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## E<sub>∞</sub>-orientations for K-theory

Maps between K-theory spectra

Thm (Adams - ~~Adams~~ Harris-Switzer):

$$K_0 K \cong \{ h(x) \in \mathbb{Q}[x, x^{-2}] \mid h(n) \in \mathbb{Z}[\frac{1}{n}] \text{ for all } n \neq 0, n \in \mathbb{Z} \}$$

Corollary:  $(K_0 K)_p^1 = \pi_0 (K_1 K)_p^1$   
 $\cong \text{mapcts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p)$

So  $[K_{p,1}^1, K_p^1] \cong \mathbb{Z}_p[\mathbb{Z}_p^{\times}]$  where  $\mathbb{Z}_p^{\times}$  corresponds to Adams-Ops  
 $\cong \text{hom}(\text{mapcts}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p), \mathbb{Z}_p)$  "functionals"

$\pi_* K \cong \mathbb{Z}[u, u^{-2}]$  with  $|u|=2$ , so any  $f \in [K_{p,1}^1, K_p^1]$   
gives a sequence of elements  $(c_n)$  in  $\mathbb{Z}_p$  such that

$$f_*(u^n) = c_n \cdot u^n.$$

Question: can you construct  $f$  from the sequence  $(c_n)$ ?

Proposition -  $f \in [K_{p,1}^1, K_p^1] \mapsto (c_n)_{n \geq 0} \in \prod \mathbb{Z}_p$   
is injective, the image consists of those  $(c_n)$  such that:

for any polynomial  $h(x) \in \mathbb{Q}_p[x] \cdot x^{-n_0}$  such that  $h(\mathbb{Z}_p^{\times}) \subseteq \mathbb{Z}_p$ ,  
then  $\sum a_i c_i \in \mathbb{Z}_p$ , where  $h(x) = \sum a_i x^i$ .

Congruences determine sequences  $(c_n)$  which come from maps.

Example:  $h(x) = \frac{1}{p^k} (x^m - x^n)$  for certain  $k, m, n$ .

For odd  $p$  this gives:

$$c_m \equiv c_n \pmod{p^k} \iff m \equiv n \pmod{(p-2)p^{k-2}}$$

For  $p=2$  :  $C_m \equiv C_n \pmod{2}$  and

$$C_m \equiv C_n \pmod{2^k} \text{ if } m \equiv n \pmod{2^{k-2}} \text{ for } k \geq 3.$$

These are "Kummer congruences"; the more general ones (involving  $h(x) \in \mathbb{Q}_p(x)$ ) are "extended Kummer congruences".

One should be able to recover the beginning of an admissible sequence  $(c_n)$  from the tail of the sequence. The Kummer congruences say how:

$$c_n = \lim_{k \rightarrow \infty} c_{n + (p-1)p^{k-1}} \in \mathbb{Z}_p.$$

Remark :

$$\pi_* K \xrightarrow{\cong} \mathbb{Z}[u, u^{-1}]$$

$$\uparrow \qquad \qquad \uparrow$$

$$\pi_* KO \rightarrow \mathbb{Z}[u^2, u^{-2}] \quad \text{isomorphism away from } 2.$$

$$K_*^1 KO = \text{map}(\mathbb{Z}_p^\times / (p-2), \mathbb{Z}_p) = \text{even functions on } \mathbb{Z}_p^\times$$

$$[KO_p^1, KO_p^1] = \text{hom}(\downarrow, \mathbb{Z}_p) = \text{functionals on functions.}$$

$$[K_p^1, KO_p^1] = \text{functionals on } \text{map}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \text{ which vanish on odd functions.}$$

Example :

$\mu(h(x)) = h(2) + h(-2)$  is a functional which vanishes on odd functions

$$\mu(x^n) = \begin{cases} 0 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even} \end{cases}$$

This functional corresponds to complexification  $K \rightarrow KO$ .

Adams operations  $\Leftrightarrow$  "Dirac measure" functional

$$\mu_{\psi} (h(x)) = h(\lambda)$$

Atiyah - Bott - Shapiro orientation

$$M^n \text{ Spin manifold} \rightsquigarrow \varphi(M) \in \pi_n KO$$

corresponds to Hirzebruch characteristic series

$$K(x) = \frac{x/2}{\sinh(x/2)} = \frac{x}{e^{x/2} - e^{-x/2}} = \exp \left( \sum_{n \geq 2} g_n \cdot \frac{x^n}{n!} \right)$$

where  $g_n = \frac{-B_n}{n}$  for even  $n$ , and  $g_n = 0$  for odd  $n$ .

