extends over $bo\langle 4\rangle^\wedge_p \longrightarrow J(c)$

At $p=2$ this does not hold, but the composite

$$\begin{tikzcd} bu\langle 4\rangle \arrow[r, "r\circ(1-\psi^c)"] \arrow[d] & bo\langle 4\rangle^\wedge_p \arrow[r] & \Sigma gl_1(S^\wedge_p) \\ J/O \arrow[urr, dashed] & & \end{tikzcd}$$

is null

$p \geq 3$

$$\begin{tikzcd}
\Sigma^{-1} bo\langle k\rangle \arrow[r, "1-\psi^\lambda"] & \Sigma^{-1} bo\langle k\rangle \arrow[r, "j"] \arrow[d] & gl_1(S^1_p) \arrow[r] \arrow[d] & gl_1(R) \\
& J(\lambda) \arrow[ur, "F"] \arrow[d] & C \arrow[ur, dashed] & \\
& bo\langle k\rangle \arrow[ur, "T"] \arrow[urr, bend right, "\varphi"'] & &
\end{tikzcd}$$

Suppose we are given an E_∞-orientation $MO\langle k\rangle \longrightarrow R$, or equivalently an extension $\varphi : bo\langle k\rangle \longrightarrow gl_1 R$ as above.

Invariant : $\varphi(u^n) = c_n \in \pi_{2n}\, gl_1(R) \cong \pi_{2n} R$ (modulo torsion

(This is Toda bracket. There is no indeterminacy since we fix nullhomotopies)

Exercise : $c_n = \frac{g_n}{2}(1-\lambda^n)$ in $\pi_{2n} R$ mod torsion

(the denominator 2 comes from realification $K \to KO$)

Now take $R = KO^1_p$ and use the "logarithm" $\ell : gl_1(KO^1_p) \longrightarrow KO^1_p$

$$u^n \longmapsto (1-p^{n-1})u^n$$

Proposition : On π_*, $\ell\varphi : bo\langle k\rangle \longrightarrow KO^1_p$ is given by the sequence

$$z_n = -\frac{B_n}{2n}(1-p^{n-1})(1-\lambda^n)$$

Necessary condition for E_∞-orientation

(1) (z_n) must satisfy the extended Kummer sequences

(2) After $K(1)$-localization

$$\begin{tikzcd}
L_{K(1)} J(\lambda) \arrow[r] \arrow[d] & L_{K(1)} S^0 \arrow[r] & KO^1_p \\
KO^1_p \arrow[urr, "\varphi"] & &
\end{tikzcd}$$

gives on π_0

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r, "\alpha(t)"] \arrow[d, "\cong"'] & \mathbb{Z}_p \\ \mathbb{Z}_p \arrow[ur, "z_0"'] & \end{tikzcd}$$

so we get condition

$$z_0 = \alpha(t) = \lim_{k \to \infty} z_{(p-1)p^{k-1}}$$

For p odd can take λ a generator of $\mathbb{Z}_p^\times$ and get that Condition (1) and (2) are sufficient for specifying an E_∞-orientation (they are also sufficient for $p=2$, but that's more complicated).

<u>Proposition</u>: (1) The numbers $-\frac{B_n}{2n}(1-p^{n-1})(1-\lambda^n)$ for $\lambda \in \mathbb{Z}_p^\times$ satisfy extended Kummer congruences.

(2) $$z_0 = \lim = \frac{1}{2p} \log(\lambda^{p-1})$$

~~Ando-Hopkins-Strickland~~

McClure : ABS orientation is H_∞
M. Joachim: ABS — " — is E_∞ $\Big\}$ can determine the number z_0

p-adic zeta-functions $\zeta^*(s)$ defined on $\mathbb{Z}_p$

$$\zeta^*(1-n) = -\frac{B_n}{n}(1-p^{n-1})$$

$$\zeta^*(s) = \left(1-\frac{1}{p}\right)\frac{1}{s-1} + \cdots$$

Mazur: constructed ζ^* via a p-adic measure

$$\mu(x^n) = -\frac{B_n}{2n}(1-p^{n-1})(1-\lambda^n) \text{ for some } \lambda \in \mathbb{Z}_p^\times$$

15.10.03

Units. R = E_∞ ring spectrum

$$\begin{tikzcd} GL_1(R) \subseteq \Omega^\infty R \arrow[d, squiggly] \\ gl_1(R) \end{tikzcd}$$

$gl_1(R)$ connective spectrum of units

$\Sigma^\infty_+ \Omega^\infty$: Connective Spectra $\rightleftarrows$ E_∞ rings : gl_1

Write $H^q(X, gl_1(R)) = [\Sigma^\infty_+ X, gl_1(R)]^q$

e.g. $H^0(X, gl_1(R)) \cong H^0(X, R)^\times$

$\pi_n gl_1 R \cong \tilde{H}^0(S^n, gl_1(R)) \cong (1 + \tilde{H}^0(S^n, R))^\times = (1 + \pi_n R)^\times$.

