N = 2 FOUR-DIMENSIONAL GAUGE THEORIES FROM FRACTIONAL BRANES

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This is a pedagogical and extended version of the results published in Refs. 1, 2 and presented by the authors in various talks during the last year. We discuss the type II D-branes (both regular and fractional) of the orbifold $R^{1,5} \otimes R^4/Z_2$, we determine their corresponding supergravity solution and show how this can be used to study the properties of $\mathcal{N}=2$ super Yang-Mills. Supergravity is able to reproduce the perturbative moduli space of the gauge theory, while it does not encode the non-perturbative corrections. The short distance region of space-time, which corresponds to the infrared region of the gauge theory, is excised by an enhançon mechanism, and more states should be included in the low energy effective action in order to enter inside the enhançon and recover the instanton corrections.

Contents

1 Introduction

Since the observation made by 't Hooft[3] of studying QCD by using the large N expansion it has been a dream of many particle physicists to use it for studying with analytical methods the non-perturbative properties of QCD as for instance confinement and chiral symmetry breaking. Moreover, since the large N expansion is an expansion in the topology of the diagrams as string theory, it has been a long standing hope to get a string theory for hadrons coming out in the non-perturbative analysis of QCD. This has crashed, however, with the fact that all known string theories contain gravity, while QCD is a theory in flat Minkowski space.

The Maldacena conjecture[4] provides for the first time a strong evidence that a string theory comes out from a gauge theory. It states that four-dimensional $\mathcal{N} = 4$ super Yang-Mills in flat space is equivalent to type IIB string theory compactified on $AdS_5 \times S^5$ and, since the two theories live in two completely different spaces, one does not run in the contradiction mentioned above. On the other hand one expects that the emergence of a string theory be related to confinement, while $\mathcal{N}=4$ super Yang-Mills is a conformal invariant theory in a Coulomb phase and therefore does not confine. Nevertheless, by means of the Maldacena conjecture one has been able to obtain nontrivial informations[5−8] on the strong coupling behaviour of $\mathcal{N}=4$ super Yang-Mills.

In the last few years many attempts have been made to use brane dynamics for studying more realistic gauge theories. In particular, the next in order of difficulty, namely $\mathcal{N}=2$ super Yang-Mills (analyzed also at a non-perturbative level by Seiberg and Witten[9]) has been studied in terms of classical solutions of the supergravity equations of motion corresponding to wrapped branes[1,2,10−13] of various type.[a]

One of these approaches[1,10] is based on using fractional D3-branes[16−19] of the orbifold $R^{1,5} \otimes R^4/Z_2$. In this case the corresponding complete classical solution of the equations of motion of type IIB supergravity has been obtained and has been used in a probe analysis for deriving the moduli space of $\mathcal{N}=2$ super Yang-Mills that is known from Ref. 9. Although the classical solution has a naked singularity at short distances, it turns out that this does not cause any problem because there is a distance, known as the enhançon,[21] which is bigger than the one where the singularity arises, where brane probes become tensionless and where, correspondingly, the classical supergravity solution looses meaning and the singularity is excised. From the point of view of the gauge theory living on the world-volume of fractional D3-branes, the enhançon corresponds to the scale where the gauge coupling constant diverges

[a] For other earlier approaches see Refs. 14 and 15.

(the analogue of $\Lambda_{\rm QCD}$ in QCD). This means that using supergravity one can indeed reproduce the perturbative region of the moduli space, obtaining, for instance, the correct β-function. But, since at the enhançon the classical solution becomes inconsistent, it is not possible to go further and use it for getting the non-perturbative instanton corrections of the Seiberg-Witten moduli space.

Another approach[12,13] is based on D5-branes wrapped on supersymmetric two-cycles of non compact four dimensional Calabi-Yau manifold, as ALE spaces.[b] The classical solution is obtained by lifting to ten dimensions a solution found in 7-dimensional gauged supergravity. Although this approach is meant to give directly the near-horizon limit of the brane, providing the supergravity dual à la Maldacena, it turns out that it, as the one based on fractional branes, is again only able to reproduce the perturbative behaviour of the gauge theory living on the brane, since the enhançon locus is present also in these cases.

The previous results, that seem to be in strong contrast with a duality interpretation à la Maldacena where the supergravity solution gives a good description of the gauge theory for *large* 't Hooft coupling, can instead be easily understood if we regard the classical supergravity solution as an effective way of summing over all open string loops, as explained in detail in Ref. 20. From this point of view, in fact, one does not take the near-horizon limit (*i.e.* $r \to 0$, where r is the distance from the source branes) that anyway cannot be taken because of the enhançon, but rather expands the classical solution around $r \to \infty$, where the metric is almost flat and the supergravity approximation is valid. This expansion corresponds to summing closed string diagrams at tree level, but, because of the open/closed string duality, it is also equivalent to summing over open string loops. Therefore, expanding the supergravity solution around $r \to \infty$ is equivalent to perform an expansion for *small* 't Hooft coupling.

In view of these considerations, it is then not surprising that the supergravity solutions corresponding to fractional and wrapped branes encode the perturbative properties of the $\mathcal{N}=2$ gauge theory living on their world-volume, and at the same time it is also natural that this approach does not include the non-perturbative instanton corrections to the moduli space. The open and fascinating problem is then how to obtain them from the brane dynamics.

This is a pedagogical and extended version of the results published in Refs. 1 and 2 in collaboration with Marialuisa Frau, Alberto Lerda and Igor Pesando. We have written it for commemorating Michael Marinov. One of us (PdV) has met him few times in Soviet Union during the Meetings organized by Nordita that allowed the physicists from the Soviet Union and those from

[b] A different approach based on wrapped branes is discussed in Refs. 22 and 23.

the Western Countries to meet and discuss in an extremely friendly and relaxed atmosphere in the time of the cold war where many people as Michael suffered of its consequences. After his migration to Israel he visited Nordita a couple of times expressing his happiness for his new life there, but also his sadness for missing the life in Moscow.

The paper is organized as follows. In Sections 2 and 3 we discuss in great detail, respectively, the spectrum of massless closed string states and that of massless open string states having their endpoints on fractional and bulk D3-branes of the orbifold $R^{1,5} \otimes R^4/Z_2$. Section 4 is devoted to the construction of the boundary state describing fractional Dp-branes and to its use to compute their boundary action and the large distance behaviour of the corresponding classical solution. In Section 5 we show that fractional branes can be thought of as wrapped branes on vanishing exceptional two-cycles of the corresponding orbifold. Sections 6 and 7 are devoted to study the constraints imposed by supersymmetry on the classical supergravity solution corresponding to the fractional D3-branes, and to the derivation of the solution itself. Finally, in the last section, by probing the supergravity background that we have obtained, with suitable fractional D-brane probes, we derive the properties of the gauge theory living on bulk and fractional branes.

2 Massless closed string states in orbifold $\mathbf{R^{1,5} \otimes R^4/Z_2}$

Let us consider type II string theory on the orbifold $R^{1,5} \otimes R^4/Z_2$ where Z_2 acts on the four directions x^6, x^7, x^8, x^9 by changing their sign:

$$\{x^6, x^7, x^8, x^9\} \rightarrow \{-x^6, -x^7, -x^8, -x^9\} . \tag{1}$$

In this section we study the spectrum of the closed string states of both type IIA and IIB theories.

We analyze the spectrum of closed strings in the light-cone gauge where the classification group is $SO(8)$ that is obtained from the original $SO(1,9)$ by dropping the string coordinates x^0 and x^1. In the case of the orbifold R^4/Z_2 this group is broken to

$$SO(8) \rightarrow SO(4) \times SO(4)_{INT} , \tag{2}$$

where the orbifold group Z_2 acts on $SO(4)_{INT}$. Let us remember that in an orbifold we have both untwisted and twisted sectors. The former corresponds to the identity of the orbifold group, and consists of the subset of the states already present in flat space, which are even under the orbifold group. The number of twisted sectors, instead, depends on the orbifold under consideration and is equal to the number of non-trivial elements of the discrete orbifold group. In our case, where the orbifold group is Z_2, there is only one twisted sector.

Let us start looking at the spectrum of the NS-NS sector, that is the same for both type IIA and type IIB. The massless states of this sector are given by

$$\psi^M_{-1/2}\,\tilde{\psi}^N_{-1/2}\,|0,k\rangle\,, \tag{3}$$

where M and N are indices of $SO(8)$ taking the values $M,N=2,3\ldots 9$. According to the breaking in Eq. (2) we write $M=(a,m)$ and $N=(b,n)$, where $a,b=2$, 3, 4, 5 are indices of the space-time $SO(4)$, while $m,n=6$, 7, 8, 9 are indices of $SO(4)_{INT}$.

Since the orbifold acts on the fermionic coordinate ψ in the same way as on the bosonic ones, according to Eq. (1), in order to preserve world-sheet supersymmetry, it is easy to see that the only states that are even under Z_2 and that therefore survive the orbifold projection are the following:

$$\psi^a_{-1/2}\,\tilde{\psi}^b_{-1/2}\,|0,k\rangle \qquad \text{and} \qquad \psi^m_{-1/2}\,\tilde{\psi}^n_{-1/2}\,|0,k\rangle\,. \tag{4}$$

Since both $\psi^a_{-1/2}$ and $\tilde{\psi}^b_{-1/2}$ transform as the vector $(2,2)$ representation of $SO(4)$ and as the singlet $(1,1)$ of $SO(4)_{INT}$, while both $\psi^m_{-1/2}$ and $\tilde{\psi}^n_{-1/2}$ transform as the singlet $(1,1)$ of $SO(4)$ and as the vector representation $(2,2)$ of $SO(4)_{INT}$, it is easy to see that the first state in Eq. (4) transforms as

$$\begin{aligned}((2,2),(1,1))\ \otimes((2,2),(1,1)) &= ((3+1,3+1),(1,1)) = \\ &= (3,3)+(1,3)+(3,1)+(1,1),(1,1)\end{aligned} \tag{5}$$

corresponding to a graviton represented by $(3,3)$, to a 2-form potential represented by $(3,1)+(1,3)$ and to a dilaton represented by the singlet $(1,1)$. All these fields are singlets of $SO(4)_{INT}$. Since $SO(4)=SU(2)_L\times SU(2)_R$, in the previous formulæ we have labelled a representation of $SO(4)$ with (p,q) where $p\,[q]$ is the dimension of the representation of $SU(2)_L[SU(2)_R]$. Analogously, it can be seen that the second state in Eq. (4) contains only 16 scalars (singlet with respect to the first $SO(4)$) that transform according to the representations $(3,3)+(3,1)+(1,3)+(1,1)$ of $SO(4)_{INT}$. In conclusion, the untwisted NS-NS sector of both type IIA and IIB theories contains a graviton, a dilaton, a two-form potential and 16 scalars.

Let us consider now the untwisted R-R sector. In the light-cone gauge we can limit ourselves to the Dirac matrices of $SO(8)$ that satisfy the Clifford algebra:

$$\{\psi^M_0,\psi^N_0\}=\delta^{MN}, \qquad M,N=2,3\ldots 9\,. \tag{6}$$

It is convenient to introduce the raising and lowering operators

$$d_i^{\pm} = \frac{1}{\sqrt{2}} \left[\psi_0^{2i} \pm i \psi_0^{2i+1} \right] , \qquad i = 1,\, 2,\, 3,\, 4 , \tag{7}$$

satisfying the algebra:

$$\{d_i^+ , d_j^- \} = \delta_{ij} . \tag{8}$$

For each i we have two states denoted by $|s_i\rangle$ with $s_i = \pm\frac{1}{2}$ that are eigenstates of the number operator N_i:

$$N_i \equiv -\, i\, \psi_0^{2i} \psi_0^{2i+1} = d_i^+ d_i^- - 1/2 , \qquad N_i|s_i\rangle = s_i|s_i\rangle . \tag{9}$$

A spinor of $SO(8)$ can then be represented by the 16 states

$$|s_1,\, s_2,\, s_3,\, s_4\rangle . \tag{10}$$

The chirality operator Γ of $SO(8)$ is given by the product of all Gamma matrices:

$$\Gamma = 2^4 N_1 N_2 N_3 N_4 , \qquad \Gamma^2 = 1 . \tag{11}$$

The 8 states with chirality equal to $\Gamma = +1(-1)$ are characterized by the fact that:

$$\sum_{i=1}^{4} N_i = \text{even (odd)} \tag{12}$$

It is important to notice that the space-time $SO(4)$ acts only on the indices $i = 1, 2$, while $SO(4)_{INT}$ acts on the remaining indices $i = 3, 4$. If we now limit ourselves only to one of the two $SO(4)$ and we use the convention where $(1, 2)$ $[(2, 1)]$ corresponds to the eigenvalue $(+1)[(-1)]$ of the $\hat{\Gamma}$ matrix of the group $SO(4)$ (for instance, $\hat{\Gamma} \equiv -\Gamma_2\Gamma_3\Gamma_4\Gamma_5 = 4N_1N_2$ in the case of space-time $SO(4)$), it is easy to see that its two spinor representations correspond to the following states:

$$(1, 2) \sim \left(\frac{1}{2}, \frac{1}{2}\right) + \left(-\frac{1}{2}, -\frac{1}{2}\right) , \tag{13}$$

$$(2, 1) \sim \left(\frac{1}{2}, -\frac{1}{2}\right) + \left(-\frac{1}{2}, \frac{1}{2}\right) . \tag{14}$$

This implies that the 8 states with chirality $+1$ and -1 are given respectively by

$$8_s = ((1, 2), (1, 2)) + ((2, 1), (2, 1)) \text{ and } 8_c = ((1, 2), (2, 1)) + ((2, 1), (1, 2)) . \tag{15}$$

The orbifold group Z_2 acts on the spinor in Eq. (10) as follows:

$$|s_1, s_2, s_3, s_4\rangle \to e^{i\pi(s_3+s_4)}|s_1, s_2, s_3, s_4\rangle \,. \tag{16}$$

This implies that Z_2 acts on the two spinors of $SO(4)_{INT}$ as

$$(1,2) \to -(1,2)\,, \qquad (2,1) \to (2,1)\,. \tag{17}$$

and the spinors 8_s and 8_c are transformed under the orbifold action as

$$8_s \to -((1,2),(1,2)) + ((2,1),(2,1))$$

and

$$8_c \to ((1,2),(2,1)) - ((2,1),(1,2))\,, \tag{18}$$

respectively. We are now ready to study the spectrum of the untwisted R-R sector. Let us start with the type IIA theory that contains two spinors with opposite chirality. This means that we should consider the product $8_s \times 8_c$, where 8_s and 8_c correspond respectively to the left and right movers, and keep only the states that are even under the orbifold group Z_2. In this case the states that survive the orbifold projections are the following:

$$\begin{aligned}((2,1),(2,1)) \times ((1,2),(2,1)) + ((1,2),(1,2)) \times ((2,1),(1,2))\\ = ((2,2),(3+1,1)) + ((2,2),(1,3+1))\,.\end{aligned} \tag{19}$$

They correspond to 8 vectors of the space-time $SO(4)$. In conclusion, the untwisted R-R sector of type IIA contains 8 vector fields. Considering now type IIB, we should take the product of two spinors with the same chirality. The states that are even under the orbifold projections are:

$$\begin{aligned}((2,1),(2,1)) \times ((2,1),(2,1)) + ((1,2),(1,2)) \times ((1,2),(1,2))\\ = ((3+1,1),(3+1,1)) + ((1,3+1),(1,3+1))\,,\end{aligned} \tag{20}$$

corresponding to 4 two-form potential and 8 scalars.