We want an Eoo ring spectra map

$$M\text{Spin} \longrightarrow KO, \text{ where } M\text{Spin} = \text{Thom}(B\text{Spin}) \cong \text{Th}(B\mathbb{O}\langle 4 \rangle)$$

$$\Sigma^{-2} b\mathbb{O}\langle 4 \rangle \xrightarrow{j\text{-homom.}} gl_2(S) \xrightarrow{gl_2(i)} gl_2(KO)$$

Ando's talk: if the component is zero, then there exists an orientation.

Orientations are classified by null-homotopies.

Proposition There is a spectrum map  $gl_1(KO_p) \rightarrow KO_p^*$  which is an iso on homotopy groups  $\pi_*$  for  $* \geq 2$  on  $\pi_{2n}$  for even  $n$ , it is multiplication by  $(1-p^{n-1})$

Also iso on  $\pi_{2n+2}, \pi_{2n+2}$ , except for  $\pi_2$ .

$$\text{So } gl_1(KO_p^* \langle 2 \rangle) \simeq KO_p^* \langle 2 \rangle$$

(Sullivan, Adams - Priddy :  $BSO \otimes \frac{\mathbb{Z}}{p} \simeq BSO \otimes \frac{\mathbb{Z}}{p}$ )

The proof will come later.

Moreover, the above map induces an equivalence

$$L_{KO} \simeq gl_2(KO) \xrightarrow{\simeq} KO_p^*$$

and  $L_{K(2)} gl_2(S) \simeq L_{K(2)} S$

Theorem: There exists an Eoo ring map  $\Omega\text{Spin} \rightarrow KO_P^1$ .

Proof:

$$\Sigma^{-2} bo\langle 4 \rangle \xrightarrow{\cong} gl_2(S) \longrightarrow gl_2(KO_P^1) \xrightarrow{\cong \text{ after } \langle 2 \rangle} KO_P^1$$

Fact:  $\pi_n \text{ map}(KO_P^1, KO_P^1) \cong \pi_0 \text{ map}(KO_P^1, KO_P^1) \oplus \pi_n KO$   
 $= 0$  if  $n = -1, -2, -3$ .

So there are no essential maps, thus no obstructions.  
 But the orientation is (practically) not unique!  $\square$

Home-Sullivan square

For spectrum  $X$ , there is a pullback square

$$\begin{array}{ccc} X & \longrightarrow & \Pi X_P^1 \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\Pi X_P^1)_{\mathbb{Q}} \end{array}$$

We'll use this for  $X = gl_2 \mathbb{R}$ .

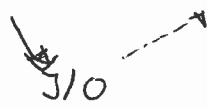
Stable Adams Conjecture (Friedlander)  $p$  odd,

The composite  $bo\langle k \rangle \xrightarrow{1-\psi^c} bo\langle k \rangle_P^1 \xrightarrow{\cong} gl_2(S_P^1)$  for  $c \in \mathbb{Z}_P^{\times}$

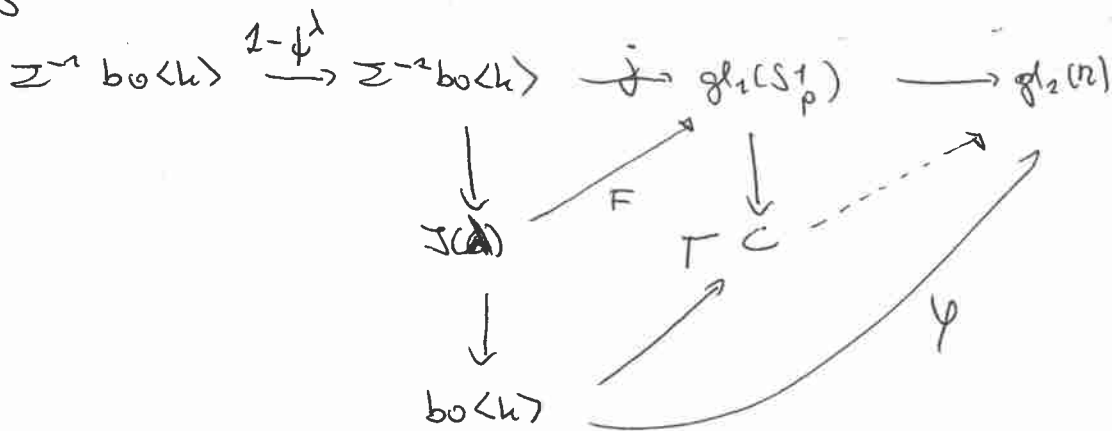
is null as a map of spectra. Equivalently, the map extends over  $bo\langle k \rangle_P^1 \rightarrow J(c)$

