GENERALIZED SUPERSYMMETRIC QUANTUM MECHANICS AND REFLECTIONLESS FERMION BAGS IN 1+1 DIMENSIONS *

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We study static fermion bags in the $1+1$ dimensional Gross-Neveu and Nambu-Jona-Lasinio models. It has been known, from the work of Dashen, Hasslacher and Neveu (DHN), followed by Shei's work, in the 1970's, that the self-consistent static fermion bags in these models are reflectionless. The works of DHN and of Shei were based on inverse scattering theory. Several years ago, we offered an alternative argument to establish the reflectionless nature of these fermion bags, which was based on analysis of the spatial asymptotic behavior of the resolvent of the Dirac operator in the background of a static bag, subjected to the appropriate boundary conditions. We also calculated the masses of fermion bags based on the resolvent and the Gelfand-Dikii identity. Based on arguments taken from a certain generalized one dimensional supersymmetric quantum mechanics, which underlies the spectral theory of these Dirac operators, we now realize that our analysis of the asymptotic behavior of the resolvent was incomplete. We offer here a critique of our asymptotic argument.

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1 Introduction

Many years ago, Dashen, Hasslacher and Neveu (DHN),[1] and following them Shei,[2] used inverse scattering analysis[3] to find static fermion-bag[5,6] soliton solutions to the large-N saddle point equations of the Gross-Neveu (GN)[7] and of the $1+1$ dimensional, multi-flavor Nambu-Jona-Lasinio (NJL)[8] models. In the GN model, with its discrete chiral symmetry, a topological soliton, the so called Callan-Coleman-Gross-Zee (CCGZ) kink,[9] was discovered prior to the work of DHN.

One version of writing the action of the $1+1$ dimensional NJL model is

$$S = \int d^2x \left\{ \sum_{a=1}^{N} \bar{\psi}_a \left[i\partial\!\!\!/ - (\sigma + i\pi\gamma_5) \right] \psi_a - \frac{1}{2g^2} \left(\sigma^2 + \pi^2 \right) \right\} , \tag{1.1}$$

where the $\psi_a \, (a = 1, \ldots, N)$ are N flavors of massless Dirac fermions, with Yukawa couplings to the scalar and pseudoscalar auxiliary fields $\sigma(x), \pi(x)$.[a]

The remarkable discovery DHN made was that all these static bag configurations were *reflectionless*. More precisely, the static $\sigma(x)$ and $\pi(x)$ configurations, that solve the saddle point equations of the NJL model, are such that the Dirac equation

$$\left[i\partial\!\!\!/ - \sigma(x) - i\pi(x)\gamma_5 \right] \psi(x) = 0 \tag{1.2}$$

in these backgrounds has scattering solutions, whose reflection amplitudes at momentum k vanish identically for **all** values of k. In other words, a fermion wave packet impinging on one side of the potential well $\sigma(x) + i\pi(x)\gamma_5$, will be totally transmitted through the well (up to phase shifts, of course).

We note in passing that besides their role in soliton theory,[3,4] reflectionless potentials appear in other diverse areas of theoretical physics.[10,11,12] For a review, which discusses reflectionless potentials (among other things) in the context of supersymmetric quantum mechanics, see Ref. 13.

Since the works of DHN and of Shei, these fermion bags were discussed in the literature several other times, using alternative methods.[14] For a recent review on these and related matters, see Ref. 15. Very recently, static chiral fermion bag solitons[16] in a $1+1$ dimensional model, as well as non-chiral (real scalar) fermion bag solitons,[17] were discussed, in which the scalar field that couples to the fermions was dynamical already at the classical level (unlike the auxiliary fields σ and π in (1.1)).

In many of these treatments, one solves the variational, saddle point equations by performing mode summations over energies and phase shifts. An

[a] The fermion bag solitons in these models arise, as is well known, at the level of the effective action, after integrating the fermions out, and not at the level of the action (1.1).

alternative to such summations is to solve the saddle point equations by manipulating the resolvent of the Dirac operator as a whole, with the help of simple tools from Sturm-Liouville operator theory. The resolvent of the Dirac operator takes care of mode summation automatically.

Some time ago, one of us had developed such an alternative to the inverse scattering method, which was based on the Gelfand-Dikii (GD) identity[18] (an identity obeyed by the diagonal resolvent of one-dimensional Schrödinger operators),[b] to study fermion bags in the GN model[19] as well as other problems.[20] That method was later applied by us to study fermion bags in the NJL model[21] and in the massive GN model.[22] Similar ideas were later used in Ref. 23 to calculate the free energy of inhomogeneous superconductors.

Application of this method in Refs. 19 and 21 reproduced the static bag results of DHN and of Shei in what seems to be a simpler manner than in the inverse scattering formalism. In Ref. 21, we followed the method introduced in Refs. 19, 20 and simply wrote down an efficient, parameter dependent, ansatz for the diagonal resolvent of the Dirac operator in a static $\sigma(x)$, $\pi(x)$ background. Construction of that ansatz was based on simple dimensional analysis, and on the Gelfand-Dikii identity. Nowhere in the construction, did we use the theory of reflectionless potentials. With the help of that ansatz, we were able to reproduce[21] Shei's inverse scattering results, in a similar manner to the reproduction of DHN's results in Ref. 19.

In addition to rederivation of bag profiles, masses and quantum numbers found by DHN and Shei, we tried in Ref. 21 to explain the reflectionless property of the static background in simple terms, by studying the expectation value of the fermion current $\langle j^1(x)\rangle = \langle\bar{\psi}(x)\gamma^1\psi(x)\rangle$ in a given background $\sigma(x) + i\gamma_5\pi(x)$, at spatial asymptotics. However, after careful reexamination, we now realize that the part of the analysis in Ref. 21 on the reflectionless nature of the background was incomplete.[c] We realized this with the help of a certain version of generalized one dimensional supersymmetric (SUSY) quantum mechanics,[24] that underlies the spectral theory of the Dirac operator in (1.2).

This paper offers critique of our asymptotic argument from Ref. 21. The rest of the results in Ref. 21, namely, bag profiles etc., remain intact, and will not be discussed here.

Before discussing this issue in detail, and in order to set our notations, let us recall some basic facts about dynamics of the NJL model. The partition

[b] For a simple derivation of the GD identity, see Refs. 20, 21.