The previous spectra for the untwisted R-R sectors can also be obtained by restricting ourselves to the states appearing in type II theories in flat space that are even under the orbifold projection. For instance in type IIA theory in flat space we have two R-R fields C_M and C_{MNP}. The even ones under the orbifold projection are C_a, C_{abc}, and C_{amn}, where (according to the notation explained at the beginning of this section) a, b, c are indices outside of the orbifold and m, n are those along the orbifold. The previous states correspond in the six-dimensional space outside the orbifold to 7 vectors and a 3-form potential that in six dimensions is dual to a vector. This means that we obtain

8 vectors as with the other method used above. The same procedure can also be applied to the case of type IIB theory containing the R-R fields C, C_{MN} and C_{MNPQ} with self-dual field strength. The states surviving the orbifold projection are C, C_{ab}, C_{mn}, $\frac{1}{2}C_{abmn}$ and $\frac{1}{2}C_{\ell mnr}$ where the factor 1/2 takes care of the self-dual field strength. Those are precisely the states found with the previous method.

Before moving to the twisted sectors let us consider the supersymmetric charges that survive the orbifold projection. The ordinary type IIA has two supercharges

$$Q \sim 8_s \qquad \text{and} \qquad \widetilde{Q} \sim 8_c\,, \tag{21}$$

that transform as 8_s and 8_c, respectively. Because of their transformation properties under Z_2 (see Eq. (18)) the states that are even under the orbifold projection are the following:

$$Q \sim ((2,1),(2,1))\,, \qquad \tilde{Q} \sim ((1,2),(2,1))\,. \tag{22}$$

This shows that the orbifold R^4/Z_2 keeps only one half of the supersymmetry of flat space. Proceeding in the same way in the case of type IIB theory we get that the supersymmetric charges surviving the orbifold projection are

$$Q,\, \widetilde{Q} \sim ((2,1),(2,1))\,. \tag{23}$$

It is interesting to notice that the supercharges in Eqs. (22) and (23) that survive the orbifold projection all transform according to the representation $(2,1)$ of $SO(4)_{INT}$ that is left invariant under the action of the following operator:

$$\Gamma^6\,\Gamma^7\,\Gamma^8\,\Gamma^9 = -\,4\,N_3 N_4\,, \qquad \Gamma^M \equiv \sqrt{2}\,\psi_0^M\,. \tag{24}$$

This follows trivially from the definition of the state $(2,1)$ given in Eq. (14).

Let us consider now the twisted sectors, starting from NS-NS fields. In this case the massless states are spinors of $SO(4)_{INT}$ that we require to be even under both the action of the orbifold group and that of the GSO operators:

$$P^L_{GSO} = P^R_{GSO} = \frac{1+\Gamma^6\,\Gamma^7\,\Gamma^8\,\Gamma^9}{2}\,, \tag{25}$$

where L and R label, respectively, the left and right movers of the closed string. As a consequence one gets

$$((1,1),(2,1)) \times ((1,1),(2,1)) = ((1,1),(3+1,1))\,, \tag{26}$$

corresponding to 4 scalars. One is a singlet of $SO(4)_{INT}$, while the other three are a triplet with respect to one of the two $SU(2)$ groups of $SO(4)_{INT}$.

Considering now the massless states in the R-R twisted sector, in type IIA theory we must consider the following GSO operators:

$$P^L_{GSO} = \frac{1+\Gamma^2\Gamma^3\Gamma^3\Gamma^4}{2} \qquad \text{and} \qquad P^R_{GSO} = \frac{1-\Gamma^2\Gamma^3\Gamma^3\Gamma^4}{2}\,. \tag{27}$$

that, together with the orbifold projection, select the following tensor product:

$$((2,1),(1,1)) \times ((1,2),(1,1)) = ((2,2),(1,1))\,, \tag{28}$$

that corresponds to a vector field. In the case of type IIB, instead, we have the following tensor product:

$$((2,1),(1,1)) \times ((2,1),(1,1)) = ((3+1,1),(1,1))\,, \tag{29}$$

corresponding to a scalar and a self-dual two-form potential.

The orbifold we are considering has a curvature singularity at its fixed point, corresponding to $x^6 = x^7 = x^8 = x^9 = 0$. It is well known that this singularity can be interpreted in terms of a vanishing two-cycle $\mathcal{C}_1$ of a smooth ALE manifold. The twisted fields can then be understood as arising from the p-form fields appearing in type II theories, dimensionally reduced on this vanishing two-cycle (notice that since the volume of the cycle is zero, there are no Kaluza-Klein states, other than the zero modes). In the NS-NS sector, the antisymmetric two-form B_{MN} gives rise to the scalar of the NS-NS twisted sector that is a singlet of $SO(4)_{INT}$, see Eq. (26). The other three scalars, transforming as a triplet of $SU(2)$, are instead geometric moduli, related to the metric tensor. In the R-R sector, we get a vector field corresponding to dimensional reduction of the three-form potential ($C_3 = \mathcal{A}_1 \wedge \omega_2$, where ω_2 is the differential form dual to the vanishing two-cycle $\mathcal{C}_1$) in type IIA theory, and in type IIB a scalar field and a two-form potential with self-dual field strength corresponding, respectively, to the dimensional reduction of $C_2 = c\,\omega_2$ and $C_4 = \mathcal{A}_2 \wedge \omega_2$. It is probably worth noticing that when we lift type IIA theory to M-theory, the NS-NS scalar singlet provides an extra component to the R-R twisted vector obtaining a vector in the 7-dimensional space orthogonal to the orbifold.

3 Massless open string states in orbifold $\mathbf{R^{1,5} \otimes R^4/Z_2}$

In this section we determine the spectrum of open strings ending on the D3-branes of the orbifold $R^{1,5} \otimes R^4/Z_2$. The group Z_2 consists of two generators: g acting on the coordinates of R^4 as in Eq. (1), and its square that is the identity e. If we consider a D3-brane located at a generic point of the orbifold covering space, we must also include its image and consequently we have four kinds of open strings. Two kinds corresponding to open strings having both their end-points on the brane or on its image and two other kinds corresponding to open strings having one endpoint on the brane and the other on its image and vice-versa. These four kinds of open strings are described by a two by two Chan-Paton matrix that we denote by:

$$\lambda = \begin{pmatrix} \text{D3-D3} & \text{D3-D3}' \\ \text{D3}'\text{-D3} & \text{D3}'\text{ -D3}' \end{pmatrix}, \tag{30}$$

where each entry describes one of the four kinds of open strings. A generic open string state in the NS sector will then be described by the product of a Chan-Paton matrix that we denote by λ and an oscillator state with a certain momentum along the world-volume of the D3-brane. In particular, a massless state of the NS sector will have the following form:

$$\lambda\, \psi^M_{-1/2}\, |0,k\rangle\,, \qquad M = 0,\, 1\, \ldots\, 9\,. \tag{31}$$

The open string states that are allowed in an orbifold are those that are left invariant under the action of Z_2 that acts on both the oscillators and the Chan-Paton factors. Since, in order to keep world-volume supersymmetry, Z_2 acts on the fermionic coordinates in the same way as on the bosonic ones, the oscillator part of the state in (31) transforms under g as

$$\psi^{\alpha,i}_{-1/2}\, |0,k\rangle \to \psi^{\alpha,i}_{-1/2}\, |0,k\rangle\,, \qquad \alpha = 0,\, 1,\, 2,\, 3\,, \qquad i = 4,\, 5 \tag{32}$$

and

$$\psi^{m}_{-1/2}\, |0,k\rangle \to -\psi^{m}_{-1/2}\, |0,k\rangle\,, \qquad m = 6,\, \ldots\, 9\,, \tag{33}$$

where we have denoted the world-volume directions of the D3-brane with α, the four directions along the orbifold with m, and the transverse ones outside the orbifold with i. On the other hand, the Chan-Paton factors transform under Z_2 as

$$\lambda \to \gamma\,(h)\,\lambda\,\gamma\,(h)^{-1}\,, \quad \gamma\,(e) = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma\,(g) = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{34}$$

The matrix $\gamma\,(g)$ can be determined by requiring that it exchanges an open string ending on the D3-brane with an open string ending on its image and vice-versa. It is easy to check that the matrix σ_1 in Eq. (34) satisfies this property.

Taking into account the action of the orbifold group on both the oscillators and the Chan-Paton factors, one gets the following invariant states that survive the orbifold projection:

$$\frac{1+\sigma_1}{2} \otimes \psi^{\alpha,i}_{-1/2}|0,k\rangle\,, \qquad \frac{1-\sigma_1}{2} \otimes \psi^{\alpha,i}_{-1/2}|0,k\rangle\,, \tag{35}$$

corresponding to two gauge fields living on the world-volume of the D3-brane represented by the index α, and four real Higgs fields represented by the index i,

$$\text{and} \quad \frac{\sigma_3+i\sigma_2}{2} \otimes \psi^{m}_{-1/2}|0,k\rangle\,, \quad \frac{\sigma_3-i\sigma_2}{2} \otimes \psi^{m}_{-1/2}|0,k\rangle\,, \quad m=6\,,7\,,8\,,9\,, \tag{36}$$

corresponding to 8 scalars. At the orbifold fixed point all these fields are massless and are grouped together in two $\mathcal{N}=2$ vector multiplets, containing a gauge and two real Higgs fields each, and two hypermultiplets, containing 4 scalars each.

The action of Z_2 on the Chan-Paton factors given in Eq. (34) corresponds to the regular representation of Z_2 defined by the relation:

$$[R(h)]_{h_1 h_2} = \delta_{hh_1,h_2}\,. \tag{37}$$

It turns out that it is a reducible representation as any non one-dimensional representation of a discrete abelian group, as Z_2 is. In order to see this directly, it is convenient to perform a change of basis in the space of the Chan-Paton factors λ by means of the following transformation:

$$\lambda \to A^{-1}\lambda A\,, \qquad A = \frac{1-i\sigma_2}{\sqrt{2}}\,. \tag{38}$$

In this new basis the regular representation becomes

$$\gamma\,(e) = \mathbb{1}\,, \qquad \gamma\,(g) = \sigma_3\,. \tag{39}$$

The massless states are given by

$$A_1^{\alpha,i} \equiv \frac{1+\sigma_3}{2} \otimes \psi^{\alpha,i}_{-1/2}\,|0,k\rangle\,, \qquad A_2^{\alpha,i} \equiv \frac{1-\sigma_3}{2} \otimes \psi^{\alpha,i}_{-1/2}\,|0,k\rangle\,, \tag{40}$$

corresponding to the two gauge fields and the four Higgs scalar of the two vector multiplets,

$$\text{and by} \quad \Phi_1^m \equiv \frac{\sigma_1+i\sigma_2}{2} \otimes \psi^{m}_{-1/2}\,|0,k\rangle\,, \quad \Phi_2^m \equiv \frac{\sigma_1-i\sigma_2}{2} \otimes \psi^{m}_{-1/2}\,|0,k\rangle\,, \tag{41}$$

corresponding to the two hypermultiplets. They can be grouped together in the 2×2 matrix

$$\begin{pmatrix} A_1 & \Phi_1 \\ \Phi_2 & A_2 \end{pmatrix} . \tag{42}$$

The charge of the scalar hypermultiplets can be determined by the commutation relations between the Chan-Paton factors of the gauge vectors and those of the scalar fields. From the commutators:

$$\left[\frac{1+\sigma_3}{2}, \frac{\sigma_1 \pm i\sigma_2}{2}\right] = \pm\frac{\sigma_1 \pm i\sigma_2}{2} , \qquad \left[\frac{1-\sigma_3}{2}, \frac{\sigma_1 \pm i\sigma_2}{2}\right] = \mp\frac{\sigma_1 \pm i\sigma_2}{2} , \tag{43}$$

one gets that Φ_1 has charges $(1, -1)$ and Φ_2 has opposite charges $(-1, 1)$ with respect to the two gauge fields A_1 and A_2. Summarizing, the low energy effective theory living on N D3-branes is four-dimensional $\mathcal{N}=2$ super Yang-Mills with gauge group $U(N) \otimes U(N)$ and with two hypermultiplets transforming in the bifundamental representation of the two gauge groups:

$$\Phi_1 \sim (N, \bar{N}) \quad \text{and} \quad \Phi_2 \sim (\bar{N}, N) . \tag{44}$$

Such a theory is conformal invariant as it can be easily checked, since the two β-functions are indeed vanishing. Hence the gauge theory living on a D3-brane transforming according to the regular representation of the orbifold group, is conformal invariant. Notice that the hypermultiplets scalars are associated with the possibility of moving the D3-brane in the orbifold directions, while the vector multiplet scalars are associated to displacements along the fixed plane (x^4, x^5). Bulk branes on orbifolds are then not much different from usual D-branes in flat space. Indeed, when moving a bound state of N bulk D3-branes from the orbifold fixed point, only the diagonal gauge group survives, and the corresponding low energy effective theory is equivalent to the Coulomb phase of $SU(N)$ $\mathcal{N}=4$ super Yang-Mills, as it is the case for D3-branes in flat space.