At  $p=2$  this does not hold, but the composite

$bu\langle k \rangle \xrightarrow{\text{not } 1-\psi^c} bo\langle k \rangle_P^1 \xrightarrow{\cong} gl_2(S_P^1)$  is null



$p \neq 3$



Suppose we are given an Eoo-orientation  $MO\langle k \rangle \rightarrow \mathbb{R}$   
 or equivalently an extension  $\varphi: bo\langle k \rangle \rightarrow gl_2 \mathbb{R}$  as above.

Invariant:  $\varphi(u^n) = c_n \in \pi_{2n} gl_2(\mathbb{R}) \cong \pi_{2n} \mathbb{R}$  (modulo torsion)

(This is Toda bracket. There is no indeterminacy since we fix nullhomotopies)

Exercise:  $c_n = \frac{g_n}{2} (1 - \lambda^n)$  in  $\pi_{2n} \mathbb{R}$  mod torsion  
 (the denominator 2 comes from realification  $K \rightarrow KO$ )

Now take  $R = KO^1_p$  and use the "logarithm"  $l: gl_2(KO^1_p) \rightarrow KO^1_p$   
 $u^n \mapsto (1 - p^{n-2})u^n$

Proposition: On  $\pi_{2n}$ ,  $l \circ \varphi: bo\langle k \rangle \rightarrow KO^1_p$  is given by  
 the sequence  $z_n = -\frac{B_n}{2n} (1 - p^{n-2})(1 - \lambda^n)$

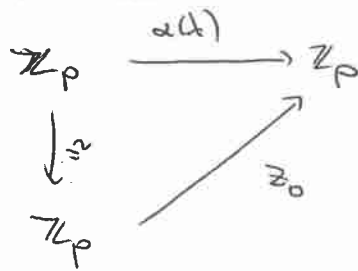
Necessary condition for Eoo-orientation

(1)  $(z_n)$  must satisfy the extended Kummer sequences

(2) After  $K(2)$ -localisation  $L_{K(2)} J(\mathbb{A}) \rightarrow L_{K(2)} S^0 \rightarrow KO^1_p$



gives an  $\pi_0$



So we get condition

$$z_0 = \alpha(t) = \lim_{k \rightarrow \infty} z_{(p-2)p^{k-1}}$$

For  $p$  odd can take  $\lambda$  a generator of  $\mathbb{Z}_p^\times$  and get that conditions (1) and (2) are sufficient for specifying an  $E_{\infty}$ -orientation (they are also sufficient for  $p=2$ , but that's more complicated).

Proposition: (1) The numbers  $-\frac{B_n}{2n} (1-p^{n-2})(1-\lambda^n)$  for  $\lambda \in \mathbb{Z}_p^\times$  satisfy extended Kummer congruences.

$$(2) \quad z_0 = \lim = \frac{1}{2p} \log(\lambda^{p-2})$$

~~Ando-Hopkins-Strom~~

Mellor: ABS orientation is  $H_{\infty}$  } can determine the number  $z_0$   
 M. Joachim: ABS ——— is  $E_{\infty}$

$p$ -adic zeta-functions  $\zeta^+(s)$  defined on  $\mathbb{Z}_p$

$$\zeta^+(1-n) = \int -\frac{B_n}{n} (1-p^{n-1})$$

$$\zeta^+(s) = (1-\frac{1}{p}) \frac{1}{s-1} + \dots$$

Mazur: constructed  $\zeta^+$  via a  $p$ -adic measure

$$\mu(\lambda^n) = -\frac{B_n}{2n} (1-p^{n-2})(1-\lambda^n) \text{ for some } \lambda \in \mathbb{Z}_p^\times$$