[c] We thank N. Graham and R. Jaffe for useful correspondence on this point.

function associated with (1.1) is [d]

$$\mathcal{Z} = \int \mathcal{D}\sigma\,\mathcal{D}\pi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\exp i\int d^2x\left\{\bar{\psi}\left[i\partial\!\!\!/ - (\sigma + i\pi\gamma_5)\right]\psi - \frac{1}{2g^2}\left(\sigma^2+\pi^2\right)\right\} \tag{1.3}$$

Integrating over the grassmannian variables leads to

$$\mathcal{Z} = \int \mathcal{D}\sigma\,\mathcal{D}\pi\,\exp\left\{iS_{eff}[\sigma,\pi]\right\}\,, \tag{1.4}$$

where the bare effective action is

$$S_{eff}[\sigma,\pi] = -\frac{1}{2g^2}\int d^2x\left(\sigma^2+\pi^2\right) - iN\,\mathrm{Tr}\,\log\left[i\partial\!\!\!/ - (\sigma + i\pi\gamma_5)\right] \tag{1.5}$$

and the trace is taken over both functional and Dirac indices.

This theory has been studied in the limit $N \to \infty$ with Ng^2 held fixed.[7] In this limit (1.3) is governed by saddle points of (1.5) and the small fluctuations around them. The most general saddle point condition reads

$$\begin{aligned}
\frac{\delta S_{\mathrm{eff}}}{\delta\sigma(x,t)} &= -\frac{\sigma(x,t)}{g^2} + iN\,\mathrm{tr}\left[\quad\langle x,t|\,\frac{1}{i\partial\!\!\!/ - (\sigma+i\pi\gamma_5)}\,|x,t\rangle\right] = 0\\
\frac{\delta S_{\mathrm{eff}}}{\delta\pi(x,t)} &= -\frac{\pi(x,t)}{g^2} - \;N\,\mathrm{tr}\left[\gamma_5\,\langle x,t|\,\frac{1}{i\partial\!\!\!/ - (\sigma+i\pi\gamma_5)}\,|x,t\rangle\right] = 0\,.
\end{aligned} \tag{1.6}$$

In particular, the non-perturbative vacuum of (1.1) is governed by the simplest large N saddle points of the path integral associated with it, where the composite scalar operator $\bar{\psi}\psi$ and the pseudoscalar operator $i\bar{\psi}\gamma_5\psi$ develop space-time independent expectation values.

These saddle points are extrema of the effective potential V_{eff} associated with (1.1), namely, the value of $-S_{eff}$ for space-time independent σ, π configurations per unit time per unit length. The effective potential V_{eff} depends only on the combination $\rho^2 = \sigma^2 + \pi^2$ as a result of chiral symmetry. V_{eff} has a minimum as a function of ρ at $\rho = m \neq 0$ that is fixed by the (bare) gap equation[7]

$$-m + iNg^2\,\mathrm{tr}\int\frac{d^2k}{(2\pi)^2}\,\frac{1}{k\!\!\!/ - m} = 0 \tag{1.7}$$

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[d] From this point to the end of this paper flavor indices are usually suppressed. Thus $i\bar{\psi}\partial\!\!\!/\psi$ should be understood as $i\sum_{a=1}^{N}\bar{\psi}_a\partial\!\!\!/\psi_a$. Similarly $\bar{\psi}\Gamma\psi$ stands for $\sum_{a=1}^{N}\bar{\psi}_a\Gamma\psi_a$, where $\Gamma = 1, \gamma_5$.

which yields the dynamical mass

$$m = \Lambda\, e^{-\frac{\pi}{Ng^2(\Lambda)}}\,. \tag{1.8}$$

Here Λ is an ultraviolet cutoff. The mass m must be a renormalization group invariant. Thus, the model is asymptotically free. We can get rid of the cutoff at the price of introducing an arbitrary renormalization scale μ. The renormalized coupling $g_R\,(\mu)$ and the cut-off dependent bare coupling are then related through $\Lambda \exp[-\pi/Ng^2(\Lambda)] = \mu \exp[1 - (\pi/Ng_R^2(\mu))]$ in a convention where $Ng_R^2\,(m) = \pi$. Trading the dimensionless coupling g_R^2 for the dynamical mass scale m represents the well known phenomenon of dimensional transmutation.

The vacuum manifold of (1.1) is therefore a circle $\rho = m$ in the σ, π plane, and the equivalent vacua are parametrized by the chiral angle $\theta = \arctan(\pi/\sigma)$. Therefore, small fluctuations of the Dirac fields around the vacuum manifold develop dynamical chiral mass $m \exp(i\theta\gamma_5)$.

Note in passing that the massless fluctuations of θ along the vacuum manifold decouple from the spectrum[25] so that the axial $U(1)$ symmetry does not break dynamically in this two dimensional model,[26] in accordance with the Coleman-Mermin-Wagner theorem.

Non-trivial excitations of the vacuum, on the other hand, are described semiclassically by large N saddle points of the path integral over (1.1) at which σ and π develop space-time dependent expectation values.[27,4] These expectation values are the space-time dependent solution of (1.6). Saddle points of this type are important also in discussing the large order behavior[28,29] of the $1/N$ expansion of the path integral over (1.1).

These saddle points describe sectors of (1.1) that include scattering states of the (dynamically massive) fermions in (1.1), as well as a rich collection of bound states thereof.

These bound states result from the strong infrared interactions, which polarize the vacuum inhomogeneously, causing the composite scalar $\bar{\psi}\psi$ and pseudoscalar $i\bar{\psi}\gamma_5\psi$ fields to form finite action space-time dependent condensates. These condensates are stable because of the binding energy released by the trapped fermions and therefore cannot form without such binding. This description agrees with the general physical picture drawn in Ref. 30. We may regard these condensates as one dimensional chiral bags[5,6] that trap the original fermions ("quarks") into stable finite action extended entities ("hadrons").

If we set $\pi(x)$ in (1.1) to be identically zero, we recover the Gross-Neveu model, defined by

$$S_{GN} = \int d^2x \left\{ \bar{\psi}\, [i\partial\!\!\!/ - \sigma]\, \psi - \frac{\sigma^2}{2g^2} \right\}\,. \tag{1.9}$$

In spite of their similarities, these two field theories are quite different, as is well-known from the field theoretic literature of the seventies. The crucial difference is that the Gross-Neveu model possesses a discrete symmetry, $\sigma \to -\sigma$, rather than the continuous axial $U(1)$ symmetry $\sigma + i\gamma_5\pi \to e^{-i\gamma_5\alpha}(\sigma + i\gamma_5\pi)$ in the NJL model (1.1). This discrete symmetry is dynamically broken by the non-perturbative vacuum, and thus there is a kink solution,[9,1,19] the CCGZ kink mentioned above, $\sigma(x) = m\tanh(mx)$, interpolating between $\pm m$ at $x = \pm\infty$ respectively. Therefore, topology insures the stability of these kinks.

In contrast, the NJL model, with its continuous symmetry, does not have a topologically stable soliton solution. The solitons arising in the NJL model can only be stabilized by binding fermions, namely, stability of fermion bags in the NJL model is not due to topology, but to dynamics.