Bousfield–Kuhn construction

$$\begin{tikzcd} \text{Spectra} \arrow[rr, "L_{K(n)}"] \arrow[dr, "\Omega^\infty"'] & & \text{Spectra} \\ & \text{Spaces}_* \arrow[ur, dashed, "\Phi_n"'] & \end{tikzcd} \qquad \Phi_n \circ \Omega^\infty \simeq L_{K(n)}$$

$$\Phi_n \left[Gl_1(R) \xrightarrow{\text{“}x \mapsto x-1\text{”}} \Omega^\infty R \right] = L_{K(n)} gl_1 R \longrightarrow L_{K(n)} R$$

The composite

$$\begin{tikzcd} gl_1(R) \arrow[r] \arrow[rr, bend right, "\ell"'] & L_{K(n)} gl_1(R) \arrow[r] & L_{K(n)} R \end{tikzcd}$$

is the map from the previous talk ℓ.

Question: What does ℓ do on π_*? What does it do as a cohomology operation?

X a space,

$$R^0(X)^\times = H^0(X, gl_1(R)) \xrightarrow{\ell} H^0(X, L_{K(n)} R) = (L_{K(n)} R)^0(X)$$

logarithmic, $\ell(xy) = \ell(x) + \ell(y)$

<u>Thm</u>: $n=1$, p prime. Then

$\ell_1 : R^0(X)^\times \longrightarrow (L_{K(1)}R)^0(X)$ is given by the formula

$$\ell_1(x) = \left(1-\tfrac{1}{p}\psi\right)(\log(x))$$
$$= \frac{1}{p}\log\left(\frac{x^p}{\psi(x)}\right)$$
$$= \sum_{k\geq 1} \frac{p^{k-1}}{k}\left(\frac{\theta(x)}{x^p}\right)^k$$

Here ψ and θ are the operations in $K(1)$-local E_∞ ring spectra discussed by Hopkins, satisfying $\psi(x) + p\,\theta(x) = x^p$.

<u>Example</u>: $R = K, KO$, then $\psi = \psi^p$ Adams operation

$K^0(S^{2n}) = \mathbb{Z}[\varepsilon]/\varepsilon^2$, so

$$\ell_1(1+\varepsilon) = \left(1-\tfrac{1}{p}\psi\right)(\log(1+\varepsilon))$$
$$= \left(1-\tfrac{1}{p}\psi\right)(\varepsilon) = \varepsilon - \frac{p^n}{p}\cdot\varepsilon = (1-p^{n-1})\varepsilon$$

<u>Thm</u>. $n=2$, $R = E_2$ Lubin-Tate spectrum

$$\ell_2(x) = (1-T(p)+R)(\log(x))$$

$T(p)$ "=" classical Hecke operator if E_2 is an elliptic spectrum

If $f \in \pi_{2n} E_2$, then $R(f) = p^{n-1}\cdot f$ mod torsion.

So on π_{2n}

$$\ell_2(\text{"}1+f\text{"}) = (1-T(p)+p^{n-1})\cdot f$$

<u>Power operations</u>: A abelian group, given $\alpha : X \to \Omega^\infty R$

$$\begin{tikzcd}
EA \times_A X^{\times A} \arrow[r, "P_A(\alpha)"] & \Omega^\infty R \\
X^A \arrow[u] \arrow[ur, "\alpha^{|A|}"'] &
\end{tikzcd}$$

uses <u>multiplicative</u> E_∞-structure

Note that $P_A(\alpha)$ factors over $E\Sigma_d \times_{\Sigma_d} X^d$ for $d = |A|$.