Units:  $R = E_{\infty}$  ring spectrum

15.10.03

$$GL_2(R) \subseteq \Omega^{\infty} R$$

$\{$   
 $gl_2(R)$  connective spectrum of units

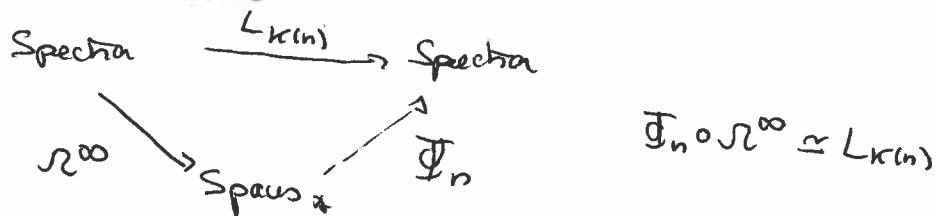
$\Sigma_+^{\infty} \Omega^{\infty}$ : connective spectra  $\rightleftarrows$   $E_{\infty}$  rings:  $gl_1$

Write  $H^q(X, gl_2(R)) = [\Sigma_+^{\infty} X, gl_2(R)]_q$

e.g.  $H^0(X, gl_2(R)) \cong H^0(X, R)^{\times}$

$$\pi_n gl_2 R \cong \tilde{H}^0(S^n, gl_2(R)) \cong (1 + \tilde{H}^0(S^n, R))^{\times} = (1 + \pi_n R)^{\times}$$

Bousfield-Kuhn construction



$$\mathcal{J}_n [GL_2(R) \xrightarrow{x \mapsto x^{-1}} \Omega^{\infty} R] = L_{K(n)} gl_2 R \longrightarrow L_{K(n)} R$$

The composite  $gl_2(R) \longrightarrow L_{K(n)} gl_2(R) \longrightarrow L_{K(n)} R$

is the map from the previous talk!

Question: What does  $l$  do on  $\pi_{2s}$ ? What does it do as a cohomology operation?

$X$  a space,

$$R^0(X)^{\times} = H^0(X, gl_2(R)) \xrightarrow{l} H^0(X, L_{K(n)} R) = (L_{K(n)} R)^0(X)$$

logarithmic,  $l(xy) = l(x) + l(y)$

Thm:  $n=1, p$  prime. Then

$l_2: R^0(X)^* \rightarrow (L_{\text{mod}} R)^0(X)$  is given by the formula

$$\begin{aligned} l_2(x) &= \left(1 - \frac{1}{p} \psi\right) (\log(x)) \\ &= \frac{1}{p} \log\left(\frac{x^p}{\psi(x)}\right) \\ &= \sum_{k \geq 2} \frac{p^{k-2}}{k} \left(\frac{\theta(x)}{x^p}\right)^k \end{aligned}$$

Here  $\psi$  and  $\theta$  are the operators in  $K(1)$ -local Bockstein spectra discussed by Hopkins, satisfying  $\psi(x) + p\theta(x) = x^p$ .

Example:  $R = K_{\mathbb{Z}}^0$ , then  $\psi = \psi^p$  Adams operation

$$K_{\mathbb{Z}}^0(S^{2n}) = \mathbb{Z}[E]/E^2, \text{ so}$$

$$\begin{aligned} l_2(1+E) &= \left(1 - \frac{1}{p} \psi\right) (\log(1+E)) \\ &= \left(1 - \frac{1}{p} \psi\right) (E) = E - \frac{p^n}{p} \cdot E = (1 - p^{n-2}) E \end{aligned}$$

Thm,  $n=2, R = E_2$  Lubin-Tate spectrum

$$l_2(x) = (1 - T(p) + R) (\log(x))$$

$T(p)$  = "dormal Hecke operator if  $E_2$  is an elliptic spectrum

For  $f \in \pi_{2n} E_2$ , then  $R(p) = p^{n-2} \cdot f \text{ mod torsion.}$

So on  $\pi_{2n}$

$$l_2("1+f") = (1 - T(p) + p^{n-2}) \cdot f$$



Power operations: A abelian group, given  $\alpha: X \rightarrow \mathcal{R}^0 \mathcal{R}$

$$\begin{array}{ccc} \mathbb{E}A \times A & X^{\times A} & \xrightarrow{P_A(\alpha)} \mathcal{R}^0 \mathcal{R} \\ \uparrow & & \nearrow \\ X^A & & \alpha^{|A|} \end{array}$$

uses multiplicative  
 $\mathbb{E}^{\infty}$ -structure

Note that  $P_A(\alpha)$  factors over  $\mathbb{E} \sum_d \times_{\Sigma_d} X^d$  for  $d = |A|$ .

Now suppose  $\alpha$  is invertible, so  $\alpha: X \rightarrow GL_1 \mathcal{R} \subseteq \mathcal{R}^0 \mathcal{R}$ .