The rest of the paper is organized as follows: In Sec. 2 we review the results of Ref. 24. We study the resolvent of the Dirac operator in a given static $\sigma(x) + i\gamma_5\pi(x)$ background. The Dirac equation in any such background has special properties. In fact, we show that it is equivalent to a pair of two isospectral Sturm-Liouville equations in one dimension, which generalize the well known one-dimensional supersymmetric quantum mechanics. We use this generalized supersymmetry to express all four entries of the space-diagonal Dirac resolvent (i.e., the resolvent evaluated at coincident spatial coordinates) in terms of a single function. As a result, we can prove that each frequency mode of the spatial current $\langle\bar{\psi}(x)\gamma^1\psi(x)\rangle$ vanishes identically, contrary to the argument we made in Ref. 21. The findings of Sec. 2 are then used in Sec. 3 to simplify the saddle point equations (1.6). We then study the spatial asymptotic behavior of the simplified equations. We use the spatial asymptotic expression of the resolvent of the Dirac operator (summarized in the Appendix) to generate an asymptotic expansion of the quantities $\delta S_{\mathit{eff}}/\delta\sigma\,(x,t)$ and $\delta S_{\mathit{eff}}/\delta\pi\,(x,t)$, evaluated on a static background $\sigma(x), \pi(x)$ (consistent with the physical boundary conditions at spatial infinity). We prove that these asymptotic expansions vanish term by term to any power in $1/x$, for *any* static $\sigma(x)$ and $\pi(x)$ that are consistent with the physical boundary conditions, and not just for reflectionless backgrounds, as we have claimed in Ref. 21. In the Appendix we recall the asymptotic behavior of resolvents of Sturm-Liouville operators and use them to derive the asymptotic behavior of the resolvent of the Dirac operator in a static bag background.

2 Resolvent of the Dirac operator with static background fields

As was explained in the introduction, we are interested in static space dependent solutions of the extremum condition on S_{eff}. To this end we need to invert the Dirac operator

$$D \equiv i\partial\!\!\!/ - (\sigma(x) + i\pi(x)\gamma_5) \tag{2.1}$$

in a given background of static field configurations $\sigma(x)$ and $\pi(x)$. In particular, we have to find the diagonal resolvent of (2.1) in that background. We stress that inverting (2.1) has nothing to do with the large N approximation, and consequently our results in this section are valid for any value of N. For example, our results may be of use in generalizations of supersymmetric quantum mechanics.

For the usual physical reasons, we set boundary conditions on our static background fields such that $\sigma(x)$ and $\pi(x)$ start from a point on the vacuum manifold $\sigma^2 + \pi^2 = m^2$ at $x = -\infty$, wander around in the σ-π plane, and then relax back to another point on the vacuum manifold at $x = +\infty$. Thus, we must have the asymptotic behavior

$$\begin{aligned} &\sigma \underset{x\to\pm\infty}{\longrightarrow} m\cos\theta_\pm\,, \qquad \sigma' \underset{x\to\pm\infty}{\longrightarrow} 0\,, \\ &\pi \underset{x\to\pm\infty}{\longrightarrow} m\sin\theta_\pm\,, \qquad \pi' \underset{x\to\pm\infty}{\longrightarrow} 0\,, \end{aligned} \tag{2.2}$$

where $\theta_\pm$ are the asymptotic chiral alignment angles. Only the difference $\theta_+ - \theta_-$ is meaningful, of course, and henceforth we use the axial $U(1)$ symmetry to set $\theta_- = 0$, such that $\sigma(-\infty) = m$ and $\pi(-\infty) = 0$. We also omit the subscript from θ_+ and denote it simply by θ from now on. As typical of solitonic configurations, we expect, that $\sigma(x)$ and $\pi(x)$ tend to their asymptotic boundary values (2.2) on the vacuum manifold at an exponential rate which is determined, essentially, by the mass gap m of the model. It is in the background of such fields that we wish to invert (2.1).

In this paper we use the Majorana representation

$$\gamma^0 = \sigma_2\,,\ \gamma^1 = i\sigma_3 \quad \text{and} \quad \gamma^5 = -\gamma^0\gamma^1 = \sigma_1 \tag{2.3}$$

for γ matrices. In this representation (2.1) becomes

$$D = \begin{pmatrix} -\partial_x - \sigma & -i\omega - i\pi \\ i\omega - i\pi & \partial_x - \sigma \end{pmatrix} = \begin{pmatrix} -Q & -i\omega - i\pi \\ i\omega - i\pi & -Q^\dagger \end{pmatrix}\,, \tag{2.4}$$

where we introduced the pair of adjoint operators

$$Q = \sigma(x) + \partial_x\,, \qquad Q^\dagger = \sigma(x) - \partial_x\,. \tag{2.5}$$

(To obtain (2.4), we have naturally transformed $i\partial\!\!\!/ - (\sigma(x) + i\pi(x)\gamma_5)$ to the ω plane, since the background fields $\sigma(x), \pi(x)$ are static.)

Inverting (2.4) is achieved by solving

$$\begin{pmatrix} -Q & -i\omega - i\pi(x) \\ & \\ i\omega - i\pi(x) & -Q^\dagger \end{pmatrix} \cdot \begin{pmatrix} a(x,y) & b(x,y) \\ & \\ c(x,y) & d(x,y) \end{pmatrix} = -i\,\mathbf{1}\,\delta(x-y) \tag{2.6}$$

for the Green's function of (2.4) in a given background $\sigma(x), \pi(x)$. By dimensional analysis, we see that the quantities a, b, c and d are dimensionless.

2.1 Generalized "Supersymmetry" in a chiral bag background

Interestingly, the spectral theory of the Dirac operator (2.4) is underlined by a certain generalized one dimensional supersymmetric quantum mechanics.[24] This generalized supersymmetry is very helpful in simplifying various calculations involving the Dirac operator and its resolvent. In the remaining part of this section, we review the discussion in Ref. 24.

The diagonal elements $a(x,y),\ d(x,y)$ in (2.6) may be expressed in term of the off-diagonal elements as

$$a(x,y) = \frac{-i}{\omega - \pi(x)}\, Q^\dagger\, c(x,y)\,, \qquad d(x,y) = \frac{i}{\omega + \pi(x)}\, Q\, b(x,y) \tag{2.7}$$

which in turn satisfy the second order partial differential equations

$$L_b(\omega)\, b(x,y) = \delta(x-y)\,, \qquad L_c(\omega)\, c(x,y) = -\delta(x-y)\,, \tag{2.8}$$

where $L_b(\omega),\ L_c(\omega)$ are the second order Sturm-Liouville operators [e]

$$\begin{aligned} L_b(\omega) &= Q^\dagger\, \frac{1}{\omega + \pi(x)}\, Q - [\omega - \pi(x)]\,, \\ L_c(\omega) &= Q\, \frac{1}{\omega - \pi(x)}\, Q^\dagger - [\omega + \pi(x)]\,. \end{aligned} \tag{2.9}$$

[e] Note that ω plays here a dual role: in addition to its role as the spectral parameter (the ω^2 terms in (2.9)), it also appears as a parameter in the definition of these operators — hence the explicit ω dependence in our notations for these operators in (2.9). However, in order to avoid notational cluttering, from now on we will denote these operators simply as L_b and L_c.