In the new basis, where the transformations of the Z_2 group are given in Eqs. (39), it is easy to see that the regular representation is reducible, implying that the bulk branes transform according to a reducible representation of the orbifold group. One could look for more elementary branes transforming according to the one-dimensional irreducible representations of the orbifold group. The group Z_2 has only two irreducible representations $\mathcal{D}_I$ $(I = 1, 2)$ given by

$$\gamma_1(e) = 1 , \qquad \gamma_1(g) = 1 ; \tag{45}$$

and

$$\gamma_2(e) = 1 , \qquad \gamma_2(g) = -1 . \tag{46}$$

The branes transforming according to one of the two previous irreducible representations are called *fractional* branes. The regular representation is of course the direct sum of the two irreducible representations above , namely:

$$R = \oplus \mathcal{D}_I \,, \qquad I = 1, 2 \,. \tag{47}$$

This simple mathematical formula has in fact a very interesting physical interpretation which will become clear when we will discuss the closed string interpretation of bulk and fractional branes in the next section.

Since Z_2 has only two irreducible representations in this case there are only two kinds of fractional branes. Furthermore, being the Chan-Paton factors one-dimensional, the fractional branes have the property of living at the orbifold fixed plane $x^6 = x^7 = x^8 = x^9 = 0$, since they do not have, by construction, an image. Let us see which is the low energy effective theory living on their world-volume. In this case the massless open string states surviving the orbifold projection are

$$\psi^{\alpha}_{-1/2} \, |0, k\rangle \qquad \text{and} \qquad \psi^{i}_{-1/2} \, |0, k\rangle \,, \tag{48}$$

corresponding in four dimensions to a gauge field and two real scalar fields belonging to an $\mathcal{N} = 2$ vector gauge multiplet. In the case of a fractional brane the additional scalars belonging to the hypermultiplets are projected out by the orbifold projection (this implying that fractional branes are stuck on the orbifold fixed plane, as already noticed). In conclusion the gauge theory living on N fractional D3-branes of the orbifold $R^{1,5} \otimes R^4/Z_2$ is pure $\mathcal{N} = 2$ super Yang-Mills with $U(\mathrm{N})$ gauge group, which is not conformal invariant. Therefore fractional branes have the advantage with respect to bulk branes that they allow for the study of non-conformal gauge theories.

The previous analysis can be extended to any orbifold of the ADE series.[24,25] For a generic orbifold of the kind R^4/Γ, (Γ being a Kleinian subgroup of $SU(2)$), bulk branes are defined as D-branes whose Chan-Paton factors transform under the regular representation of Γ (and hence, by construction, they have images). Fractional branes, on the other hand, are defined as D-branes whose Chan-Paton factors transform under the irreducible representations of Γ (and do not have images). Hence, for a generic orbifold theory, there are as many different kinds of fractional branes, as the number of different irreducible representations of Γ. While for abelian orbifolds (A series, corresponding to Z_N) the dimension of the irreducible representations is one, for non abelian orbifolds (DE series), this is not true anymore. In these cases, the number of different fractional branes is then less than the order of Γ. The generalization of Eq. (47) is indeed

$$R = \oplus\, d_I \mathcal{D}_I\,, \quad \text{with} \quad \sum_{I=0}^{n-1} d_I = |\Gamma|\,, \quad I = 0,\, 2,\, \ldots\,,\, n-1\,, \tag{49}$$

where $|\Gamma|$ is the order of the discrete group Γ, d_I is the dimension of the I-th irreducible representation and n is their number, this also being equal to the number of different types of fractional branes. Eqs. (47) and (49) seem to suggest that bulk branes can somehow be thought as the 'sum' of fractional branes. This näive idea turns out to be correct, as it will become apparent in the next section, when discussing fractional branes from the boundary state point of view. One can generalize the analysis of the massless open string spectrum performed for Z_2 to the case of a general orbifold group Γ and easily see that the gauge theory living on N bulk branes corresponds to the following group:

$$U(d_0 N) \times U(d_1 N) \times\, \ldots\, \times U(d_{n-1} N)\,, \tag{50}$$

with hypermultiplets transforming in the bifundamental of any given couple of gauge groups. Also in these more general cases, as for the Z_2 orbifold, it can be shown that the gauge theory living on bulk branes is conformal invariant, namely that all the n β-functions are vanishing. The hypermultiplets correspond, again, to open string stretched between a D-brane and its images and therefore are present only in the low energy spectrum of bulk branes. On the contrary, fractional branes, which do not have images, are described as before by pure $\mathcal{N}$=2 and are stuck at the orbifold fixed plane. They are free to move only on the fixed plane (x^4, x^5), the corresponding degrees of freedom being described by the two scalars of the $\mathcal{N}$=2 vector multiplet.

All previous considerations, which we have done for D3-branes, can be easily extended to a general bulk and fractional Dp-brane. The only essential difference, at this level, is that the low energy effective theory living on them is in general a $p+1$ dimensional gauge theory. Since we are mainly interested in four dimensional gauge theories, we will not spend more time discussing Dp-branes here. Nevertheless, when discussing fractional branes from the closed string point of view, we will make a more general treatment which will be valid for a generic value of p.

4 Boundary state description of fractional branes

In this section we analyze in some detail the Dp-branes of type II string theories in the background of the orbifold $R^{1,5} \otimes R^4/Z_2$ using the formalism of the boundary state.

The starting point in string theory for describing a fractional Dp-brane is the vacuum energy Z of the open strings stretched between two fractional Dp-branes which is given by

$$Z = \int_0^\infty \frac{ds}{s} \, \mathrm{Tr}_{\mathrm{NS-R}} \left[\left(\frac{1 + (-1)^F}{2} \right) \left(\frac{e+g}{2} \right) \mathrm{e}^{-2\pi s (L_0 - a)} \right] , \tag{51}$$

where the first term under the trace performs the GSO projection, e and g are the two elements of orbifold Z_2, and $a=1/2$ in the NS sector and $a=0$ in the R sector. When one takes the e inside the bracket, one gets half of the contribution of the open strings stretched between two Dp-branes in flat space, whereas when one takes the g inside the bracket, one obtains the contribution of the twisted sectors of the fractional Dp-branes. Let us consider in general a Dp-brane with $r+1$ directions of its world-volume outside and $s = p - r$ directions along the orbifold R^4/Z_2. To be more specific, we divide both the world-volume and the transverse directions in directions that are outside and along the orbifold. As far as the transverse directions are concerned we have then $4-s$ along the orbifold and $5-r$ outside it. In this case the vacuum amplitude is equal to

$$Z = Z_e + Z_g \, , \tag{52}$$

with

$$\begin{aligned} Z_e &= \frac{1}{2} \int_0^\infty \frac{ds}{s} \, \mathrm{Tr}_{\mathrm{NS-R}} \left[P_{\mathrm{GSO}} \, \mathrm{e}^{-2\pi s \, (L_0 - a)} \right] \\ &= \frac{1}{2} \frac{V_{p+1}}{(8\pi^2 \alpha')^{(p+1)/2}} \int_0^\infty \frac{ds}{s^{(p+3)/2}} \, \frac{1}{2} \left[\frac{f_3^8(q) - f_4^8(q) - f_2^8(q)}{f_1^8(q)} \right] , \end{aligned} \tag{53}$$

$$\begin{aligned} Z_g &= \frac{1}{2} \int_0^\infty \frac{ds}{s} \, \mathrm{Tr}_{\mathrm{NS-R}} \left[g \, P_{\mathrm{GSO}} \, \mathrm{e}^{-2\pi s \, (L_0 - a)} \right] \\ &= \frac{V_{r+1}}{2^s \, (8\pi^2 \alpha')^{(r+1)/2}} \int_0^\infty \frac{ds}{s^{(r+3)/2}} \left[\frac{f_3^4(q) \, f_4^4(q)}{f_1^4(q) \, f_2^4(q)} - \frac{f_3^4(q) \, f_4^4(q)}{f_1^4(q) \, f_2^4(q)} \right] , \end{aligned} \tag{54}$$

where P_{GSO} is the GSO projection, $q = \mathrm{e}^{-\pi s}$ and the f's are the standard one-loop modular functions. Notice the appearance of the important factor 2^{-s} in Eq. (54) that is due to the integration over the bosonic zero modes along the orbifolded directions.

After performing the modular transformation $s \to t = 1/s$, Z_e and Z_g can be interpreted as tree level closed string amplitudes between two untwisted and two twisted boundary states, respectively, i.e.

$$Z_e = \frac{\alpha'\pi}{2}\int_0^\infty dt \; {}^U\langle \mathrm{D}p\,|\, \mathrm{e}^{-\pi t(L_0+\widetilde{L}_0-2a)}\,|\,\mathrm{D}p\rangle^U , \tag{55}$$

$$Z_g = \frac{\alpha'\pi}{2}\int_0^\infty dt \; {}^T\langle \mathrm{D}p\,|\, \mathrm{e}^{-\pi t(L_0+\widetilde{L}_0)}\,|\,\mathrm{D}p\rangle^T . \tag{56}$$

From Eq. (55) it is immediate to realize that Z_e is one half of the amplitude for Dp-branes in flat space, and therefore the untwisted part of the boundary state is simply

$$|\mathrm{D}p\rangle^U = \frac{T_p}{2\sqrt{2}}\left(|\mathrm{D}p\rangle^U_{\mathrm{NS}} + |\mathrm{D}p\rangle^U_{\mathrm{R}}\right), \qquad T_p = \sqrt{\pi}\left(2\pi\sqrt{\alpha'}\right)^{3-p}, \tag{57}$$

where $|\mathrm{D}p\rangle^U_{\mathrm{NS}}$ and $|\mathrm{D}p\rangle^U_{\mathrm{R}}$ are the usual boundary states for a bulk Dp-brane.[26,27] Notice that in the previous equation we have explicitly extracted from the boundary state of a bulk brane in flat space the usual normalization factor $T_p/2$.

From Eq. (54) we can see that the twisted amplitude for a fractional Dp-brane with s directions along the orbifold is the same as the one for a fractional Dr-brane entirely outside the orbifold, apart from a factor 2^{-s}. Therefore, using Eq. (56), we can deduce that the boundary state $|\mathrm{D}p\rangle^T$ is similar to the boundary state for a fractional Dr-brane transverse to the orbifold, but with an extra factor of $2^{-s/2}$ in its normalization. In conclusion, we get

$$|\mathrm{D}p\rangle^T = -\frac{1}{2^{s/2}}\,\frac{T_r}{2\sqrt{2}\,\pi^2\alpha'}\left(|\mathrm{D}p\rangle^T_{\mathrm{NS}} + |\mathrm{D}p\rangle^T_{\mathrm{R}}\right), \tag{58}$$

where $$|\mathrm{D}p\rangle^T_{\mathrm{NS,R}} = \frac{1}{2}\left(|\mathrm{D}p,+\rangle^T_{\mathrm{NS,R}} + |\mathrm{D}p,-\rangle^T_{\mathrm{NS,R}}\right). \tag{59}$$

Here, the Ishibashi states are

$$|\mathrm{D}p,\eta\rangle^T_{\mathrm{NS}} = |\mathrm{D}p_X\rangle^T|\mathrm{D}p_\psi,\eta\rangle^T_{\mathrm{NS}} \quad \text{(in the NS}-\text{NS twisted sector)} \tag{60}$$

and $$|Dp,\eta\rangle^T_{\mathrm{R}} = |Dp_X\rangle^T|Dp_\psi,\eta\rangle^T_{\mathrm{R}} \quad \text{(in the R}-\text{R twisted sector}^{\,c}\text{)}, \tag{61}$$

where

$$|\mathrm{D}p_X\rangle^T = \delta^{(5-r)}(\widehat{q}^i - y^i)\prod_{n=1}^{\infty}\exp\left[-\frac{1}{n}\alpha_{-n}\cdot S\cdot\tilde{\alpha}_{-n}\right]$$

$$\times \prod_{r=1/2}^{\infty} \exp\left[-\frac{1}{r}\alpha_{-r}\cdot S\cdot\tilde{\alpha}_{-r}\right] \prod_{\alpha}{}' |p_\beta = 0\rangle \prod_{i}{}' |p_i\rangle \,, \tag{62}$$

$$|\mathrm{D}p_\psi,\eta\rangle^T_{NS} = \prod_{r=1/2}^{\infty} \exp\left[\mathrm{i}\,\eta\,\psi_{-r}\cdot S\cdot\tilde{\psi}_{-r}\right] \prod_{n=1}^{\infty} \exp\left[\mathrm{i}\,\eta\,\psi_{-n}\cdot S\cdot\tilde{\psi}_{-n}\right] |\mathrm{D}p_\psi,\eta\rangle^{(0)\,T}_{\mathrm{NS}}, \tag{63}$$

$$|\mathrm{D}p_\psi,\eta\rangle^T_{R} = \prod_{n=1}^{\infty} \exp\left[\mathrm{i}\,\eta\,\psi_{-n}\cdot S\cdot\tilde{\psi}_{-n}\right] \prod_{r=1/2}^{\infty} \exp\left[\mathrm{i}\,\eta\,\psi_{-r}\cdot S\cdot\tilde{\psi}_{-r}\right] |\mathrm{D}p_\psi,\eta\rangle^{(0)\,T}_{\mathrm{R}}, \tag{64}$$

where $S = (\eta_{\alpha\beta}, -\delta_{ij})$, with the longitudinal indices α, β taking values $0, 1, \ldots, p$, and the transverse indices i, j taking values $p+1, \ldots, 9$. The prime in the vacuum of Eq. (62) indicates that the indices β and i run only over the longitudinal and transverse directions not included in the orbifold because there is no zero mode on the orbifold directions.