Then get extension over group completion

$$\begin{array}{ccc} \left( \prod_d \mathbb{B} \Sigma_d \right) \times X & \xrightarrow{\Delta} & \prod_d \mathbb{E} \sum_d \times_{\Sigma_d} X^d \longrightarrow GL_1(\mathcal{R}) \\ \downarrow & \text{group completion} \downarrow & \uparrow \\ \mathcal{R}^0 \mathcal{S} \times X & \xrightarrow{\mathcal{J}} & \mathcal{R}^0 \sum_+^{\infty} X \xrightarrow{\tilde{P}(\alpha)} GL_1 \mathcal{R} \end{array}$$

adjoint to  $\alpha$  w.r.t.  $(\mathbb{E} \Sigma_d)$   
 $GL_1 \mathcal{R} \cong \mathcal{R}^0(gl_1 \mathcal{R})$

$P(\alpha)$

So we have a function  $\mathcal{R}^0(X) \rightarrow \mathcal{R}^0(\mathcal{R}^0 \mathcal{S} \times X) \cong \mathcal{R}^0(\mathcal{R}^0 \mathcal{S}) \otimes_{\mathcal{R}^0} \mathcal{R}^0(X)$

For  $u \in \mathcal{R}_0^1(\mathcal{R}^0 \mathcal{S})$  can consider slant product

$$\begin{array}{ccc} \mathcal{R}^0(X)^{\times} & \longrightarrow & \mathcal{R}^0(\mathcal{R}^0 \mathcal{S} \times X)^{\times} \subseteq \mathcal{R}^0(\mathcal{R}^0 \mathcal{S} \times X) \\ & \searrow \psi_u(\alpha) & \downarrow \text{slant} \\ & & \mathcal{R}^0(X) \end{array}$$

Thm: There exist  $u \in \mathcal{R}_0^1(\mathcal{R}^0 \mathcal{S})$  such that  $l(x) = \psi_u(x)$ .  
Furthermore,  $u$  is the unique special element, i.e. satisfying

(a)  $\tau(u) = 1$  and

(b)  $x \circ u = \tau(x) \cdot u$  for all  $x \in \mathcal{R}_0^1(\mathcal{R}^0 \mathcal{S})$

In the above we have used the following notation:

$$R_q^1 X = \pi_q L_{K(n)}(R_1 X) \quad \text{"completed homology"}$$

(we can assume that  $R$  is  $K(n)$ -local, by locality of logarithm).

Construction of  $\lambda$ : There is a natural transformation of functors

$$\text{Spectra} \rightarrow \text{spectra}$$

$$\lambda: X \rightarrow L_{K(n)} \Sigma_+^{\infty} \mathcal{R}^{00} X$$

defined as follows:

$$\mathcal{R}^{00} X \xrightarrow{\mathcal{R}^{00} X} (\mathcal{R}^{00} \mathcal{R}^{00} X) (\mathcal{R}^{00} X)$$

not  $\mathcal{R}^{00}$ -map

↓ inclusion determined by the  
splitting  $\Sigma_+^{\infty} K \rightarrow S_+^0 \rightarrow \Sigma_+^{\infty} K$   
for pointed  $K$ .

$$(\Sigma_+^{\infty} \Sigma_+^{\infty} \mathcal{R}^{00} X)$$

Now apply Bousfield-Kuhn  $\Phi_n$ :

$$X \rightarrow L_{K(n)} X \rightarrow L_{K(n)} \Sigma_+^{\infty} \mathcal{R}^{00} X$$

λ

Fact: the logarithm  $l$  comes from  $\lambda$ :

For space  $K$  and  $\alpha: K \rightarrow \mathcal{R}^{00} \mathcal{R}_2(\mathbb{R})$

adjoin  $\mathcal{Q}: \Sigma_+^{\infty} K \rightarrow \mathcal{R}_2(\mathbb{R})$

adjoin  $\mathcal{Q}': \Sigma_+^{\infty} \mathcal{R}^{00} \Sigma_+^{\infty} K \rightarrow \mathbb{R}$

$K(n)$ -localize to get

$$\Sigma_+^{\infty} K \xrightarrow{\lambda_{\Sigma_+^{\infty} K}} L_{K(n)} \Sigma_+^{\infty} \mathcal{R}^{00} \Sigma_+^{\infty} K \rightarrow L_{K(n)} \mathbb{R} = \mathbb{R}$$

l(α)