We can write these two operators more explicitly as

$$L_{b,c}(\omega) = \tag{2.10}$$
$$-\partial_x \frac{1}{\omega \pm \pi(x)}\,\partial_x + \left[\sigma^2(x) + \pi^2(x) \mp \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega \pm \pi(x)}\right]\frac{1}{\omega \pm \pi(x)}\,.$$

Thus, $b(x,y)$ and $-c(x,y)$ are simply the Green's functions of these operators, namely,

$$\begin{aligned} b(x,y) &= \frac{\theta\,(x-y)\,b_2(x)b_1(y) + \theta\,(y-x)\,b_2(y)b_1(x)}{W_b} \\ c(x,y) &= -\frac{\theta\,(x-y)\,c_2(x)c_1(y) + \theta\,(y-x)\,c_2(y)c_1(x)}{W_c}\,. \end{aligned} \tag{2.11}$$

Here $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$ are pairs of independent fundamental solutions of the two equations $L_b b(x) = 0$ and $L_c c(x) = 0$, subjected to the boundary conditions

$$b_1(x)\,, c_1(x) \underset{x\to-\infty}{\longrightarrow} A^{(1)}_{b,c}(k)e^{-ikx}\,, \qquad b_2(x)\,, c_2(x) \underset{x\to+\infty}{\longrightarrow} A^{(2)}_{b,c}(k)e^{ikx} \tag{2.12}$$

with some possibly k dependent coefficients $A^{(1)}_{b,c}(k)$, $A^{(2)}_{b,c}(k)$ and with [f]

$$k = \sqrt{\omega^2 - m^2}\,, \quad \mathrm{Im}\,k \geq 0\,. \tag{2.13}$$

The purpose of introducing the (yet unspecified) coefficients $A^{(1)}_{b,c}(k)$, $A^{(2)}_{b,c}(k)$ will become clear following Eqs. (2.16) and (2.17). The boundary conditions (2.12) are consistent, of course, with the asymptotic behavior (2.2) of σ and π due to which both L_b and L_c tend to a free particle hamiltonian $[-\partial_x^2 + m^2 - \omega^2]$ as $x \to \pm\infty$.

The wronskians of these pairs of solutions are

$$\begin{aligned} W_b(k) &= \frac{b_2(x)b_1'(x) - b_1(x)b_2'(x)}{\omega + \pi(x)}\,, \\ W_c(k) &= \frac{c_2(x)c_1'(x) - c_1(x)c_2'(x)}{\omega - \pi(x)}\,. \end{aligned} \tag{2.14}$$

As is well known, $W_b(k)$ and $W_c(k)$ are independent of x.

[f] We see that if $\mathrm{Im}\,k > 0$, b_1 and c_1 decay exponentially to the left, and b_2 and c_2 decay to the right. Thus, if $\mathrm{Im}k > 0$, both $b(x,y)$ and $c(x,y)$ decay as $|x-y|$ tends to infinity.

Note in passing that the canonical asymptotic behavior assumed in the scattering theory of the operators L_b and L_c corresponds to setting $A_{b,c}^{(1)} = A_{b,c}^{(2)} = 1$ in (2.12). Thus, the wronskians in (2.14) are *not* the canonical wronskians used in scattering theory. As is well known in the literature,[3] the *canonical* wronskians are proportional (with a k independent coefficient) to $k/t(k)$, where $t(k)$ is the transmission amplitude of the corresponding operator L_b or L_c. Thus, on top of the well-known features of $t(k)$, the wronskians in (2.14) will have additional spurious k-dependence coming from the amplitudes $A_{b,c}^{(1)}(k)$, $A_{b,c}^{(2)}(k)$ in (2.12).

Substituting the expressions (2.11) for the off-diagonal entries $b(x,y)$ and $c(x,y)$ into (2.7), we obtain the appropriate expressions for the diagonal entries $a(x,y)$ and $d(x,y)$. We do not bother to write these expressions here. It is useful however to note, that despite the ∂_x's in the Q operators in (2.7), that act on the step functions in (2.11), neither $a(x,y)$ nor $d(x,y)$ contain pieces proportional to $\delta(x-y)$. Such pieces cancel one another due to the symmetry of (2.11) under $x \leftrightarrow y$.

We will now prove that the spectra of the operators L_b and L_c are essentially the same. Our proof is based on the fact that we can factorize the eigenvalue equations $L_b b(x) = 0$ and $L_c c(x) = 0$ as

$$\frac{1}{\omega - \pi(x)}\, Q^\dagger\, \frac{1}{\omega + \pi(x)}\, Q\, b = b\,, \qquad \frac{1}{\omega + \pi(x)}\, Q\, \frac{1}{\omega - \pi(x)}\, Q^\dagger\, c = c\,, \quad (2.15)$$

as should be clear from (2.8) and (2.9).

The factorized equations (2.15) suggest a mapping between their solutions. Indeed, given that $L_b\, b(x) = 0$, then clearly

$$c(x) = \frac{1}{\omega + \pi(x)}\, Q\, b(x) \quad (2.16)$$

is a solution of $L_c c(x) = 0$. Similarly, if $L_c c(x) = 0$, then

$$b(x) = \frac{1}{\omega - \pi(x)}\, Q^\dagger\, c(x) \quad (2.17)$$

solves $L_b\, b(x) = 0$.

Thus, in particular, given a pair $\{b_1(x), b_2(x)\}$ of independent fundamental solutions of $L_b b(x) = 0$, we can obtain from it a pair $\{c_1(x), c_2(x)\}$ of independent fundamental solutions of $L_c c(x) = 0$ by using (2.16), and vice versa. Therefore, with no loss of generality, we henceforth assume, that the two pairs of independent fundamental solutions $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$, are related by (2.16) and (2.17).