The zero-mode part of the boundary state has a nontrivial structure in both sectors; in the NS-NS sector it is given by [27]

$$|\mathrm{D}p_\psi,\eta\rangle^{(0)\,T}_{\mathrm{NS}} = \left(\widehat{C}\,\widehat{\gamma}^6 \,\ldots\, \widehat{\gamma}^{5+s}\,\frac{1+\mathrm{i}\,\eta\,\widehat{\gamma}}{1+\mathrm{i}\,\eta}\right)_{LM} |L\rangle|\widetilde{M}\rangle \,, \tag{65}$$

where $\widehat{\gamma}^\ell$ are the gamma matrices and $\widehat{C}$ the charge conjugation matrix of $SO(4)$, $\widehat{\gamma} = \widehat{\gamma}^6 \ldots \widehat{\gamma}^9$, and, finally, $|L\rangle$ and $|\widetilde{M}\rangle$ are spinors of $SO(4)$. The matrices of $SO(4)$ satisfy the following relations under transposition:

$$\widehat{C}^t = \widehat{C}\,, \qquad \widehat{\gamma}^{\ell\,t} = \widehat{C}\,\widehat{\gamma}^\ell\,\widehat{C}^{-1}. \tag{66}$$

In the R-R sector, instead, we have

$$|\mathrm{D}p_\psi,\eta\rangle^{(0)\,T}_{\mathrm{R}} = \left(\overline{C}\,\bar{\gamma}^0 \ldots \bar{\gamma}^r\,\frac{1+\mathrm{i}\,\eta\,\bar{\gamma}}{1+\mathrm{i}\eta}\right)_{AB} |A\rangle\,|\widetilde{B}\rangle \,, \tag{67}$$

where $\bar{\gamma}^\alpha$ are the gamma matrices and $\overline{C}$ is the charge conjugation matrix of $SO(1,5)$, $\bar{\gamma} = \bar{\gamma}^0 \ldots \bar{\gamma}^5$, and, finally, $|A\rangle$ and $|\widetilde{B}\rangle$ are spinors of $SO(1,5)$. The matrices of $SO(1,5)$ satisfy the following relations under transposition:

$$\overline{C}^{\,t} = -\,\overline{C}\,, \qquad \bar{\gamma}^{\alpha\,t} = -\,\overline{C}\,\bar{\gamma}^\alpha\,\overline{C}^{\,-1}. \tag{68}$$

In order to compute the fermionic zero-mode contribution to Z_g in Eq. (56), it is convenient to write explicitly the conjugate vacuum states, which are given for the twisted NS-NS sector and for the twisted R-R sector by[27]

$$ {}^{(0)T}_{\rm NS}\langle {\rm D}p_\psi, \eta| = \langle \widetilde{M}|\,\langle L|\left(\widehat{C}\,\widehat{\gamma}^6\ldots\widehat{\gamma}^{5+s}\,\frac{1-{\rm i}\,\eta\,\widehat{\gamma}}{1-{\rm i}\,\eta}\right)_{LM} \tag{69} $$

and
$$ {}^{(0)T}_{\rm R}\langle {\rm D}p_\psi, \eta| = \langle \widetilde{B}|\,\langle A|\left(\overline{C}\bar{\gamma}^0\ldots\bar{\gamma}^r\,\frac{1+{\rm i}\,\eta\,\bar{\gamma}}{1-{\rm i}\,\eta}\right)_{AB}, \tag{70} $$

respectively. By using the previous expressions and performing some straightforward algebra, it is possible to show that

$$ {}^{(0)T}_{\rm NS}\langle {\rm D}p_\psi, \eta_1|{\rm D}p_\psi, \eta_2\rangle^{(0)\,T}_{\rm NS} = 4\,\delta_{\eta_1\eta_2;1} \tag{71} $$

and
$$ {}^{(0)T}_{\rm R}\langle {\rm D}p_\psi, \eta_1|{\rm D}p_\psi, \eta_2\rangle^{(0)\,T}_{\rm R} = -\,4\,\delta_{\eta_1\eta_2;1} \tag{72} $$

for the NS-NS sector and for the R-R sector, respectively.

The previous twisted and untwisted boundary states are the building blocks for constructing the boundary state associated with the two kinds of fractional branes corresponding to the two irreducible representations of the orbifold group Z_2 given in Eqs. (45) and (46) and that associated to a bulk brane. Since, as one can see comparing Eqs. (45) and (46), the only difference between the two fractional Dp-branes of Z_2 is the sign for the generator $\gamma(g)$, the boundary states associated to them will just differ for a sign in front of the twisted sector. This means that the boundary states associated to the two fractional D-branes will be given by

$$ |{\rm D}p\rangle_1 \;=\; |{\rm D}p\rangle^U + |{\rm D}p\rangle^T, \tag{73} $$

$$ |{\rm D}p\rangle_2 \;=\; |{\rm D}p\rangle^U - |{\rm D}p\rangle^T. \tag{74} $$

On the other hand, a bulk brane is not coupled to the twisted sector and the corresponding boundary state can be obtained by simply summing-up the boundary states of a fractional Dp-brane of type 1 and one of type 2. Indeed, by summing Eqs. (73) and (74) one sees that the twisted contribution cancel and one is left with 2 times the untwisted boundary state, which is precisely that of a bulk brane:

$$ |{\rm D}p\rangle_b = |{\rm D}p\rangle_1 + |{\rm D}p\rangle_2 = 2\,|{\rm D}p\rangle^U, \tag{75} $$

where the subscript b in the last equation stands for *bulk*. Since the tension of a brane is proportional to the normalization of the corresponding boundary state, Eq. (75) shows that a fractional brane has a tension that is 1/2 of that of a bulk brane. All previous considerations can be generalized to any orbifold of the ADE series. In particular, in these more general cases Eq. (75) becomes

$$|\mathrm{D}p\rangle_b = \sum_I |\mathrm{D}p\rangle_I \,, \tag{76}$$

where the convention on the index I is the same as in the previous section. Eqs. (75) and (76) are nothing else than the closed string counterpart of Eqs. (47) and (49). Once again we see that in an orbifold theory the bulk Dp-branes can be thought of as bound states of more elementary Dp-branes, the so-called fractional branes.

Having determined the boundary state for both the untwisted and the twisted sectors of a fractional Dp-brane, we will use it in the following for computing the couplings of the brane with the closed string fields. That will help us to determine the world-volume action of a fractional Dp-brane and the large distance behaviour of the classical supergravity solution corresponding to it. These are well known things by now and the reader is urged to consult Refs. 26, 20 and 28 for details and explanations. In particular, it is important to stress that by saturating the previously constructed boundary states with the closed string states one gets the couplings of the D-brane with the closed string states that are canonically normalized in the bulk action written in the orbifold covering space. We want, however, to write the couplings corresponding to the fields defining in the physical space. In Refs. 1 and 2 we normalized the bulk action with an overall factor $1/2\kappa^2_{orb}$, where $\kappa_{orb} = \sqrt{2}\,\kappa$ and we took ω_2 in Eqs. (94) to be normalized in such a way that the first integral in Eqs. (94) is equal to $\sqrt{2}$ instead of 1 and the second integral in Eqs. (94) is equal to 1 instead of $1/2$. In this paper we normalize the bulk action with an overall factor $1/(2\kappa^2)$ and we use an ω_2 satisfying the relations in Eqs. (94).

By saturating the boundary state $|\mathrm{D}p\rangle$ with the massless closed string states of the various sectors, one can determine which are the fields that couple to the fractional Dp-brane. In particular, following the procedure found in Ref. 26 and reviewed in Ref. 28, one can find that in the untwisted sector the Dp-brane emits the graviton[d] $h_{\mu\nu}$, the dilaton ϕ and the (p+1)-form potential C_{p+1}. For the case $s = 0$, as for instance, is the case for a fractional D3-brane, the couplings of these fields with the boundary state are explicitly given[29] by

$$\begin{aligned}
\langle \mathrm{D}p \,|\, h\rangle &= -\frac{T_p}{2}\, h_\alpha^{\ \alpha} V_{p+1} \,, \\
\langle \mathrm{D}p \,|\, \phi\rangle &= \frac{T_p}{2\kappa}\,\frac{3-p}{4}\,\phi\, V_{p+1} \,, \\
\langle \mathrm{D}p \,|\, C_{p+1}\rangle &= \frac{T_p}{2\kappa}\, C_{01\ldots p} V_{p+1} \,,
\end{aligned} \tag{77}$$

[d] We recall that the graviton field and the metric are related by $G_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, where $2\kappa^2 = (2\pi)^7(\alpha')^4 g_s^2$.

where $T_p = \sqrt{\pi}\,(2\pi\sqrt{\alpha'})^{(3-p)}$, appearing in the normalization of the boundary state, is related to the brane tension in units of the gravitational coupling constant,[26,27] V_{p+1} is the (infinite) world-volume of the Dp-brane, and the index α labels its $p+1$ longitudinal directions.

By doing this same analysis in the twisted sectors, we find that the boundary state $|\mathrm{D}p\rangle$ emits a massless scalar $\widetilde{b}$ (the modulus related to B_2, namely the singlet under $SO(4)_{INT}$ in Eq. (26)) from the NS-NS sector, and a $(p+1)$-form potential A_{p+1} from the R-R sector. These fields exist only at the orbifold fixed point $x^6 = x^7 = x^8 = x^9 = 0$ and

$$\begin{aligned} \langle \mathrm{D}p \,|\, \widetilde{b}\rangle &= \mp \frac{T_r}{2^{s/2}\,2\kappa}\,\frac{1}{2\pi^2\alpha'}\,\widetilde{b}Vr+1\,, \\ \langle \mathrm{D}p \,|\, A_{p+1}\rangle &= \pm \frac{T_r}{2^{s/2}\,2\kappa}\,\frac{1}{2\pi^2\alpha'}\,A_{01\ldots p}\,V_{r+1}\,, \end{aligned} \tag{78}$$

where V_{r+1} is the (infinite) world-volume of the Dp-brane that lies outside the orbifold, represent their couplings with the boundary state.[29] The upper sign refers to fractional branes of type 1 while the lower sign to fractional branes of type 2 (we recall, again, that on the orbifold we are considering there are two types of fractional branes).

From the explicit couplings (77) and (78), it is possible to infer the form of the world-volume action of a fractional Dp-brane. Of course, the boundary state approach allows to obtain only the terms of the world-volume action that are linear in the bulk fields. However, terms of higher order can be determined with other methods.[2] Numerically, both couplings (77) and (78) are the same as those we derived in Refs. 1 and 2, the only difference being, as already stressed, that here we are expressing them in terms of κ instead of κ_{orb}. Therefore, we obtain (in the Einstein frame) that

$$S^{\mathrm{Dp}}_{\mathrm{b}}\Big|_{\mathrm{U}} = -\frac{\tau_p}{2}\int d^{p+1}x\, \mathrm{e}^{(p-3)/4\phi}\sqrt{-\det G_{\alpha\beta}} + \frac{\tau_p}{2}\int C_{p+1}\,, \tag{79}$$

where $G_{\alpha\beta}$ is the induced metric and $\tau_p \equiv T_p/\kappa = (2\pi\sqrt{\alpha'})^{-p}/(g_s\sqrt{\alpha'})$ is the tension of the bulk branes which is also equal to that of the branes in flat space. It is easy to check that this action correctly accounts for the couplings from Eqs. (77).

Equation (79) shows that fractional branes have a tension that in the case of the orbifold under consideration is just a half of that of a bulk brane. That is the reason of the name fractional branes. Similarly we can see that its charge with respect to the R-R field C_{p+1} is a half of that carried by bulk Dp-branes. As already noticed, the same conclusion holds from Eq. (75) and, for more general orbifolds, from Eq. (76). Summarizing, for a generic orbifold theory, calling τ_p and μ_p the tension and the charge of a bulk Dp-brane respectively, those of fractional branes are

$$\tau_{p,I} = \frac{d_I}{|\Gamma|}\,\tau_p = \frac{d_I}{|\Gamma|}\,\frac{T_p}{\kappa}\,, \tag{80}$$

$$\mu_{p,I} = \frac{d_I}{|\Gamma|}\,\mu_p = \frac{d_I}{|\Gamma|}\,\frac{T_p}{\kappa}\,. \tag{81}$$

For the twisted fields, instead, things are slightly more complicated. Using the couplings in Eqs. (78) one can write

$$\begin{aligned} S_{\mathrm{b}}^{\mathrm{Dp}}\Big|_{\mathrm{T}} =& \pm \frac{1}{2\pi^2\alpha'}\,\frac{\tau_r}{2\cdot 2^{s/2}} \\ &\times \left\{ -\int d^{r+1}x\, \mathrm{e}^{(p-3)/4\phi}\sqrt{-\det G_{\alpha\beta}}\,\widetilde{b} + \int \mathcal{A}_{r+1} + \dots \right\}, \end{aligned} \tag{82}$$

where in the first term the four-dimensional induced metric has been inserted to enforce reparametrization invariance on the world-volume, while the ellipses stand for terms of higher order which are not accounted by the boundary state approach but which, in principle, can be present.