So the special element  $u$  is the composite

$$\lambda_S: S \longrightarrow L_{\text{kin}} \Sigma_{+}^{\infty} \mathcal{R}^{\infty} S$$

$$\in \pi_0 \left( L_{\text{kin}} \Sigma_{+}^{\infty} \mathcal{R}^{\infty} S \xrightarrow{\text{Hurewicz}} R_0^{\wedge}(\mathcal{R}^{\infty} S) \right)$$

On the "special" conditions:

(a)  $\tau: R_0^{\wedge}(\mathcal{R}^{\infty} S) \rightarrow R_0$  induced by

$$\Sigma_{+}^{\infty} \mathcal{R}^{\infty} S \longrightarrow S \quad \text{coint of adjunction}$$

Fact:  $X \xrightarrow{\lambda} L_{\text{kin}} \Sigma_{+}^{\infty} \mathcal{R}^{\infty} X \xrightarrow{L_{\text{kin}}(\text{coint})} L_{\text{kin}} X$   
 (underbrace) augmentation of  $L_{\text{kin}}$

(b) means:  $X \xrightarrow{\alpha} GL_1(\mathbb{R})$

$$\rightsquigarrow P(\alpha): \mathcal{R}^{\infty} \Sigma_{+}^{\infty} X \longrightarrow GL_1(\mathbb{R})$$

and have  $\beta: X \longrightarrow \mathcal{R}^{\infty} \mathbb{R}$

$$\rightsquigarrow S(\alpha): \mathcal{R}^{\infty} \Sigma_{+}^{\infty} X \longrightarrow \mathcal{R}^{\infty} \mathbb{R}$$

and (b) means  $\ell(P(\alpha)) = S(\ell(\alpha))$

(b) says:  $\cdot$  in  $\tau(x) \cdot u$  is multiplication in  $R_0^{\wedge}(\mathcal{R}^{\infty} S)$   
 coming from ~~product~~ <sup>additive</sup> H-space structure on  $\mathcal{R}^{\infty} S \times \mathcal{R}^{\infty} S \xrightarrow{+} \mathcal{R}^{\infty} S$   
 and  $\circ$  from multiplicative H-space structure.

Uniqueness of special elements: suppose  $u, u'$  are both special, then

$$\tau(u) \cdot u' = u \circ u' = u' \circ u = \tau(u') \circ u$$

$$\begin{matrix} \parallel & & \parallel \\ 1 \cdot u' = u' & & u = 1 \cdot u \end{matrix}$$

$K(1)$ -local

$$\pi_0 L_{K(1)}(VB\Sigma_k^+) = (\pi_0 L_{K(1)}S^0) [x, \theta(x), \theta^2(x), \dots]$$

$$\pi_0 L_{K(1)}(\mathbb{R}^{\infty}S) = \text{same with } x \text{ marked and capped at } p.$$

In high dimensions we use Hopkins-Kuhn-Ravenel character theory.