The coefficients $A^{(1)}_{b,c}(k), A^{(2)}_{b,c}(k)$ in (2.12) are to be adjusted according to (2.16) and (2.17), and this was the purpose of introducing them in the first place. Thus, with no loss of generality, we may make the standard choice

$$A^{(1)}_b = A^{(2)}_b = 1 \tag{2.18}$$

in (2.12). The coefficients $A^{(1)}_c, A^{(2)}_c$ are then determined by (2.16):

$$A^{(1,2)}_c = \frac{\sigma(\mp\infty) \mp ik}{\pi(\mp\infty) + \omega}\,. \tag{2.19}$$

We note that these $b(x) \leftrightarrow c(x)$ mappings can break only if

$$Q\,b = 0 \qquad \text{or} \qquad Q^\dagger\, c = 0\,, \tag{2.20}$$

for $b(x)$ or $c(x)$ that *solve* (2.15). Do such solutions exist? Let us assume, for example, that $Q\,b = 0$ and that $L_b b = 0$. From the first equation in (2.15) (or in (2.8)), we see that this is possible if and only if $\omega \pm \pi(x) \equiv 0$, which clearly cannot hold if $\partial_x \pi(x) \neq 0$. A similar argument holds for $Q^\dagger\, c = 0$. Thus, if $\partial_x \pi(x) \neq 0$, the mappings (2.16) and (2.17) are one-to-one. In particular, a bound state in L_b implies a bound state in L_c (at the same energy) and vice-versa.

An interesting related result concerns the wronskians W_b and W_c. From (2.14), and from (2.16) and (2.17) it follows immediately that for pairs of independent fundamental solutions $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$ we have

$$W_c = \frac{c_2\partial_x c_1 - c_1\partial_x c_2}{\omega - \pi(x)} = c_1 b_2 - c_2 b_1 = \frac{b_2\partial_x b_1 - b_1\partial_x b_2}{\omega + \pi(x)} = W_b\,. \tag{2.21}$$

The wronskians of pairs of independent fundamental solutions of L_b and L_c, which are related via (2.16) and (2.17) are equal !

To summarize, if $\partial_x \pi(x) \neq 0$, L_b and L_c have the same set of energy eigenvalues and their eigenfunctions are in one-to-one correspondence.

If, however, π =const., then we are back to the familiar "supersymmetric" factorization

$$Q^\dagger\, Q\, b = (\omega^2 - \pi^2)\, b\,, \qquad Q\, Q^\dagger\, c = (\omega^2 - \pi^2)\, c\,, \tag{2.22}$$

and mappings

$$c(x) = \frac{1}{\omega + \pi}\, Q\, b(x)\,, \qquad b(x) = \frac{1}{\omega - \pi}\, Q^\dagger\, c(x)\,. \tag{2.23}$$

As is well known from the literature on supersymmetric quantum mechanics, the mappings (2.23) break down if either $Qb = 0$ or $Q^\dagger c = 0$, in which case the two operators $Q^\dagger Q$ and $QQ^\dagger$ are isospectral, but only up to a "zero-mode" (or rather, an $\omega^2 = \pi^2$ mode), which belongs to the spectrum of only one of the operators.[g] The case $\pi(x) \equiv 0$ brings us back to the GN model. Supersymmetric quantum mechanical considerations were quite useful in the study of fermion bags in Ref. 19.

The "Witten index" associated with the pair of isospectral operators L_b and L_c, is always null for backgrounds in which $\partial_x \pi(x) \neq 0$, since they are absolutely isospectral, and not only up to zero modes. There is no interesting topology associated with spectral mismatches of L_b and L_c. This is not surprising at all, since, as we have already stressed in the introduction, the NJL model, with its continuous axial symmetry, does not support topological solitons. This is in contrast to the GN model, for which $\pi \equiv 0$, which contains topological kinks, whose topological charge is essentially the Witten index of the pair of operators (2.22).

We note in passing that isospectrality of L_b and L_c which we have just proved, is consistent with the γ_5 symmetry of the system of equations in (2.6), which relates the resolvent of D with that of $\tilde{D} = -\gamma_5 D \gamma_5$. Due to this symmetry, we can map the pair of equations $L_b b(x,y) = \delta(x-y)$ and $L_c c(x,y) = -\delta(x-y)$ (Eqs. (2.8)) on each other by

$$b(x,y) \leftrightarrow -c(x,y) \quad \text{together with} \quad (\sigma, \pi) \to (-\sigma, -\pi)\,. \tag{2.24}$$

(Note that under these reflections we also have $a(x,y) \leftrightarrow -d(x,y)$, as we can see from (2.7).) The reflection $(\sigma,\pi) \to (-\sigma,-\pi)$ just shifts both asymptotic chiral angles $\theta_\pm$ by the same amount π, and clearly does not change the physics. Since this reflection interchanges $b(x,y)$ and $c(x,y)$ without affecting the physics, these two objects must have the same singularities as functions of ω, consistent with isospectrality of L_b and L_c.

Isospectrality of L_b and L_c realizes a concrete generalization of SUSY quantum mechanics. We should mention, in passing, the recent discussion,[32] made in a different context, with other generalizations of SUSY quantum mechanics.

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[g] This is true for short range decaying potentials on the whole real line. For periodic potentials both operators may have that $\omega^2 = \pi^2$ mode in their spectrum.[31] Strictly speaking, (to the best of our knowledge) only the case $\pi = 0$ appears in the literature on supersymmetric quantum mechanics.

2.2 The diagonal resolvent

Following[20,21] we define the diagonal resolvent $\langle x\,|iD^{-1}|x\,\rangle$ symmetrically as

$$\begin{aligned}\langle x\,|-iD^{-1}|x\,\rangle &\equiv \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x)\end{pmatrix} \\ &= \frac{1}{2}\lim_{\epsilon\to 0+}\begin{pmatrix} a(x,y)+a(y,x) & b(x,y)+b(y,x) \\ c(x,y)+c(y,x) & d(x,y)+d(y,x)\end{pmatrix}_{y=x+\epsilon} \end{aligned} \tag{2.25}$$

Here $A(x)$ through $D(x)$ stand for the entries of the diagonal resolvent, which following (2.7) and (2.11) have the compact representation[h]

$$\begin{aligned} B(x) &= \frac{b_1(x)b_2(x)}{W_b}\,, \qquad D(x) = \frac{i}{2}\,\frac{[\partial_x + 2\sigma(x)]\,B\,(x)}{\omega+\pi(x)}\,, \\ C(x) &= -\frac{c_1(x)c_2(x)}{W_c}\,, \qquad A(x) = \frac{i}{2}\,\frac{[\partial_x - 2\sigma(x)]\,C\,(x)}{\omega-\pi(x)}\,. \end{aligned} \tag{2.26}$$

We now use the generalized "supersymmetry" of the Dirac operator, which we discussed in the previous subsection, to deduce some important properties of the functions $A(x)$ through $D(x)$.