In the case of a fractional Dp-brane with no world-volume directions along the orbifold, we finally get the boundary actions[30]

$$\begin{aligned} S_1 =& -\frac{\tau_p}{2}\int d^{p+1}x\, \mathrm{e}^{(p-3)/4\phi}\sqrt{-\det G_{\alpha\beta}}\left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) \\ &+ \frac{\tau_p}{2}\int_{V_{p+1}} \left[C_{p+1}\left(1 + \frac{\tilde{b}}{2\pi^2\alpha'}\right) + \frac{1}{2\pi^2\alpha'}\,\mathcal{A}_{p+1}\right], \end{aligned} \tag{83}$$

$$\begin{aligned} S_2 =& -\frac{\tau_p}{2}\int d^{p+1}x\, \mathrm{e}^{(p-3)/4\phi}\sqrt{-\det G_{\alpha\beta}}\left(1 - \frac{\tilde{b}}{2\pi^2\alpha'}\right) \\ &+ \frac{\tau_p}{2}\int_{V_{p+1}} \left[C_{p+1}\left(1 - \frac{\tilde{b}}{2\pi^2\alpha'}\right) - \frac{1}{2\pi^2\alpha'}\,\mathcal{A}_{p+1}\right], \end{aligned} \tag{84}$$

for fractional branes of type 1 and type 2, respectively. The world-volume action of a bulk brane is given by the sum of the two previous ones, namely $S_b = S_1 + S_2$, and is coincident with the world-volume action of a Dp-brane in flat space. In fact, the bulk Dp-branes of an orbifold are pretty much similar to the normal branes in flat space and indeed are only coupled to closed string fields of the untwisted sector as the metric, the dilaton and the R-R field C_{p+1}.

Fractional Dp-branes are instead characterized by the fact that they couple also to the fields of the twisted sector. As already noticed, from the supergravity point of view twisted fields are the zero modes of the usual NS-NS and R-R forms present in the type II spectrum dimensionally reduced on the shrinking cycles of the orbifold. Indeed, the orbifolds R^4/Γ can be seen as singular limit of smooth ALE spaces where the compact 2-cycles characterizing these smooth spaces shrink to zero size. This can suggest some relation between fractional branes and higher dimensional branes wrapped on these exceptional cycles. We will come back on this issue in the next section.

Let us end this section by using the boundary state formalism to compute the asymptotic behaviour of the various fields the fractional branes couple to, in the corresponding classical brane solution (see Ref. 26 for an explanation of this technique). Considering a stack of N_1 fractional Dp-branes of type 1 and N_2 branes of type 2, we find that, to leading order in $N_1 g_s$ and $N_2 g_s$, the metric is

$$ds^2 \sim \left(1 - \frac{Q_p}{r^{7-p}} \times \frac{7-p}{8}\right)\eta_{\alpha\beta}dx^\alpha dx^\beta + \left(1 + \frac{Q_p}{r^{7-p}} \times \frac{p+1}{8}\right)\delta_{ij}dx^i dx^j\,, \quad (85)$$

the dilaton is

$$\phi \sim \frac{3-p}{4}\,\frac{Q_p}{r^{7-p}}\,, \quad (86)$$

and the R-R untwisted field is

$$C_{p+1} \sim -\frac{Q_p}{r^{7-p}}\, dx^0 \wedge dx^1 \ldots \wedge dx^p\,, \quad (87)$$

where

$$Q_p \equiv \frac{\kappa T_p}{(7-p)\Omega_{8-p}}\,(N_1+N_2)\,, \quad \Omega_q = \frac{2\pi^{(q+1)/2}}{\Gamma(\frac{p+1}{2})}\,, \quad r^2 = \sum_i (x^i)^2\,. \quad (88)$$

The large distance behaviour of the twisted fields that are stuck at the orbifold fixed point and therefore depend only on the transverse directions outside of the orbifold, is given by

$$\tilde{b} = \frac{K_p}{\rho^{3-r}}\,, \qquad K_p = \frac{2\kappa T_r}{2^{s/2}(r-3)\,\Omega_{4-r}}\,\frac{1}{2\pi^2\alpha'}\,(N_1-N_2)\,, \quad (89)$$

for the fluctuation of the b field, and by

$$\mathcal{A}_{p+1} = -\frac{K_p}{\rho^{3-r}}\, dx^0 \wedge dx^1 \ldots \wedge dx^p\,, \qquad \rho^2 = x_{p+1}^2 + \ldots + x_5^2\,, \quad (90)$$

for the R-R twisted field. In Sec. 7 we will write the equations of motion of type IIB supergravity, and restricting us to the case $p = 3$ (and $r = 0$), we will determine the complete supergravity solution describing a bound state of N_1 fractional D3-branes of type 1 and N_2 of type 2.

5 Fractional branes as wrapped branes

In this section we investigate at a deeper level the idea that we have just anticipated, namely that fractional branes are just a certain kind of wrapped branes. Let us first state this correspondence in a precise mathematical fashion. Later, we will test its validity in more concrete terms by comparing with the results obtained in the previous section. The orbifolds R^4/Γ are singular limits of ALE spaces, the latter being non compact four dimensional manifolds uniquely characterized, for any given Γ, by the presence of compact holomorphic 2-cycles (which topologically are spheres) which shrink to zero size in the orbifold limit. A well established mathematical result (known as the McKay correspondence[31]) states that for any given ALE space these 2-cycles are in one-to-one correspondence with the simple roots α_I of a simply-laced Lie algebra (the ADE *extended* Dynkyn diagrams) and these roots correspond to the irreducible representations D_I of Γ. Actually, the number of cycles equals the number of roots of the non-extended Dynkyn diagrams, and hence is one less than the full number of roots and irreducible representations. Indeed the trivial irreducible representation, $\mathcal{D}_0$ (the one defined by Eq. (45), for the Z_2 orbifold) is associated with a cycle $\mathcal{C}_0$ which is minus the sum of all other cycles $\mathcal{C}_i$, i.e. $\mathcal{C}_0 = -\sum_{i=1}^{n-1} d_i \mathcal{C}_i$. The corresponding simple root, α_0, is the extra root of the extended Dynkyn diagram. Schematically, one has

$$\alpha_I \leftrightarrow \mathcal{C}_I \leftrightarrow \mathcal{D}_I\,, \qquad \text{with} \quad I = 0,\, 1,\, \cdots,\, n-1\,. \tag{91}$$

Recalling from Sec. 3 that fractional branes are uniquely identified by the irreducible representations $\mathcal{D}_I$ of Γ, one can then suspect the existence of a relation between fractional branes and the shrinking cycles of the orbifold R^4/Γ. This is indeed the case. The precise statement is as follows. A fractional Dp-brane is a D$(p+2)$-brane wrapped on a compact 2-cycle of a ALE manifold, in the limit in which the volume of such cycle vanishes and the ALE space degenerates to the orbifold R^4/Γ. These branes can exist in the orbifold limit because, although the size of the compact cycle shrinks to zero, a non vanishing B_2-flux persists on it, keeping the brane tensionful. For a general orbifold the precise value of this flux is

$$\int_{\mathcal{C}_i} B_2 = \left(2\pi\sqrt{\alpha'}\right)^2 \frac{d_i}{|\Gamma|}\,. \tag{92}$$

As already noticed, in the case of the Z_2 orbifold one has $d_i/|\Gamma| = 1/2$. This non-vanishing background flux is not put by hand but, as shown in Ref. 32, is required to keep string theory conformal on the orbifold. It is this requirement that makes the existence of fractional D-branes as stable non-perturbative states of the string spectrum possible. As already noticed, there is one cycle less than the number of irreducible representations. In fact, the fractional Dp-brane associated to the trivial representation $\mathcal{D}_0$ is obtained by wrapping a D$(p+2)$-brane on $\mathcal{C}_0$, with an additional background flux of the world-volume gauge field $\mathcal{F}$ such that $\int_{\mathcal{C}_0}\mathcal{F}_2 = 2\pi$. As we will explicitly show in the case of $\Gamma = Z_2$, this assures that such a brane gets an untwisted Dp-brane charge of the same sign of that of the branes associated to the non-trivial representations. This in fact guarantees that it is a brane and not an anti-brane.

Let us now explicitly verify all these statements by considering our working example, the orbifold Z_2. As already discussed, in this case we have just one shrinking cycle, $\mathcal{C}_1$, and just two different kinds of fractional Dp-branes. The fractional brane of type 1 should correspond to a D$(p+2)$-brane wrapped on $\mathcal{C}_1$. The fractional brane of type 2 to a D$(p+2)$-brane wrapped on $\mathcal{C}_0 = -\,\mathcal{C}_1$ and with a non vanishing $\mathcal{F}$-flux on it. Let us then consider the world-volume action of a wrapped D$(p+2)$-brane and see how it actually gives rise to the actions in Eqs. (83) and (84) in the limit of shrinking cycle. In the Einstein frame a D$(p+2)$-brane world-volume action has the form

$$\begin{aligned} S = &-\,\tau_{p+2}\int d^{p+3}x\,\exp\left(\frac{p-1}{4}\,\phi\right)\sqrt{-\det\left[G_{\alpha\beta} + \mathrm{e}^{-\phi/2}\left(B_{\alpha\beta} + 2\pi\alpha'\mathcal{F}_{\alpha\beta}\right)\right]} \\ &+\,\tau_{p+2}\int\left(C\wedge \mathrm{e}^{B+2\pi\alpha'\mathcal{F}}\right)_{p+3}\,. \end{aligned} \tag{93}$$

The smooth limit of the Z_2 orbifold is the well known Eguchi-Hanson space, which has an antiself-dual two-form ω_2 which is associated to the compact 2-sphere $\mathcal{C}_1$ whose radius we define as a. We use conventions where ω_2 satisfies the following properties:

$$\omega_2 = -\,{}^*\omega_2\,, \qquad \int_{\mathcal{C}_1}\omega_2 = 1\,, \qquad \int_{R^4/Z_2}{}^*\omega_2\wedge\omega_2 = \frac{1}{2}\,. \tag{94}$$

The compact cycle vanishes in the orbifold limit $a \to 0$ but, as already said, a non-zero B_2-flux persists on it. In order to obtain, from the action in Eq. (93), the world-volume actions of the two fractional Dp-branes given in Eqs. (83) and(84) we should start from an action with no world-volume fields switched-on along the $p+1$ non-compact directions of the world-volume. That is to say, both B and $\mathcal{F}$ are non-vanishing only on the cycle $\mathcal{C}_1$. The action in Eq. (93)

describes a brane wrapped on $\mathcal{C}_1$ by considering the world-volume V_{p+3} as a product of the $p+1$-dimensional volume V_{p+1} times the volume of the cycle $\mathcal{C}_1$ and keeping only those fields that are left in the limit of $a \to 0$:

$$V_{p+3} = V_{p+1} \times \mathcal{C}_1 \,, \quad B_2 = b\,\omega_2 \,, \quad C_{p+3} = \mathcal{A}_{p+1} \wedge \omega_2 \,. \tag{95}$$

By noticing that the metric has no support on the vanishing cycle, one can easily factorize the matrix in the determinant in the action (93) as a direct product of a $(p+1) \times (p+1)$ matrix $G_{\alpha\beta}$ times a 2×2 matrix where only B and $\mathcal{F}$ are present. Let us consider the case of a fractional brane of type 1 first. We want to show that it corresponds to a D$(p+2)$-brane wrapped on $\mathcal{C}_1$ with no $\mathcal{F}$-flux. Inserting the expressions (95) into Eq. (93) one gets

$$\begin{aligned} S &= \tau_{p+2} \left\{ -\int d^{p+1}x \, \exp\left(\frac{p-3}{4}\,\phi\right) \sqrt{-\det G_{\alpha\beta}} \int_{\mathcal{C}_1} B \right. \\ &\qquad \left. + \int C_{p+1} \int_{\mathcal{C}_1} B + \int \mathcal{A}_{p+1} \right\} = \\ &= \tau_p \left\{ -\int d^{p+1}x \, \exp\left(\frac{p-3}{4}\,\phi\right) \sqrt{-\det G_{\alpha\beta}} \left(\frac{1}{2} + \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right) \right. \\ &\qquad \left. + \int C_{p+1} \left(\frac{1}{2} + \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right) + \frac{1}{(2\pi\sqrt{\alpha'})^2} \int \mathcal{A}_{p+1} \right\} . \end{aligned} \tag{96}$$

In the second step we have used the fact that

$$(2\pi\sqrt{\alpha'})^2\,\tau_{p+2} = \tau_p \quad \text{and} \quad \int_{\mathcal{C}_1} B = b = (2\pi\sqrt{\alpha'})^2 \left(\frac{1}{2} + \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right), \tag{97}$$

where $\tilde{b}$ is the fluctuation of the B_2-flux around the background value given in Eq. (92). The above action precisely coincides with that in Eq. (83), as anticipated. By repeating the same reasoning for a D$(p+2)$-brane which is wrapped on $\mathcal{C}_0 = -\mathcal{C}_1$ but with an additional $\mathcal{F}$-flux such that $\int_{\mathcal{C}_0} \mathcal{F}_2 = 2\pi$ one easily gets

$$\begin{aligned} S &= \tau_{p+2} \left\{ -\int d^{p+1}x \, \exp\left(\frac{p-3}{4}\,\phi\right) \sqrt{-\det G_{\alpha\beta}} \int_{\mathcal{C}_0} (B + 2\pi\alpha'\mathcal{F}) \right. \\ &\qquad \left. + \int C_{p+1} \int_{\mathcal{C}_0} (B + 2\pi\alpha'\mathcal{F}) + \int \mathcal{A}_{p+1} \right\} \end{aligned}$$

$$= \tau_p \left\{ - \int d^{p+1}x \, \exp\left(\frac{p-3}{4}\,\phi\right) \sqrt{-\det G_{\alpha\beta}} \left(\frac{1}{2} - \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right) \right.$$
$$\left. + \int C_{p+1} \left(\frac{1}{2} - \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right) - \frac{1}{(2\pi\sqrt{\alpha'})^2} \int \mathcal{A}_{p+1} \right\}, \tag{98}$$

which is just the action in Eq. (84). In deriving the above equation it is worth noting that

$$\int_{\mathcal{C}_0} (B + 2\pi\alpha'\mathcal{F}) = - \int_{\mathcal{C}_1} B + 2\pi\alpha' \int_{\mathcal{C}_0} \mathcal{F} \tag{99}$$
$$= -\,(2\pi\sqrt{\alpha'})^2 \left(\frac{1}{2} + \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right) + (2\pi\sqrt{\alpha'})^2 = (2\pi\sqrt{\alpha'})^2 \left(\frac{1}{2} - \frac{1}{(2\pi\sqrt{\alpha'})^2}\,\tilde{b}\right)$$

From the last equation it is clear that, as anticipated, the presence of the $\mathcal{F}$-flux has the effect of letting the asymptotic value of the *untwisted* charge to be unchanged. By summing up the two actions (96) and (98) one gets back the world-volume action of a bulk brane, according to the idea that bulk branes, in an orbifold theory, can be thought of as bound states of fractional branes of different kinds. Again, the procedure described throughout this section for the the Z_2 orbifold, can be easily extended to more general Γ's. We refer to Ref. 11 for a complete treatment of these more general cases.