16.10.03

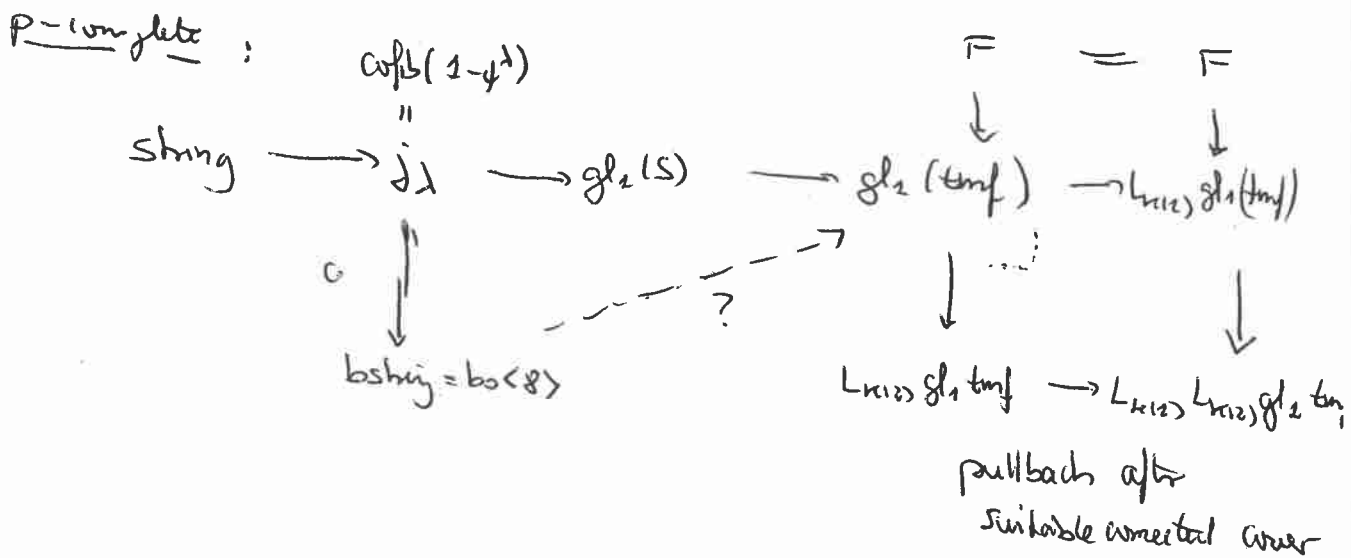
The string cobordism of tmf

$$E_{\infty}(M\text{string}, \mathbb{R}) \simeq \text{space of null homotopies of} \\ [\text{string} \xrightarrow{j} gl_2(S^0) \xrightarrow{gl_2(i)} gl_2(\mathbb{R})]$$

Choosing a basepoint, the homotopy type of this space is  $\text{map}(b\text{string}, gl_2(\mathbb{R}))$

We'll do:

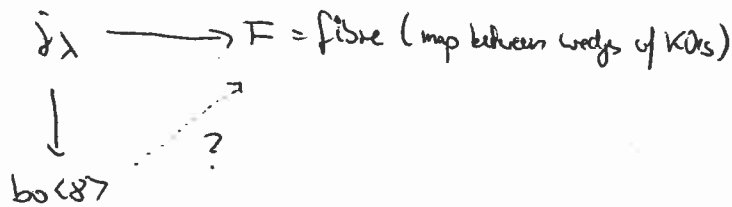
- (1) p-complete case      (2) put together with arithmetic square



If  $K(1)_* X = 0$ , then  $\text{map}(X, L_{K(1)} Y) = *$ .

So the extension problem lifts equivalently to the fibre  $F$ .

Reduction to extension problem:



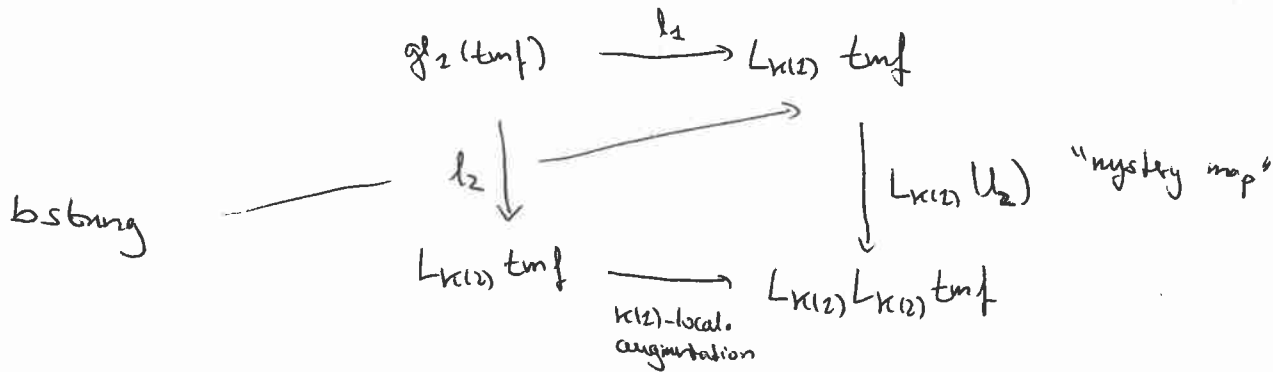
Given an Eoo orientation whose Hurwicz series has the form

$$K(x) = \exp\left(\sum g_n \frac{x^n}{n!}\right)$$

then the corresponding extension  $bo\langle 8 \rangle \rightarrow gl_2(tmf)$  sends

$$\pi_{2n} bo\langle 8 \rangle \cong U^n \mapsto \frac{g_n}{2} (1 - \lambda^n)$$

Bousfield-Kuhn: can rewrite the square as



From Hopkins' talks:  $L_{K(2)} tmf \cong KO \otimes A$  for  $A = \text{ring of } p\text{-adic modular forms of wt } 0$ .