From (2.26) and from (2.5) we we have

$$A(x) = \frac{i}{2}\frac{\partial_x - 2\sigma(x)}{\omega-\pi(x)}\left(-\frac{c_1c_2}{W_c}\right) = \frac{i}{2W_c}\frac{c_2Q^\dagger c_1 + c_1Q^\dagger c_2}{\omega-\pi(x)}\,.$$

Using (2.17) first, and then (2.16), we rewrite this expression as

$$A(x) = \frac{i}{2W_c}(c_2b_1 + c_1b_2) = \frac{i}{2W_c}\frac{b_1Qb_2 + b_2Qb_1}{\omega+\pi(x)}\,.$$

Then, using the fact that $W_c = W_b$ (Eq. (2.21)) and (2.26), we rewrite the last expression as

$$A(x) = \frac{i}{2}\frac{\partial_x + 2\sigma}{\omega+\pi(x)}\left(\frac{b_1b_2}{W_b}\right) = \frac{i}{2}\frac{(\partial_x+2\sigma)B(x)}{\omega+\pi(x)}\,.$$

Thus, finally,

$$A(x) = D(x)\,. \tag{2.27}$$

[h] A, B, C and D are obviously functions of ω as well. For notational simplicity we suppress their explicit ω dependence.

Supersymmetry renders the diagonal elements A and D equal.

Due to (2.26), $A = D$ is also a first order differential equation relating B and C. We can also relate the off diagonal elements B and C to each other more directly. From (2.26) and from (2.16) we find

$$C(x) = -\frac{c_1 c_2}{W_c} = -\frac{(Qb_1)(Qb_2)}{(\omega+\pi)^2 W_c}\,. \tag{2.28}$$

After some algebra, and using (2.21), we can rewrite this as

$$-(\omega+\pi)^2 C = \sigma^2 B + \sigma B' + \frac{b_1' b_2'}{W_b}$$

The combination $b_1'b_2'/W_b$ appears in $B'' = (b_1 b_2/W_b)''$. After using $L_b b_{1,2} = 0$ to eliminate b_1'' and b_2'' from B'', we find

$$\frac{b_1' b_2'}{W_b} = \frac{1}{2}B'' - \frac{\pi' B'}{2(\omega+\pi)} - \left(\sigma^2 + \pi^2 - \sigma' - \omega^2 + \frac{\sigma\pi'}{\omega+\pi}\right) B$$

Thus, finally, we have

$$-(\omega+\pi)^2 C = \frac{1}{2}B'' + \left(\sigma - \frac{\pi'}{2(\omega+\pi)}\right) B' - \left(\pi^2 - \sigma' - \omega^2 + \frac{\sigma\pi'}{\omega+\pi}\right) B\,. \tag{2.29}$$

In a similar manner we can prove that

$$(\omega-\pi)^2 B = -\frac{1}{2}C'' + \left(\sigma - \frac{\pi'}{2(\omega-\pi)}\right) C' + \left(\pi^2 + \sigma' - \omega^2 + \frac{\sigma\pi'}{\omega-\pi}\right) C\,. \tag{2.30}$$

We can simplify (2.29) and (2.30) further. After some algebra, and using (2.26) we arrive at

$$\begin{aligned} C(x) &= \frac{i}{\omega+\pi(x)}\,\partial_x D(x) - \frac{\omega-\pi(x)}{\omega+\pi(x)}\,B(x) \\ B(x) &= \frac{i}{\omega-\pi(x)}\,\partial_x A(x) - \frac{\omega+\pi(x)}{\omega-\pi(x)}\,C(x)\,. \end{aligned} \tag{2.31}$$

Supersymmetry, namely, isospectrality of L_b and L_c, enables us to relate the diagonal resolvents of these operators, B and C, to each other.

Thus, we can use (2.26), (2.27) and (2.31) to eliminate three of the entries of the diagonal resolvent in (2.26), in terms of the fourth.

Note that the two relations in (2.31) transform into each other under

$$B \leftrightarrow -C \quad \text{simultaneously with} \quad (\sigma,\pi) \to (-\sigma,-\pi)\,, \tag{2.32}$$

in consistency with (2.24). The relations in (2.31) are linear and homogeneous, with coefficients that for $\partial_x \pi(x) \neq 0$ do not introduce additional singularities in the ω plane. Thus, we see, once more, that B and C have the same singularities in the ω plane. We refer the reader to Sec. 4 in Ref. 21 for concrete examples of such resolvents.

The case $\pi(x) \equiv 0$ brings us back to the GN model. In the GN model, our B and C, coincide, respectively, with ωR_- and $-\omega R_+$, defined in Eqs. (9) and (10) in Ref. 19. With these identifications, the relation $A = D$ (Eq. (2.27)) coincides essentially with Eq. (18) of Ref. 19. The relations (2.29) and (2.30) were not discussed in Ref. 19 but one can verify them, for example, for the resolvents corresponding to the kink case $\sigma(x) = m \tanh mx$ (Eq. (29) in Ref. 19) for which

$$C = -\frac{\omega}{2\sqrt{m^2-\omega^2}}\,, \quad B = \left[\left(\frac{m\,\mathrm{sech}\,mx}{\omega}\right)^2 - 1\right] C\,.$$

2.3 Bilinear fermion condensates and vanishing of the spatial fermion current

Following basic principles of quantum field theory, we may write the most generic flavor-singlet bilinear fermion condensate in our static background as

$$\langle \bar\psi_{a\alpha}(t,x)\,\Gamma_{\alpha\beta}\,\psi_{a\beta}(t,x)\rangle_{\mathrm{reg}} = N\!\int \frac{d\omega}{2\pi}\,\mathrm{tr}\left[\Gamma\langle x|\frac{-i}{\omega\gamma^0 + i\gamma^1\partial_x - (\sigma + i\pi\gamma_5)}|x\rangle_{\mathrm{reg}}\right]$$
$$= N\!\int \frac{d\omega}{2\pi}\,\mathrm{tr}\left\{\Gamma\left[\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x)\end{pmatrix} - \begin{pmatrix} A & B \\ C & D\end{pmatrix}_{VAC}\right]\right\}, \tag{2.33}$$

where we have used (2.25). Here $a = 1, \cdots, N$ is a flavor index, and the trace is taken over Dirac indices α, β. As usual, we regularized this condensate by subtracting from it a short distance divergent piece embodied here by the diagonal resolvent

$$\langle x\,| - iD^{-1}|x\,\rangle_{VAC} = \begin{pmatrix} A & B \\ C & D\end{pmatrix}_{VAC} \tag{2.34}$$
$$= \frac{1}{2\sqrt{m^2-\omega^2}}\begin{pmatrix} i\,m\cos\theta & \omega + m\sin\theta \\ -\omega + m\sin\theta & i\,m\cos\theta\end{pmatrix}$$

of the Dirac operator in a vacuum configuration $\sigma_{VAC} = m\cos\theta$ and $\pi_{VAC} = m\sin\theta$.