6 Requirements of supersymmetry

The goal of this section and the subsequent one is to obtain the supergravity solution describing (a bound state of) fractional D3-branes on the Z_2 orbifold. In this section we study the constraints that supersymmetry imposes on the solution. We do not do this just for completeness, but because, as we shall see, supersymmetry actually imposes certain conditions on the fields entering the solution which drastically simplify the structure of the equations of motion and, correspondingly, the derivation of the solution itself.

We are interested, as usual, in classical supersymmetric backgrounds where the dilatino λ and the gravitino ψ_M are consistently put to zero. Moreover, in order to insure supersymmetry, we require that the supersymmetry variations of both λ and ψ_M be vanishing. In this way we will obtain some constraints on the ansatz. The following[33] is a compact way of writing the gravitino and dilatino variation:

$$\kappa\,\delta\psi_M = \Big(D_M - \frac{i}{2}\,Q_M\Big)\epsilon + \frac{i}{16\cdot 5!}\,F_{M_1\dots M_5}\Gamma^{M_1\dots M_5}\Gamma_M\epsilon$$
$$- \frac{1}{16}\left(2\,\not{\tilde{G}}_{(3)}\Gamma_M + \Gamma_M\,\not{\tilde{G}}_{(3)}\right)\epsilon^*, \qquad (100)$$

$$\kappa\,\delta\lambda = i\left(\not{P}\epsilon^* - \frac{1}{4}\,\not{\tilde{G}}_{(3)}\epsilon\right), \qquad (101)$$

where $\not{\tilde{G}}_{(3)} = (1/3!)\,\Gamma^{MNP}\tilde{G}_{MNP}$, $D_M = \partial_M + (1/4)\,\omega_{Mrs}\Gamma^r\Gamma^s$ is the covariant derivative with respect to the metric g_{MN},

$$P_M = \frac{\partial_M B}{1 - BB^*}\,, \qquad Q_M = \frac{Im\,(B\,\partial_M B^*)}{1 - BB^*}\,,$$
$$B = \frac{1 + i\tau}{1 - i\tau}\,, \qquad \tilde{G}_{(3)} = i\left(\frac{1 + i\tau^*}{1 - i\tau}\right)^{1/2} e^{\phi/2}G_3\,, \qquad (102)$$

and ϵ is a complex ten-dimensional spinor with definite chirality: $\Gamma_{11}\epsilon = -\epsilon$, $\Gamma_{11} = \Gamma^{\underline{0}}\dots\Gamma^{\underline{9}}$, denoting flat indices by the underlined indices. Finally, the complex scalar τ and the complex 3-form G_3 are equal to

$$\tau = C_0 + i\,e^{-\phi} \qquad \text{and} \qquad G_3 = F_3 + \tau H_3\,, \qquad (103)$$

where $F_3 = dC_2$ and $H_3 = dB_2$. Notice that fractional branes actually couple to G_3 (or better to say, to the twisted fields arising from its dimensional reduction on the vanishing cycle of the orbifold, see previous sections), this being a specific feature of a general class of supergravity solution recently discussed in the literature (for an explicit example on smooth ALE spaces, see for instance, Ref. 34). For this reason, in the following we closely follow the approach discussed in Refs. 35 and 36, where the supersymmetry constraints for solutions with non trivial G_3-flux have been discussed. As far as our orbifold is concerned, these include both fractional D3 and D7 branes while D1 and D5 branes belong to a different class of solutions (see the above cited references for details). An ansatz compatible with the symmetries of the system is

$$ds^2 = Z^{-1/2}\eta_{\alpha\beta}dx^\alpha dx^\beta + Z^{1/2}e^{-\phi(x^i)}\delta_{ij}dx^i dx^j + Z^{1/2}\delta_{mn}dx^m dx^n\,, \qquad (104)$$

and

$$\tilde{F}_5 = dH^{-1}\wedge V_4 +^* \left(dH^{-1}\wedge V_4\right)\,, \qquad (105)$$

with $\alpha,\beta = 0\dots 3$, $i,j = 4,\,5$, $m,n = 6\dots 9$. As far as the transverse space is concerned, it is convenient to introduce complex coordinates z^i ($i = 1,\,2,\,3$) as

$$z_1 = x^4 + ix^5, \quad z_2 = x^6 + ix^7, \quad z_3 = x^8 + ix^9. \qquad (106)$$

Let us start studying the dilatino variation. The vanishing of the dilatino equation is obtained by a separate cancellation between the two terms appearing in Eq. (101):

$$\not{P}\epsilon^* = 0\,, \qquad \not{\tilde{G}}_{(3)}\epsilon = 0\,. \tag{107}$$

In the case of a D3-brane the first condition is simply solved by considering a constant τ. This is, however, not a good solution for the D7-brane since it is coupled to τ. On the other hand, τ can depend only on the coordinates transverse to the world-volume of the D7-brane, namely on z_1 and $\bar{z}_1$. But, if we assume that in this case τ is a holomorphic function of z_1 (namely $\partial_{\bar{1}}\tau = 0$) and impose the condition $\Gamma^1\epsilon^* = 0$ (the index 1 corresponds to the first complex variable introduced in Eq. (106)) on the spinor ϵ, it is easy to check that the first condition in Eq. (107) is again satisfied.

The second of Eqs. (107) fixes some components of $\tilde{G}_{(3)}$ to be zero. In order to satisfy it we have to extend the previous condition $\Gamma^1\epsilon^* = 0$ to the other two values of i. Moreover we assume that $\tilde{G}_3$ has only non-zero components if the indices are along the six-dimensional space transverse to the D3-brane. With these two assumptions the second equation in (107) is satisfied if we impose[36]

$$\tilde{G}_{ijk} = \tilde{G}_{i\bar{i}k} = 0\,, \qquad i,j,k = 1,\,2,\,3\,. \tag{108}$$

Let us now study the gravitino variation. By imposing, again, a separate cancellation between the terms depending on $\not{\tilde{G}}_{(3)}$ and the other ones in Eq. (100), we arrive at

$$\left(D_M - \frac{i}{2}Q_M\right)\epsilon + \frac{i}{5!\cdot 16}\,F_{M_1\ldots M_5}\Gamma^{M_1,\ldots M_5}\Gamma_M\,\epsilon = 0\,, \tag{109}$$

$$\left(2\not{\tilde{G}}_{(3)}\Gamma_M + \Gamma_M\not{\tilde{G}}_{(3)}\right)\epsilon^* = 0\,. \tag{110}$$

Once specified for the longitudinal components of the D3-brane, Eq. (109) can be reduced by using Eqs. (104) and (105) to

$$\partial_\alpha\epsilon - \frac{1}{8}\,\Gamma_w\Gamma_\alpha\left[1 - \frac{1}{2}\left(\frac{Z}{H} + \frac{H}{Z}\right)\Gamma_{(5)}\right]\epsilon = 0\,, \tag{111}$$

where $\Gamma_w = (\gamma^i\partial_i + \gamma^m\partial_m)\ln Z$, $\Gamma_{(5)} = i\,\Gamma^{\underline{0}}\ldots\Gamma^{\underline{3}}$. Equation (111) is clearly satisfied by choosing $Z = H$ and by taking a spinor ϵ that does not depend on the coordinates of the longitudinal directions of the D3-brane and that has positive four-dimensional chirality: $\Gamma_{(5)}\epsilon = \epsilon$. This equation, together with the condition $\Gamma_{11}\epsilon = -\,\epsilon$ implies that

$$\Gamma_{(7)}\epsilon = -\,i\,\epsilon\,, \qquad \Gamma_{(7)} = \Gamma^{\underline{4}}\ldots\Gamma^{\underline{9}}\,. \tag{112}$$

This condition leads to the identity $\not{\tilde{G}}_{(3)}\epsilon = i\not{\tilde{G}}_{(3)}\Gamma_{(7)}\epsilon$, which, when solved, gives the important relation[37,38]

$$-i\,\tilde{G}_{(3)} =^{*6} \tilde{G}_{(3)}\,, \tag{113}$$

where $*6$ denotes the Hodge dual in the six-dimensional transverse space of a D3-brane.

In order to study the conditions that follow from the components transverse to the D3-brane of Eq. (109), it is convenient to decompose the 10-dimensional Dirac matrices in terms of the 4 and 6-dimensional ones by writing $\epsilon = \xi\otimes\eta$, where ξ and η are spinors in four and six dimensions, respectively, satisfying the conditions

$$\gamma^5\xi = \xi\,, \qquad \gamma^7\eta = -\,i\eta\,, \qquad \Gamma_{11} = -\,i\,\gamma^5\otimes\gamma^7. \tag{114}$$

By using the previous decomposition in the transverse components in Eq. (109) we obtain

$$\partial_i\eta + \frac{1}{4}\,\partial_i\,\alpha(z_1)\,\eta - \frac{1}{8}\,\partial_i\ln H\eta = 0 \tag{115}$$

and

$$\partial_{\bar{i}}\,\eta - \frac{1}{4}\,\partial_{\bar{i}}\,\bar{\alpha}(\bar{z}_1)\,\eta - \frac{1}{8}\,\partial_{\bar{i}}\,\ln H\eta = 0\,, \tag{116}$$

with $\alpha(z_1) + \bar{\alpha}(\bar{z}_1) = \phi + \ln(1 - BB^*)$. The previous system of equations is simply solved by choosing $\eta = H^{1/8}\exp\big[-(\alpha(z_1) - \bar{\alpha}(\bar{z}_1))/4\big]\chi$, with χ being a constant spinor.

Finally, Eq. (110) fixes that some other components of $\tilde{G}_{(3)}$ are zero. In particular, it imposes[36]

$$\tilde{G}_{\bar{i}\bar{j}\bar{k}} = \tilde{G}_{\bar{i}\bar{j}k} = 0\,. \tag{117}$$

Collecting together Eqs. (108) and (117) we conclude that the only non-vanishing components of $\tilde{G}_3$ are $\tilde{G}_{\bar{i}jk}$ with $i \neq j, k$. This implies that $\tilde{G}_{(3)}$ is a (2,1) form. In the next section we will see that this property simplifies the equation of motion.

7 Classical solution for fractional D-branes

In this section, by considering type IIB supergravity on the Z_2 orbifold, we will derive the complete classical solution describing a bound state of N_1 fractional branes of type 1 and N_2 of type 2. We will see that this solution belongs to a class of type IIB supersymmetric solutions all characterized by the presence of a non-trivial G_3-flux.

Let us start by considering the action (in the Einstein frame) of type IIB supergravity in ten dimensions which can be written as[e]

$$S_{\rm IIB} = \frac{1}{2\kappa^2}\left\{\int d^{10}x\,\sqrt{-\det G}\,R - \frac{1}{2}\int\Big(d\phi\wedge{}^*d\phi + {\rm e}^{-\phi}H_3\wedge{}^*H_3 + {\rm e}^{2\phi}F_1\wedge{}^*F_1 + {\rm e}^{\phi}\widetilde{F}_3\wedge{}^*\widetilde{F}_3 + \frac{1}{2}\widetilde{F}_5\wedge{}^*\widetilde{F}_5 - C_4\wedge H_3\wedge F_3\Big)\right\}, \tag{118}$$

where $$\widetilde{F}_3 = F_3 + C_0\wedge H_3\,,\qquad \widetilde{F}_5 = F_5 + C_2\wedge H_3\,, \tag{119}$$

and $$H_3 = dB_2\,,\quad F_1 = dC_0\,,\quad F_3 = dC_2\,,\quad F_5 = dC_4 \tag{120}$$

are the field strengths of the NS-NS 2-form and of the 0-, 2-, and 4-form potentials of the R-R sector, respectively. As usual, the self-duality constraint ${}^*\widetilde{F}_5 = \widetilde{F}_5$ has to be implemented on shell.