Now suppose α is invertible, so $\alpha : X \to GL_1 R \subseteq \Omega^\infty R$.
Then get extension over group completion

$$\begin{tikzcd}
\left(\coprod_d B\Sigma_d\right) \times X \arrow[r, "\Delta"] \arrow[d] & \coprod_d E\Sigma_d \times_{\Sigma_d} X^d \arrow[r] \arrow[d, "\text{group completion}"'] & GL_1(R) \\
\Omega^\infty S \times X \arrow[r, "\delta"'] \arrow[urr, bend right=40, "P(\alpha)"'] & \Omega^\infty \Sigma^\infty_+ X \arrow[ur] & \tilde{P}(\alpha) \arrow[u]
\end{tikzcd}$$

adjoint to α w.r.t. $GL_1 R \simeq \Omega^\infty(gl_1 R)$.

So we have a function $R^0(X) \longrightarrow R^0(\Omega^\infty S \times X) \cong R^0(\Omega^\infty S) \otimes_{R^0} R^0(X)$

For $u \in R_0^\wedge(\Omega^\infty S)$ can consider slant product

$$\begin{tikzcd}
R^0(X)^\times \arrow[r] \arrow[dr, "\psi_u(\alpha)"'] & R^0(\Omega^\infty S \times X)^\times \arrow[d] & \subseteq R^0(\Omega^\infty S \times X) \arrow[dl, "u \backslash -"] \\
 & R^0(X) &
\end{tikzcd}$$

<u>Thm</u>: There exist $u \in R_0^\wedge(\Omega^\infty S)$ such that $\ell(x) = \psi_u(x)$.
Furthermore, u is the unique <u>special element</u>, i.e. satisfying

(a) $\varepsilon(u) = 1$ and

(b) $x \circ u = \varepsilon(x) \cdot u$ for all $x \in R_0^\wedge(\Omega^\infty S)$

In the above we have used the following notation:

$$R^\wedge_q X = \pi_q \, L_{K(n)}(R \wedge X) \qquad \text{"completed homology"}$$

(we can assume that R is $K(n)$-local, by naturality of logarithm).

Construction of u: There is a natural transformation of functors Spectra $\to$ spectra

$$\lambda : X \longrightarrow L_{K(n)} \Sigma^\infty_+ \Omega^\infty X$$

defined as follows:

$$\begin{tikzcd}
\Omega^\infty X \arrow[r, "\eta_{\Omega^\infty X}"', "\text{not } \Omega^\infty\text{-map}"] & (\Omega^\infty \Sigma^\infty)(\Omega^\infty X) \arrow[d, "\text{inclusion determined by the splitting } \Sigma^\infty_+ K \simeq S^0 \vee \Sigma^\infty K \text{ for pointed } K."] \\
& (\Omega^\infty \Sigma^\infty_+ \Omega^\infty X)
\end{tikzcd}$$

Now apply Bousfield-Kuhn Φ_n:

$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend right, "\lambda"'] & L_{K(n)} X \arrow[r] & L_{K(n)} \Sigma^\infty_+ \Omega^\infty X
\end{tikzcd}$$

<u>Fact:</u> the logarithm ℓ comes from λ:

For space K and $\alpha : K \longrightarrow \Omega^\infty gl_1(R)$

adjoin $\tilde{\alpha} : \Sigma^\infty_+ K \longrightarrow gl_1(R)$

adjoin $\tilde{\tilde{\alpha}} : \Sigma^\infty_+ \Omega^\infty \Sigma^\infty_+ K \longrightarrow R$

$K(n)$-localize to get

$$\begin{tikzcd}
\Sigma^\infty_+ K \arrow[r, "\lambda_{\Sigma^\infty_+ K}"] \arrow[rr, bend right, "\ell(\alpha)"'] & L_{K(n)} \Sigma^\infty_+ \Omega^\infty \Sigma^\infty_+ K \arrow[r] & L_{K(n)} R = R
\end{tikzcd}$$

So the special element u is the composite

$$\lambda_S : S \longrightarrow L_{K(n)} \Sigma^\infty_+ \Omega^\infty S$$

$$\in \pi_0 \left(L_{K(n)} \Sigma^\infty_+ \Omega^\infty S \xrightarrow{\text{Hurewicz}\cdots} R_0^\wedge(\Omega^\infty S) \right)$$

On the "special" conditions:

(a) $\tau : R_0^\wedge(\Omega^\infty S) \longrightarrow R_0$ induced by

$\Sigma^\infty_+ \Omega^\infty S \longrightarrow S$ counit of adjunction

Fact: $\underbrace{X \xrightarrow{\lambda} L_{K(n)} \Sigma^\infty_+ \Omega^\infty X \xrightarrow{L_{K(n)}(\text{counit})} L_{K(n)} X}_{\text{augmentation of } L_{K(n)}}$

(b) means: $X \xrightarrow{\alpha} Gl_1(R)$

$\rightsquigarrow$ $P(\alpha) : \Omega^\infty \Sigma^\infty_+ X \longrightarrow GL_1(R)$

and have $\beta : X \longrightarrow \Omega^\infty R$

$\rightsquigarrow$ $S(\alpha) : \Omega^\infty \Sigma^\infty_+ X \longrightarrow \Omega^\infty R$

and (b) means $\ell(P(\alpha)) = S(\ell(\alpha))$

(b) says: "$\circ$" in $\tau(x) \cdot u$ is multiplication in $R_0^\wedge(\Omega^\infty S)$
coming from ~~product~~ additive H-space structure on $\Omega^\infty S \times \Omega^\infty S \xrightarrow{+} \Omega^\infty$
and $\circ$ from multiplicative H-space structure.

Uniqueness of special elements: suppose u, u' are both special, then

$$\begin{array}{ccccccc} \tau(u) \circ u' & = & u \circ u' & = & u' \circ u & = & \tau(u') \circ u \\ \| & & & & & & \| \\ 1 \cdot u' = u' & & & & & & u = 1 \cdot u \end{array}$$

K(1)-local

$$\pi_0 L_{K(1)}(\bigvee B\Sigma_k^+) = (\pi_0 L_{K(1)} S^0)[x, \theta(x), \theta^2(x), \dots]$$

$$\pi_0 L_{K(1)}(\Omega^\infty S) = \text{ same with } x \text{ inverted and completed at } p.$$

In higher dimensions we use Hopkins-Kuhn-Ravenel character theory.

16.10.03

The string orientation of tmf

$E_\infty(MString, R) \simeq$ space of null homotopies of

$$[\,string \xrightarrow{j} gl_1(S^0) \xrightarrow{gl_1(\iota)} gl_1(R)\,]$$

Choosing a basepoint, the homotopy type of this space is

$map(bstring, gl_1(R))$

We'll do:

(1) p-complete case (2) put together with arithmetic square

p-complete:

$$\begin{tikzcd}
 & \mathrm{cofib}(1-\psi^\ell) \arrow[d, equal] & & F \arrow[d] \arrow[r, equal] & F \arrow[d] \\
string \arrow[r] & j_\ell \arrow[r] \arrow[d, "c"'] & gl_1(S) \arrow[r] & gl_1(tmf) \arrow[r] \arrow[d] & L_{K(1)} gl_1(tmf) \arrow[d] \\
 & bstring = bo\langle 8\rangle \arrow[urr, dashed, "?"] & & L_{K(2)} gl_1 tmf \arrow[r] & L_{K(2)} L_{K(1)} gl_1 tmf
\end{tikzcd}$$

pullback after suitable connected cover

If $K(2)_* X = 0$, then $map(X, L_{K(2)} Y) = *$.

So the extension problem lifts equivalently to the fibre F.

Reduction to extension problem:

$$\begin{tikzcd} j_\lambda \arrow[r] \arrow[d] & F = \text{fibre (map between wedges of KO's)} \\ bo\langle 8\rangle \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

Given an E_∞ orientation whose Hirzebruch series has the form

$$K(x) = \exp\left(\sum g_n \frac{x^n}{n!}\right)$$

then the corresponding extension $bo\langle 8\rangle \longrightarrow gl_1(tmf)$ sends

$$\pi_{2n}\, bo\langle 8\rangle \ni u^n \longmapsto \frac{g_n}{2}(1-\lambda^n)$$

Bousfield-Kuhn: can rewrite the square as

$$\begin{tikzcd} & gl_1(tmf) \arrow[r, "l_1"] \arrow[d, "l_2"'] & L_{K(1)}\, tmf \arrow[d, "L_{K(1)}(U_2)"] & \text{"mystery map"} \\ bstring \arrow[urr] & L_{K(2)}\, tmf \arrow[r, "\substack{K(1)\text{-local.} \\ \text{augmentation}}"'] & L_{K(1)}L_{K(2)}\, tmf & \end{tikzcd}$$

From Hopkins' talks: $L_{K(1)}\, tmf \simeq KO \otimes A$ for A = ring of p-adic modular forms of wt 0.