In our convention for γ matrices (2.3) we have

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = \frac{A(x)+D(x)}{2}\,\mathbf{1} + \frac{A(x)-D(x)}{2i}\,\gamma^1 + i\frac{B(x)-C(x)}{2}\,\gamma^0 + \frac{B(x)+C(x)}{2}\,\gamma_5\,. \tag{2.35}$$

An important condensate is the expectation value of the fermion current $\langle j^\mu(x)\rangle$. In particular, consider its spatial component. In our static background $(\sigma(x),\pi(x))$, it must, of course, vanish identically

$$\langle j^1(x)\rangle = 0\,. \tag{2.36}$$

Thus, substituting $\Gamma=\gamma^1$ in (2.33) and using (2.35) we find

$$\langle j^1(x)\rangle = iN\int\frac{d\omega}{2\pi}\,[A(x)-D(x)]\;. \tag{2.37}$$

But we have already proved that $A(x) = D(x)$ in *any* static background $(\sigma(x),\pi(x))$ (Eq.(2.27)). Thus, each frequency component of $\langle j^1\rangle$ vanishes separately, and (2.36) holds identically. It is remarkable that the generalized supersymmetry of the Dirac operator guarantees the consistency of any static $(\sigma(x),\pi(x))$ background.

We discussed $< j^1(x) >= 0$ in Ref. 21. However, that analysis was incomplete as it considered only the asymptotic behavior of $< j^1(x) >$, which misled us to draw an overrestrictive necessary consistency condition on the background.

Expressions for other bilinear condensates may be derived in a similar manner to the derivation of $< j^1(x) >$ (here we write the unsubtracted quantities). Thus, substituting $\Gamma=\gamma^0$ in (2.33) and using (2.35), (2.27) and (2.31), we find that the fermion density is

$$\langle j^0(x)\rangle = iN\int\frac{d\omega}{2\pi}\,[B(x)-C(x)] = iN\int\frac{d\omega}{2\pi}\,\frac{2\omega B(x)-i\partial_x D(x)}{\omega+\pi(x)}\,. \tag{2.38}$$

Similarly, the scalar and pseudoscalar condensates are

$$\langle\bar\psi(x)\psi(x)\rangle = N\int\frac{d\omega}{2\pi}\,[A(x)+D(x)] = 2N\int\frac{d\omega}{2\pi}\,D(x)\;, \tag{2.39}$$

and

$$\langle\bar\psi(x)\gamma^5\psi(x)\rangle = N\int\frac{d\omega}{2\pi}\,[B(x)+C(x)] = N\int\frac{d\omega}{2\pi}\,\frac{2\pi(x)B(x)+i\partial_x D(x)}{\omega+\pi(x)}\,. \tag{2.40}$$

3 The saddle point equations and reflectionless backgrounds

For static backgrounds $(\sigma(x), \pi(x))$ we have the (divergent) formal relation

$$\begin{aligned}\langle x, t|\frac{1}{i\partial\!\!\!/ - (\sigma + i\pi\gamma_5)}|x, t\rangle &= \int \frac{d\omega}{2\pi}\langle x|\frac{1}{\omega\gamma^0 + i\gamma^1\partial_x - (\sigma + i\pi\gamma_5)}|x\rangle \\ &= i\int \frac{d\omega}{2\pi}\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x)\end{pmatrix}\end{aligned}$$

(see Eq. (2.33)). Therefore, using (2.3) and (2.27), the bare saddle point equations (1.6) for static bags are

$$\begin{aligned}\frac{\delta S_{\mathit{eff}}}{\delta\sigma\,(x,t)}_{|\mathrm{static}} &= -\frac{\sigma\,(x)}{g^2} - 2N\int \frac{d\omega}{2\pi}\,A(x) = 0 \\ \frac{\delta S_{\mathit{eff}}}{\delta\pi\,(x,t)}_{|\mathrm{static}} &= -\frac{\pi\,(x)}{g^2} - iN\int \frac{d\omega}{2\pi}\,[B(x) + C(x)] = 0\,, \qquad (3.1)\end{aligned}$$

where $(\sigma(x), \pi(x))$ are subjected to the asymptotic boundary conditions (2.2). As was already mentioned following (2.2), we further assume that $\sigma(x)$ and $\pi(x)$ tend to their asymptotic boundary values on the vacuum manifold at an exponential rate which is determined, essentially, by the mass gap m of the model, as typical of solitonic configurations.

The ω-integrals in (3.1) are divergent. For bounded bag profiles which satisfy the boundary conditions (2.2), the diagonal resolvent (2.25) tends, for large ω, to that of the vacuum background (2.34). Thus, we note from (2.34), that while each of the integrals $\int d\omega\, B(x)$ and $\int d\omega\, C(x)$ diverges linearly with the ultraviolet cutoff, their sum diverges only logarithmically, as does $\int d\omega\, A(x)$. The saddle point equations for vacuum condensates, i.e., the gap equations

$$\begin{aligned}-\frac{\sigma_{VAC}}{g^2} - 2N\int \frac{d\omega}{2\pi}\,A_{VAC} &= 0 \\ -\frac{\pi_{VAC}}{g^2} - iN\int \frac{d\omega}{2\pi}\,[B_{VAC} + C_{VAC}] &= 0\,, \qquad (3.2)\end{aligned}$$

exhibit the same logarithmic divergence, of course. Thus, we can take care of the UV divergence in (3.1) by subtracting from these equations the corresponding gap equations.

We now concentrate on the subtracted saddle point equation for $\sigma(x)$

$$\frac{\sigma(x) - \sigma_{VAC}}{2Ng^2} = -\int_{\mathcal{C}} \frac{d\omega}{2\pi}\,[A(x) - A_{VAC}]\,. \qquad (3.3)$$

The integration contour $\mathcal{C}$ in (3.3) is commonly[i] taken as indicated in Fig. 1, which shows qualitatively the spectrum of the Dirac equation in a bag background. Besides the continuum states in that spectrum (the two cuts corre-

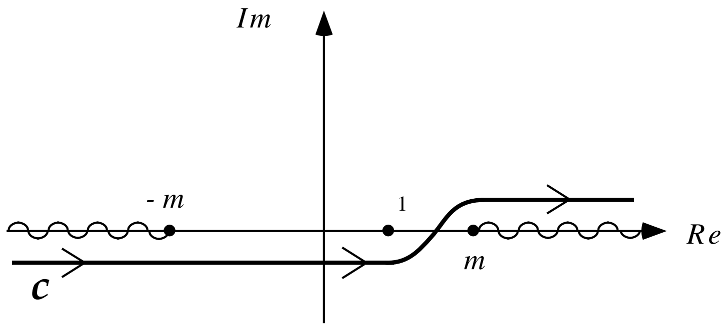

Figure 1: The contour $\mathcal{C}$ in the complex ω plane in Eq. (3.3). The continuum states appear as the two cuts along the real axis with branch points at $\pm m$ (wiggly lines), and the bound state is the pole at ω_1.

sponding to the Fermi sea of negative energy states $\omega \leq -m$, and scattering states with $\omega \geq m$), there are bound states within the gap $-m \leq \omega \leq m$, which trap fermions into the bag. One such bound state is indicated in Fig. 1 as the pole at $\omega = \omega_1$. The detailed calculation to determine bound state energies like ω_1 is discussed in Refs. 2 and 21 (after establishing the reflectionless property of the background). We stress that $\sigma_{_{VAC}}$ in (3.3) can be the σ component of *any* point on the vacuum manifold $\sigma^2 + \pi^2 = m^2$ (and similarly for $\pi_{_{VAC}}$, which appears in the subtracted equation for $\pi(x)$).