In order to find a classical solution corresponding to fractional D3-branes, we have to add to the above bulk action the corresponding world-volume action, that we call generically S_b. By varying the sum of bulk and boundary actions one can derive equations of motion for various fields of type IIB supergravity. One gets

for dilaton: $$d{}^*d\phi + \frac{1}{2}{\rm e}^{-\phi}H_3\wedge{}^*H_3 - {\rm e}^{2\phi}F_1\wedge{}^*F_1 - \frac{1}{2}\tilde{F}_3\wedge{}^*\tilde{F}_3 + 2\kappa^2\frac{\delta\mathcal{L}_b}{\delta\phi} = 0\,; \tag{121}$$

for axion: $$d\left({\rm e}^{2\phi}\,{}^*F_1\right) - {\rm e}^{\phi}H_3\wedge{}^*\tilde{F}_3 + 2\kappa^2\frac{\delta\mathcal{L}_b}{\delta C_0} = 0\,; \tag{122}$$

for $\mathrm{R-R}\ \ 2-$form: $$d\big({\rm e}^{\phi}\,{}^*\tilde{F}_3\big) + \tilde{F}_5\wedge H_3 + 2\kappa^2\frac{\delta\mathcal{L}_b}{\delta C_2} = 0\,; \tag{123}$$

for NS$-$NS 2$-$ form field: $$d\big({\rm e}^{-\phi}{}^*H_3 + {\rm e}^{\phi}C_0\tilde{F}_3\big) - \tilde{F}_5\wedge F_3 + 2\kappa^2\frac{\delta\mathcal{L}_b}{\delta B_2} = 0\,; \tag{124}$$

for R$-$R 4$-$form field: $$d\,{}^*\tilde{F}_5 + H_3\wedge F_3 + 2\kappa^2\frac{\delta\mathcal{L}_b}{\delta C_4} = 0\,; \tag{125}$$

and finally

$$R_{\mu\nu} - \frac{1}{4\ \cdot 4!}(\tilde{F}_5)_{\mu\rho\sigma\tau\delta}(\tilde{F}_5)_{\nu}{}^{\rho\sigma\tau\delta} + 2\kappa^2\frac{\delta\mathcal{L}}{\delta G^{\mu\nu}} = \frac{1}{2}\left[\partial_\mu\phi\partial_\nu\phi + {\rm e}^{2\phi}\partial_\mu C_0\partial_\nu C_0\right] + T_{\mu\nu} \tag{126}$$

[e] Our conventions for curved indices and forms are the following: $\varepsilon^{0\ldots 9} = +1$; the signature is $(-,+^9)$; $\mu,\nu = 0,\ldots,9$; $\omega_{(n)} = (1/n!)\,\omega_{\mu_1\ldots\mu_n}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$, and $*\omega_{(n)} = \big(\sqrt{-\det G}\big)/(n!\,(10-n)!\big)\ \varepsilon_{\nu_1\ldots\nu_{10-n}\,\mu_1\ldots\mu_n}\,\omega^{\mu_1\cdots\mu_n}\,dx^{\nu_1}\wedge\ldots\wedge dx^{\nu_{10-n}}$.

for the metric, where

$$T_{\mu\nu} = \frac{\mathrm{e}^{-\phi}}{2\;3!}\left[3\,H_{\mu\sigma\rho}H_{\nu}{}^{\sigma\rho} - \frac{G_{\mu\nu}}{4}H^2\right] + \frac{\mathrm{e}^{\phi}}{2\;3!}\left[3\,(\tilde{F}_3)_{\mu\sigma\rho}(\tilde{F}_3)_{\nu}{}^{\sigma\rho} - \frac{G_{\mu\nu}}{4}F_3^2\right] \tag{127}$$

and $\mathcal{L}_b$ is the Lagrangian corresponding to the boundary action. By using the quantities introduced in Eq. (103), it is possible to rewrite the four equations for the dilaton, the axion and the two 2-form potentials in terms of two complex equations as

$$d\,{}^*d\tau + i\mathrm{e}^{\phi}d\tau\wedge{}^*d\tau + \frac{i}{2}\,G_3\wedge{}^*G_3 = 2i\kappa^2\mathrm{e}^{-\phi}\left[\frac{\delta\mathcal{L}_b}{\delta\phi} + i\mathrm{e}^{-\phi}\frac{\delta\mathcal{L}_b}{\delta C_0}\right] \tag{128}$$

and

$$d\,{}^*G_3 + d\tau\wedge\left[\,i\mathrm{e}^{\phi}\,{}^*G_3 + {}^*H_3\right] - i\tilde{F}_5\wedge G_3 = -\,2i\kappa^2\left[\frac{\delta\mathcal{L}_b}{\delta B_2} - \tau\frac{\delta\mathcal{L}_b}{\delta C_2}\right]. \tag{129}$$

For a D3-brane we assume the following ansatz for the metric:

$$ds_{10}^2 = H^{-1/2}\eta_{\alpha\beta}\,dx^\alpha dx^\beta + H^{1/2}\,ds_6^2\,. \tag{130}$$

For the self-dual 5-form field strength we assume

$$\tilde{F}_5 = d\left(\frac{1}{H}\,dx^0\wedge\ldots\wedge dx^3\right) + {}^*d\left(\frac{1}{H}\,dx^0\wedge\ldots dx^3\right). \tag{131}$$

Notice that the six-dimensional space transverse to the D3-brane does not need to be flat. The fact that the warp factors for the metric and for the 5-form field strength are the same is a consequence of supersymmetry, as it has been shown in the previous section. Using Eqs. (130) and (131), together with Eq. (113) and the following equation

$${}^*G_3 = -\,{}^{*6}G_3\,\frac{1}{H}\,dx^0\wedge\ldots\wedge dx^3\,, \tag{132}$$

one can rewrite Eq. (129) as follows:

$$-\,d\,{}^{*6}G_3\wedge\frac{1}{H}\,dx^0\wedge\ldots\wedge dx^3 = -2i\kappa^2\left[\frac{\delta\mathcal{L}_b}{\delta B_2} - \tau\frac{\delta\mathcal{L}_b}{\delta C_2}\right]. \tag{133}$$

On the other hand, we can write:

$$G_3 = G_1\wedge\omega_2\,, \qquad G_1 \equiv d\gamma = dc + i\,db\,, \tag{134}$$

where for a D3-brane we have taken τ to be constant and equal to i (notice that in terms of higher dimensional fields we can write $\gamma = \int_{\mathcal{C}_1}(C_2 + iB_2)$). This is possible because Eq. (128) for τ is identically satisfied if we assume Eq. (113) as required by supersymmetry. On the other hand, since ω_2 is a (1,1)-form, the condition that G_3 should be a $(2,1)$-form implies γ to be an analytic function of the complex variable $z_1 = x^4 + ix^5$(which, from now on, we call z). By using the boundary action in Eq. (133) one arrives at the following equation for γ:

$$- d\,{}^{*6}d\gamma = i\,\frac{2\kappa^2\tau_3}{2\pi^2\alpha'}\,(N_1 - N_2)\,\delta^{(2)}(x)\,dx^4\wedge dx^5\wedge{}^{*_4}\omega_2\,, \tag{135}$$

where the two-dimensional δ-function is in the space spanned by x^4 and x^5. From Eq. (135), after some calculation, one gets

$$\partial_i\partial^i\gamma = 2\,\pi\,i\,K_3\,\delta^{(2)}(x)\,, \tag{136}$$

where K_3 is defined in Eq. (89), $K_3 = 4\pi g_s\,\alpha'(N_1 - N_2)$. The solution reads:

$$\gamma = i\,K_3\log\left(z/z_{(1)}\right), \tag{137}$$

where $z_{(1)} = \epsilon\,\exp\left[-\pi/\,2g_s(N_1 - N_2)\right]$ (this definition ensures that at $|z| = \epsilon$, which is a long-distance regulator for the logarithmic function, the field γ has its correct background value). Let us now consider the equation that determine the warp factor H. Inserting the ansatz (130)-(131) into Eq. (125), we get:

$$\delta^{ij}\partial_i\partial_j H + \frac{1}{2}\,|\partial_z\gamma|^2\,\delta(x^6)\ldots\delta(x^9) + 4\,\pi^3 Q_3\,\delta(x^4)\ldots\delta(x^9) = 0\,, \tag{138}$$

where Q_3 is defined in Eq. (88), $Q_3 = 2\pi g_s(\alpha')^2(N_1 + N_2)$. It is easy to verify that Eq. (126) gives exactly the same equation for the warp factor H. Using standard technique, it is possible to integrate Eq. (138) and obtain

$$H = 1 + \frac{Q_3}{r^4} + \frac{K_3^2}{2\,r^4}\left[\log\left(\frac{r^4}{\epsilon^2(r^2 - |z|^2)}\right) - 1 + \frac{|z|^2}{r^2 - |z|^2}\right]. \tag{139}$$

While the previous expressions, for g_sN_1, $g_sN_2 \ll 1$, reproduce the large distance behaviour obtained from the boundary state in Sec. 4, they finally provide the complete supergravity solution we were searching for.

A closer look at the form of the warp factor H, shows that the metric has a naked singularity at some point $r = r_0$ where indeed H vanishes. The singularity is of repulson type[39] because in its vicinity the gravitational force, that is related to the gradient of G_{00}, is repulsive. The appearance of these kind of singularities is quite a general feature of supergravity solutions corresponding to non-conformal sources and one expects that string theory should be able to resolve them. In this case, as we discuss in the next section, the singularity is resolved by an enhançon mechanism, similar to the one originally discussed in Ref. 21, that excises the region close to the singularity giving a regular solution in the region of the space time where it has a physical meaning.

8 The probe action and the $\mathcal{N}=2$ gauge theory

In this section we will try to see how much can supergravity tell us about the gauge theories describing the low energy effective dynamics of fractional branes. As we have discussed in Sec. 3, fractional D3-branes are described, in general, by non-conformal $\mathcal{N}=2$ super Yang-Mills, at low energy. Therefore, answering the question above could give non trivial insight on non-conformal extensions of the gauge/gravity correspondence. While we will give a precise meaning, at the gauge theory level, to all relevant physical quantities entering the supergravity solution, we will also find out that in order to get a prediction for the *full* moduli space of the $\mathcal{N}=2$ gauge theory one should go beyond the pure supergravity analysis. This is a quite general feature when considering non-conformal extensions of the gauge/gravity correspondence, and the answer, in this case, will be that supergravity does indeed encode the perturbative moduli space of the gauge theory but it is not able to include non-perturbative corrections. As it will become clear in what follows, the so-called enhançon[21] plays a crucial role in all that. Indeed, besides curing the naked singularity in a way that we are going to discuss, it will also put a limit on the range of validity of the gauge/gravity correspondence pointing to a duality where string states play a role, even at low energies. After these anticipations, let us now proceed to our analysis.

As explained in Sec. 3, the low energy theory living on N_1 fractional D3-branes of type 1 and N_2 of type 2 is $\mathcal{N}=2$ super Yang-Mills with gauge groups $SU(\mathrm{N}_1)\times SU(\mathrm{N}_2)$ and two hypermultiplets transforming in the $(N_1,\overline{N}_2)$ and $(\overline{N}_1,N_2)$, respectively. In order to get information on this gauge theory from supergravity we shall use the probe technique. For a review of this technique we refer to Ref. 40.

Let us first consider a fractional D3-brane probe of type 1, carrying a gauge field $F_{\alpha\beta}$ and slowly moving in the supergravity background produced by N_1 fractional D3-branes of type 1 and N_2 of type 2. From the gauge theory point of view this corresponds to the $SU(N_1)\times SU(N_2)\times U(1)$ broken phase of $SU(N_1+1)\times SU(N_2)$ gauge theory and the probe gauge coupling should equal the gauge coupling of the first gauge group, $SU(N_1)$, at an energy scale Λ which is related to the distance $|z|$ at which the probe brane is taken far from the other branes. We fix the static gauge and study the world-volume action of the probe, regarding the transverse coordinates as Higgs fields $\Phi^i=(2\pi\alpha')^{-1}x^i$, and expanding up to quadratic terms in derivatives. By straightforward computations we find that the probe action becomes

$$S = S_0 + S_{\text{gauge}}\,, \tag{140}$$

where S_0 is just the same action of Eq. (83) with $p = 3$, while

$$\begin{aligned} S_{\text{gauge}} = & -\frac{1}{4\pi g_s} \int d^4x \sqrt{-\det G_{\alpha\beta}} \\ & \times \left\{ \frac{1}{4} \,\mathrm{e}^{-\phi}\, G^{\alpha\gamma} G^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} + \frac{1}{2}\, G_{ij} G^{\alpha\beta} \partial_\alpha \Phi^i \partial_\beta \Phi^j \right\} \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_1} B_2 \\ & + \frac{1}{4\pi g_s} \int d^4x \, \frac{1}{4}\, F_{\alpha\beta} \widetilde{F}^{\alpha\beta} \, \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_1} C_2 \,, \end{aligned} \tag{141}$$

where $\widetilde{F}^{\alpha\beta} = (1/2)\, \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$. Inserting in S_0 the supergravity solution obtained in the Sec. 7, one can easily see that S_0 becomes independent of the distance between the probe and the source branes that yield the classical solution. This is in agreement with the fact that there is no interaction between the probe and the source since fractional branes are BPS states and do not exert any force on each other.

Considering now the above equation, we see that the dependence on the function H drops out in this case too, while the kinetic terms for the gauge field strength $F_{\alpha\beta}$ and the scalar fields Φ^i have the same coefficient, in agreement with the fact that the gauge theory living on the brane has $\mathcal{N} = 2$ supersymmetry. Indeed, one gets[f]

$$S_{\text{gauge}} = -\frac{1}{g_1(\mu)^2} \int d^4x \left\{ \frac{1}{2} \partial_\alpha \Phi^i \partial^\alpha \Phi^i + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right\} + \frac{\theta_1}{32\pi^2} \int d^4x F_{\alpha\beta} \tilde{F}^{\alpha\beta} \,, \tag{142}$$

where

$$\frac{1}{g_1(\mu)^2} = \frac{1}{4\pi g_s} \int_{\mathcal{C}_1} \hat{B}_2 = \frac{1}{g_1^2} + \frac{N_1 - N_2}{4\pi^2} \log \mu \tag{143}$$

and

$$\theta_1 = \frac{2\pi}{g_s} \int_{\mathcal{C}_1} \hat{C}_2 = 2\,(N_1 - N_2)\,\theta \tag{144}$$

are the effective Yang-Mills gauge coupling and the θ-angle, respectively. The renormalization group scale is defined by $\mu = |z|/\epsilon \equiv \Lambda/\Lambda_0$, while $g_1^2 = 8\pi g_s$ is the bare coupling, i.e. the value of the gauge coupling at the ultraviolet cutoff[g] $\Lambda_0 = (2\pi\alpha')^{-1}\epsilon$. Equation (143) correctly predicts, from supergravity,

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[f] For the sake of simplicity in this formula (and subsequent ones) we define dimensionless 2-forms $\hat{B}_2$ and $\hat{C}_2$ as $\hat{B}_2 = \left(2\pi\sqrt{\alpha'}\right)^{-2} B_2$ and $\hat{C}_2 = \left(2\pi\sqrt{\alpha'}\right)^{-2} C_2$.

[g] The probe analysis automatically fixes the gravity/gauge theory distance/energy relation to be[41] $|z| = 2\pi\alpha'\Lambda$.