The first talk explained how to build maps between p-completed KO's.

Similarly, $L_{K(1)} L_{K(2)}\, tmf \simeq KO \otimes B$.

On $\pi_{2n}\, gl_1(tmf) = MF_n \ni f$ we have

$$l_1(f) = \left(1 - \tfrac{1}{p}\psi\right)(f) = (1 - p^{n-1}\cdot V)(f)$$

where $(Vf)(q) = f(q^p)$

Moreover, $l_2(f) = (1 - T(p) + p^{n-1})f$

with $T(p)$ classical Hecke operator

There exists a factorization of $L_{K(1)} \ell_2$:

$$\begin{tikzcd}
L_{K(1)} tmf \arrow[d, "L_{K(1)}(\ell_2)"'] \arrow[rr, dashed, "1-U"] & & L_{K(1)} tmf \arrow[dll, "L_{K(1)}(\text{adj. unit})"] \\
L_{K(1)} L_{K(2)} tmf & &
\end{tikzcd}$$

$U =$ Atkin operator

On Tate curve $T(q) = \mathbb{C}^\times / q^{\mathbb{Z}}$, the p-points

$$\left(\mathbb{C}^\times / q^{\mathbb{Z}}\right)[p] = \mathbb{Z}/p \times \mathbb{Z}/p$$

generators: $q^{1/p}$, $\zeta \in \mu_p$

Look at

$$\begin{array}{ccc}
\mathbb{C}^\times / q^{\mathbb{Z}} & \longrightarrow & \mathbb{C}^\times / \zeta^i q^{1/p \, \mathbb{Z}} \\
\| & & \| \\
T(q) & \longrightarrow & T(\zeta^i q^{1/p}) \\
q & \longmapsto & \zeta^i q^{1/p} \\
u & \longmapsto & u
\end{array} \qquad i \in \{0, \dots, p-1\}$$

Hecke operator : $\quad (T(p) f)(q) = \frac{1}{p}\left[p^n f(q) + \sum_{i=0}^{p-1} f(\zeta^i q^{1/p}) \right]$

$f \in MF_n$

$$= p^{n-1} Vf + Uf$$

This defines the Atkin operator U on p-adic modular forms and one can construct a map $U : L_{K(1)} tmf \longrightarrow L_{K(1)} tmf$ which realizes the Atkin operator on homotopy groups.

To check that the diagram which factors $L_{K(2)} l_2$ actually commutes, it suffices to check that on homotopy groups.

Reduction: enough to show

$$\left(L_{K(2)} l_2\right) \circ l_1 = \left(L_{K(1)}\, \mathrm{unit}\right) \circ (1-U) \circ l_1$$

Why: $L_{K(2)} X \simeq \operatorname{holim}_r \; v_1^{-1}\left(X \wedge M(p^r)\right)$

and we have good control over $\pi_* gl_1(tmf)$ and $\pi_* L_{K(2)} tmf$.

Formulas: if $f(q) = \sum a_n \cdot q^n$, then

$$(Uf)(q) = \sum a_{np} \cdot q^n \qquad U \circ V = id$$

$$\Rightarrow \quad (1-U) \circ (1-p^{n-1} V) = 1 - U - p^{n-1} V + p^{n-1}\, U \circ V$$
$$= 1 - T(p) + p^{n-1}$$

Thus $1-U$ "is" the mystery map.

Conditions:

$$z_n = (1-p^{n-1}V)\left(\frac{g_n}{2}(1-\lambda^n)\right)$$

To build an E_∞-orientation with Hirzebruch series

$$K(x) = \exp\left(\sum g_n \frac{x^n}{n!}\right) \quad \text{and} \quad g_n = 2 \cdot G_n$$

↑ Eisenstein series

we need:

(1) at each prime p,

$z_n = (1-p^{n-1}V)\,\frac{g_n}{2}(1-\lambda^n)$ should satisfy extended Kummer congruences

(2) $z_0 = \alpha(\lambda) = \lim -\frac{B_n}{4n}$

(alternative statement: for $g_n \in MF_n \otimes \mathbb{Q} \subseteq \mathbb{Q}[[q]]$, we have $g_n \equiv -\frac{B_n}{2n} \bmod \mathbb{Z}[[q]]$)

(3) $(1-U)\, z_n = 0$.