The first talk explained how to build maps between  $p$ -completed KO's. Similarly,  $L_{K(2)} L_{K(2)} tmf \cong KO \otimes B$ .

On  $\pi_{2n} gl_2(tmf) = MF_n \cong f$  we have

$$l_1(f) = (1 - \frac{1}{p} V)(f) = (1 - p^{n-2} \cdot V)(f)$$

$$\text{where } (Vf)(q) = f(q^p)$$

Moreover,  $l_2(f) = (1 - T(p) + p^{n-2})f$   
with  $T(p)$  classical Hecke operator

There exists a factorization of  $L_{k(n)} b_2$  :

$$\begin{array}{ccc}
 L_{k(n)} \text{ tmf} & \xrightarrow{1-u} & L_{k(n)} \text{ tmf} \\
 \downarrow L_{k(n)}(1/2) & & \swarrow L_{k(n)}(\text{adj. unit}) \\
 L_{k(n)} L_{k(n)} \text{ tmf} & & 
 \end{array}$$

$u = \text{Atkin operator}$

On Tate curve  $T(q) = \mathbb{C}^x / q^{\mathbb{Z}}$ , the  $p$ -points

$$(\mathbb{C}^x / q^{\mathbb{Z}})[p] = \mathbb{Z}/p \times \mathbb{Z}/p$$

generates:  $q^{1/p}, \dots, q^{(p-1)/p}$

Look at

$$\begin{array}{ccc}
 \mathbb{C}^x / q^{\mathbb{Z}} & \longrightarrow & \mathbb{C}^x / q^{\sum_{i=0}^{p-1} q^{i/p} \mathbb{Z}} \\
 \parallel & & \parallel \\
 T(q) & \longrightarrow & T(q^{1/p}) \\
 q & \longmapsto & q^{1/p} \\
 u & \longmapsto & u
 \end{array}$$

$i \in \{0, \dots, p-1\}$

Hecke operator :

$$(T(p)f)(q) = \frac{1}{p} \left[ p^n f(q) + \sum_{i=0}^{p-1} f(q^{1/p}) \right]$$

$$= p^{n-1} V f + U f$$

This defines the Atkin operator  $U$  on  $p$ -adic modular forms and one can construct a map  $U: L_{k(n)} \text{ tmf} \rightarrow L_{k(n)} \text{ tmf}$  which realizes the Atkin operator on homotopy groups.

To check that the diagram which factors  $L_{(1,2)}$  actually commutes, it suffices to check that on homology groups.

Reduction: enough to show

$$(L_{(1,2)} l_2) \circ l_1 = (L_{(1,2)} \overset{\text{mult}}{\circ}) \circ (1-U) \circ l_1$$

Why:  $L_{(1,2)} X \neq \text{holim}_r V_1^{-2}(X, M(p^r))$

and we have good control over  $\pi_* gl_2(\text{tmf})$  and  $\pi_* L_{(1,2)} \text{tmf}$ .

Formulas: if  $f(q) = \sum a_n q^n$ , then

$$(Uf)(q) = \sum a_n p^n q^n, \quad U \circ V = \text{id}$$

$$\begin{aligned} \Rightarrow (1-U) \circ (1-p^{n-2}V) &= 1 - U - p^{n-2}V + p^{n-2}U \circ V \\ &= 1 - T(p) + p^{n-2} \end{aligned}$$

Thus  $1-U$  "is" the mystery map.

Conditions:

$$z_n = (1-p^{n-2}V) \left( \frac{g_n}{2} (1-\lambda^n) \right)$$

To build an  $E_{\infty}$ -orientation with Hurewicz series

$$h(x) = \exp \left( \sum g_n \frac{x^n}{n!} \right) \quad \text{and} \quad g_n = 2 \cdot G_n$$

we need:

$\uparrow$   
Eisenstein series

(1) at each prime  $p$ ,

$$z_n = (1-p^{n-2}V) \frac{g_n}{2} (1-\lambda^n) \quad \text{should satisfy extended Kummer congruences}$$

(2)  $z_0 = \alpha(\lambda) = \lim_{\leftarrow} -\frac{B_n}{4n}$

(alternative statement: for  $g_n \in \mathbb{Z}[F_n] \otimes \mathbb{Q} \cong \mathbb{Z}[q]$ , we have

$$g_n \equiv -\frac{B_n}{2n} \pmod{\mathbb{Z}[q]}$$

(3)  $(1-U) z_n = 0$ .