$g_1(\mu)$ to be the running coupling constant of an $\mathcal{N}=2$ supersymmetric gauge theory with gauge group $SU(\mathrm{N}_1)$ and two hypermultiplets in the $(N_1, \overline{N}_2)$ and $(\overline{N}_1, N_2)$, respectively (N_2 is a flavour index here).

By probing now the same background with a fractional D3-brane of type 2, one gets similar results. Eqs. (143) and (144), in particular, become

$$\frac{1}{g_2(\mu)^2} = \frac{1}{4\pi g_s}\left(1 - \int_{\mathcal{C}_1}\hat{B}_2\right) = \frac{1}{g_2^2} - \frac{N_1 - N_2}{4\pi^2}\log\mu \tag{145}$$

$$\theta_2 = -\frac{2\pi}{g_s}\int_{\mathcal{C}_1}\hat{C}_2 = 2\,(N_2 - N_1)\,\theta\,, \tag{146}$$

where $g_2(\mu)$ is the gauge coupling of the second gauge group $SU(N_2)$ at the scale μ and $g_2^2 = 8\pi g_s$ (the role of N_1 and N_2 is exchanged now, N_2 is a colour index while N_1 a flavour one). Again, the supergravity prediction exactly agrees with the gauge theory expectations. Notice that the two β-functions that one gets from Eqs. (143) and (145), have opposite sign, according to the sign of $N_1 - N_2$. For $N_1 > N_2$ (we will always use this convention in the remaining part of this section), the first gauge theory is UV-free while the second one is IR-free. This follows from the relative weights in the two theories of the matter present. In particular, one finds that

$$\beta\,(g_1) = -\,\frac{N_1 - N_2}{8\pi^2}\,g_1(\mu)^3\,, \qquad \beta\,(g_2) = +\,\frac{N_1 - N_2}{8\pi^2}\,g_2(\mu)^3\,. \tag{147}$$

Notice also that according to the general discussion in Sec. 3, the system we have probed actually corresponds to a bound state of say, $M \equiv N_1$–N_2 fractional branes of type 1 and N_2 bulk branes. This is the reason why the β-functions just depend on M, the net number of fractional branes present. The gauge theory living on a bulk brane is conformal invariant and hence, bulk branes are expected not to give any contribution to the β-function. This can also be seen by noticing that

$$\frac{1}{g_1(\mu)^2} + \frac{1}{g_2(\mu)^2} = \frac{1}{4\pi g_s}\,, \tag{148}$$

$$\frac{1}{g_1(\mu)^2} - \frac{1}{g_2(\mu)^2} = \frac{1}{2\pi g_s}\left(\int_{\mathcal{C}_1}\hat{B}_2 - \frac{1}{2}\right), \tag{149}$$

where the sum of the couplings, corresponding to the contribution of the bulk branes, is not running, while the difference, measuring the (net amount of) fractional branes contribution, is running according to the (fluctuation of the) B_2-flux.

The previous results show that supergravity provides the perturbative moduli space of the gauge theory (which is exact at one loop, in this case). But what about non-perturbative contributions? Here is where the enhançon comes into play. We have seen that, because of the presence of a non-vanishing background B_2-flux,[32] the fractional branes are in general tensionful. On the other hand, since the factor in front of the gauge kinetic term in Eq. (142) is the same that gives the effective tension of the brane, as one can see comparing Eqs. (141) and (83), the tension of a fractional brane is running precisely as the gauge coupling constant in Eq. (143). From it one sees that on the geometric locus defined by

$$z_{(1)} = \epsilon \exp\left[-\,\pi\,/\,2(N_1 - N_2)g_s\right], \tag{150}$$

the type 1 brane probe becomes tensionless! This locus is known as the enhançon.[21] This is the point where the fluctuation of the B_2-field cancels precisely its background value. The vanishing of the probe tension indicates that at the distance $z_{(1)}$ new light (string) degrees of freedom come into play,[10] meaning that the supergravity approximation leading to the solution described in the previous section is not valid in the region of space-time $\rho < z_{(1)}$. So, the solution we have found makes sense only at distances bigger than the enhançon and the unwanted repulson singularity, which is cloaked inside it (one can show this is the case for any choice of the parameters), is then excised.

One can immediately recognize what is the meaning of the enhançon from the gauge theory point of view. In fact Eq. (143) shows that the enhançon is the scale $\Lambda_{(1)} = (2\pi\alpha')^{-1} z_{(1)}$ where the gauge coupling $g_1(\mu)$ diverges and where non-perturbative corrections become relevant ($\Lambda_{(1)}$ then corresponds to the dynamically generated scale). This automatically implies that the supergravity solution is only able to reproduce the perturbative moduli space of the gauge theory, while the appearance of the enhançon prevents from using the classical solution to analyze the strong-coupling properties of the gauge theory, where instanton effects should become relevant.[9]

A similar reasoning can be repeated for the probe brane of type 2. The only subtle point one should bare in mind is that the β-function is now IR-free. The consequence is that the corresponding enhançon appears in the UV, which indeed now corresponds to the strongly coupled region of the theory. This region is, however, not really important in our present analysis because, for any value of the parameters, it is always bigger then the UV cut-off $\Lambda_0 = (2\pi\alpha')^{-1}\epsilon$ and hence out of reach of the supergravity solution, whose logarithmic running for twisted fields is regulated at $|z| = \epsilon$. Indeed, the expression of the type 2 enhançon is

$$z_{(2)} = \epsilon \exp\left[\pi\,/\,2(N_1 - N_2)g_s\right] > \epsilon\,. \tag{151}$$

Summarizing, the physical picture one ends up with is that supergravity reproduces the gauge theory of $SU(N_1) \times SU(N_2)$ between the UV-cutoff Λ_0 and the type 1 dynamically generated scale $\Lambda_{(1)}$ (the enhançon), while the extreme IR and UV regions are not accessible by supergravity.

From the previous considerations, one should suspect the existence of a precise relation between the twisted field γ and the period matrix τ of Seiberg-Witten.[9] To find it out we should consider, for each gauge group, the explicit expression of the prepotential $\mathcal{F}$, compute it in the corner of the moduli space consistent with our probe analysis, and finally recall the relation between $\mathcal{F}$ and τ, namely $\tau_{lm} = \partial^2\mathcal{F}/\partial a_l\,\partial a_m$, where a_l and a_m are the moduli of the gauge theory ($l, m = 1, \ldots , N_I + 1$ and $I = 1, 2$ since we have two gauge groups). For any of the two gauge groups, the corresponding perturbative prepotential reads (see for instance, Ref. 42)

$$\mathcal{F}_{pert} = \frac{i}{8\pi} \sum_{l,m=1}^{N_I+1} (a_l - a_m)^2 \log \frac{(a_l - a_m)^2}{\Lambda^2_{(I)}}$$

$$- \frac{i}{8\pi} \sum_{l=1}^{N_I+1} \sum_{k=1}^{2N_J} (a_l + M_k)^2 \log \frac{(a_l + M_k)^2}{\Lambda^2_{(I)}} , \tag{152}$$

where $\Lambda_{(I)}$ is the dynamically generated scale (see above), and M_k are the masses of the hypermultiplets corresponding to strings stretched between branes of different types (the sum over k goes up to the number of hypermultiplets which is 2 times the number of flavours which for the gauge group $U(\mathrm{N}_I)$ is indeed N_J).

The type I probe analysis corresponds to the breaking of the corresponding gauge group $SU(\mathrm{N}_I + 1) \to SU(\mathrm{N}_I) \times U(1)$, which, in terms of the moduli a_l means that one modulus, say $a_{N_I+1} \equiv a$, is taken to be much bigger than the others. This simplifies the prepotential in Eq. (152) as

$$\mathcal{F} = \frac{i}{4\pi} (N_I - N_J)\, a^2 \,\log \frac{a^2}{\Lambda^2_{(I)}} , \tag{153}$$

which implies that

$$\tau_{l,m} \sim 0 , \qquad \text{with} \quad l, m = 1, \ldots , N_I , \tag{154}$$

$$\tau_{N_I+1,N_I+1} \equiv \tau_I = \frac{4\pi}{g_I(\mu)^2}\, i + \frac{\theta_I}{2\pi} = \frac{i}{\pi} (N_I - N_J) \log a/\Lambda/\Lambda_{(I)} . \tag{155}$$

From the above equation one gets the precise relation between the complex twisted field $\hat{\gamma}(z)$ and τ_I computed above:

$$\hat{\gamma}_I(z) = g_s \tau_I(z)\,, \tag{156}$$

where we define

$$\hat{\gamma}_1 = \int_{\mathcal{C}_1} \left(\hat{C}_2 + i\hat{B}_2\right), \qquad \hat{\gamma}_2 = \int_{\mathcal{C}_0} \left(\hat{C}_2 + i\hat{B}_2 + 2\pi i\,\alpha' \mathcal{F}\right). \tag{157}$$

Again, as already stressed, the above identification holds from the supergravity solution only up to the perturbative part of τ.

For the sake of clarity, we finally summarize in table 1 all the relations between supergravity and gauge theory quantities.

GRAVITY		GAUGE THEORY
Transverse coordinates	$x^i \longleftrightarrow \Phi^i$	Higgs field
B_2-flux through $\mathcal{C}_1$	$\int_{\mathcal{C}_1} B_2 \longleftrightarrow g_1(\mu)$	$U(N_1)$ gauge coupling
B_2-flux through $\mathcal{C}_0$	$\int_{\mathcal{C}_0} B_2 \longleftrightarrow g_2(\mu)$	$U(N_2)$ gauge coupling
C_2-flux through $\mathcal{C}_1$	$\int_{\mathcal{C}_1} C_2 \longleftrightarrow \theta_1$	$U(N_1)$ θ-angle
C_2-flux through $\mathcal{C}_0$	$\int_{\mathcal{C}_0} C_2 \longleftrightarrow \theta_2$	$U(N_2)$ θ-angle
IR-regulator	$\epsilon \longleftrightarrow \Lambda_0$	UV-cutoff
Enhançon	$z_{(1)} \longleftrightarrow \Lambda_{(1)}$	Dynamically generated scale

Table 1: Correspondence between gravity and gauge theory parameters. The precise numerical relations can be found in the main text.

Let us end this section with few final observations. As it has been discussed in Ref. 11, by computing the flux of the untwisted field strength $\tilde{F}_5$ through a surface which intersects the z-plane on some given curve Σ, one gets

$$\Phi(\tilde{F}_5) = 2\pi^2 g_s \left(N_1 + N_2 + \frac{g_s}{2\pi}\,(N_1 - N_2)^2 \log\frac{|z|}{\epsilon}\right). \tag{158}$$

From the above equation one can see that the 5-form flux is running. This is a general feature of supergravity solutions generated by non-conformal sources (see for instance, Refs. 43 - 45), indicating that the effective untwisted charge is decreasing through the IR, the decrease being proportional to the net amount of fractional branes present,[h] $M = N_1 - N_2$. Qualitatively, this is a correct result since the untwisted charge corresponds to the number of degrees of freedom of the dual gauge theory, and these are expected to diminish through the IR. It is also correct that this decreasing is proportional to M, since fractional branes make the gauge theory non-conformal and hence are the ones responsible for the running. However, in order to make a more quantitative matching, one should have a better understanding of the physics of the enhançon. There have been various efforts in trying to understand at a deeper level the role of the enhançon for this and other systems (see in particular, Refs. 10, 46, and 47 for interesting discussions on this point). However, quantitative results to go beyond the above successful perturbative analysis, have not yet been obtained. The fact that fractional D-brane probes vanish at the enhançon, has suggested the idea[21] that it is not possible to build up a source made of fractional D-branes located at the origin $r = 0$. Rather, the constituent branes are smeared on the enhançon shell and branes that are coupled to the twisted fields and that therefore become tensionless at the enhançon cannot enter inside the enhançon region. In this way it is clear that, while the exterior solution, due to Gauss' theorem, is of course unchanged, the interior one could look completely different. This picture has been shown to agree both with excision criteria[48,49] and, more concretely, with some solid gauge theory consistency checks which have been done using the SW curve.[47] At the same time the enhançon cannot be the end of the story. Indeed, at the enhançon the moduli space metric, which is proportional to $g_1(\mu)^{-2}$, vanishes. According to Seiberg-Witten,[9] this cannot be the case in the full quantum moduli space of $\mathcal{N}=2$ super Yang-Mills, and in fact you should take into account the instanton corrections, which become relevant at strong coupling, and that give rise to a positive definite moduli space metric. This is consistent with the previous supergravity analysis, which indicates the presence of new light degrees of freedom at the enhançon scale. By including them in the low energy effective action, one should get back an enhançon free and singularity free solution, as discussed recently in Ref. 50 and, on the side of the gauge theory, one should recover the non-perturbative corrections.

Let us end with the following (qualitative) observation. From the gauge theory side, the effective gauge theory will receive corrections proportional to

[h] Notice that at the enhançon the fractional brane contribution vanishes and the flux becomes proportional to the amount of bulk charge contribution, i.e. $\Phi(\tilde{F}_5) = 4\pi^2 g_s N_2$ (recall that we are choosing $N_1 > N_2$ and therefore N_2, according to the discussion in Sec. 3, is the actual number of bulk branes present in the bound state).

powers of the one-instanton contribution to the partition function

$$\exp\left(2\pi\, i\, \tau_1\right) = \left(\frac{\Lambda_{(1)}}{z}\right)^{2\,(N_1-N_2)}. \tag{159}$$

As noticed in Ref. 51, on the string theory side such effects can be due to fractional D-instantons (which are D1 Euclidean branes wrapped on the vanishing cycle $\mathcal{C}_1$ and which indeed become tensionless at the enhançon), whose action is indeed

$$\exp\left(2\pi i\,\frac{\hat{\gamma}_1}{g_s}\right) \equiv \exp\left(2\pi\, i\, \tau_1\right). \tag{160}$$

Unfortunately, it has not been possible until now to make this argument quantitative and in particular to determine the coefficients of the instanton corrections.[52]

Again, all what we have been discussing in this section can be extended to more general orbifolds of the complete ADE series. We refer again to Ref. 11 for a complete treatment of these more general cases.

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