The contour integral in (3.3) is most conveniently calculated by deforming the contour $\mathcal{C}$ into the contour $\mathcal{C}'$ shown in Fig. 2. (This is allowed, since the subtracted integral in (3.3) converges.) The "hairpin" wing of $\mathcal{C}'$ picks up the contribution of the filled Fermi sea, and the little circle around the simple pole at $\omega = \omega_1$ is the contribution of fermions populating the bound state of the "bag".

Let us study the spatial asymptotic behavior of (3.3). From (A.13) we have

$$A(x) - A_{_{VAC}}(-\infty) \quad \underset{x\to-\infty}{\longrightarrow} \quad \frac{ik - \sigma(-\infty)}{2k}\, r_2(k) e^{-2ikx}$$

[i] See e.g., Sec. 4 of Ref. 33 for a detailed discussion. If the bound state at energy ω_1 traps n_f fermions, then routing the contour $\mathcal{C}$ as indicated in Fig. 1 means that we are discussing the sector of the model with fermion number n_f.

$$A(x) - A_{VAC}(+\infty) \underset{x\to+\infty}{\longrightarrow} \frac{ik + \sigma(+\infty)}{-2k}\, r_1(k) e^{2ikx}\,, \tag{3.4}$$

where $\sigma(\pm\infty)$ are the appropriate vacuum boundary values of σ from (2.2), and $r_1(k), r_2(k)$ are the reflection coefficients defined in (A.1).

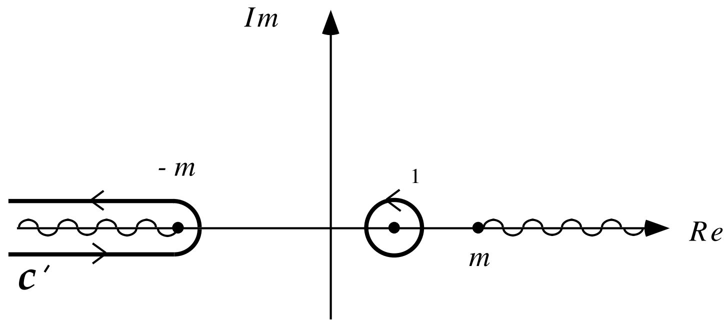

Figure 2: The deformed integration contour $\mathcal{C}'$ used in Eq. (3.3).

Thus, for example, studying (3.3) as $x \to \infty$, we see from (3.4) that

$$\frac{\sigma(x) - \sigma(+\infty)}{2Ng^2} \underset{x\to+\infty}{\longrightarrow} \int_{\text{Fermi sea}} \frac{d\omega}{2\pi}\, \frac{ik + \sigma(+\infty)}{2k}\, r_1(k) e^{2ikx} + \mathcal{O}(e^{-\text{const.}\, mx})\,. \tag{3.5}$$

The first term in (3.5) is the contribution coming from the Fermi sea (i.e., the "hairpin" wing of $\mathcal{C}'$). The second, exponentially small term on the right hand side of (3.5) comes from the bound state pole (i.e., it is proportional to the bound state wave function squared). Due to the asymptotic boundary conditions on $\sigma(x)$, the left hand side of (3.5) is also exponentially small as $x \to \infty$. Thus, the first term on the right hand side of (3.5) must have an exponentially small bound as $x \to \infty$.

We now change the variable to $k = \sqrt{\omega^2 - m^2}$. When mapping into the k-plane, the lower wing of the cut in Fig. 2 is transformed into $k = |k|e^{i\delta}$, and the upper wing is transformed into $-|k|e^{-i\delta}$, with $\delta \to 0+$.[j] Thus, we may write the dispersion integral on the right hand side of (3.5) coming from states

[j] For example, just above the cut $\omega - m = |\omega - m|e^{i(\pi-\delta)}$ and $\omega + m = |\omega + m|e^{i(\pi-\delta)}$, with $\delta \to 0+$. Thus, just above the cut, $k = \sqrt{\omega^2 - m^2} = \sqrt{|\omega^2 - m^2|e^{i(\pi-\delta)}} = -|k|e^{-i\delta}$.

in the Fermi sea as $-\Delta(x)$, where

$$\begin{aligned}\Delta(x) &= \int_0^\infty \frac{dk}{4\pi}\,\frac{(ik+\sigma(\infty))\,r_1(k)e^{2ikx}+(k\to -k)}{\sqrt{k^2+m^2}}\\ &= \mathrm{Re}\left[\int_0^\infty \frac{dk}{2\pi}\,\frac{(ik+\sigma(\infty))\,r_1(k)e^{2ikx}}{\sqrt{k^2+m^2}}\right]\,, \end{aligned} \tag{3.6}$$

where in the last equality we used the reflection property $r_1(-k) = r_1^*(k)$ (see Eq. (A.2) in the Appendix) of $r_1(k)$.

The function $\Delta(x)$ has to die off at least at an exponential rate as $x \to \infty$. Thus, we are to study the asymptotic behavior of

$$G(x) = \int_0^\infty \frac{dk}{2\pi}\,\frac{(ik+\sigma(\infty))\,r_1(k)e^{2ikx}}{\sqrt{k^2+m^2}} \tag{3.7}$$

at large x. To this end we have to invoke some of the general properties of the reflection coefficient $r_1(k)$ of the operator L_b in (2.9).

Due to the boundary conditions (2.2) on the background fields $\sigma(x)$ and $\pi(x)$, the operator L_b tends exponentially fast (in x) to its asymptotic free particle form. Thus, its "scattering potential" is localized in a finite region in space.

From the literature on scattering theory (in one space dimension)[3] we know that the reflection coefficient $r(k)$ of Schrödinger operators with short range potential wells[k] is analytic on the real k axis (and generally follows the threshold behavior $r(k) = -1 + ak + \cdots$) and dies off like $1/k$ as $k \to \infty$, i.e., at large kinetic energy. Strictly speaking, the discussion of these issues in the various references in Ref. 3 concentrates mostly on Schrödinger operators of the standard form $-\partial_x^2 + V(x)$, but the arguments leading to the conclusions about the behavior of $r(k)$ may be easily generalized to the scattering theory of the Sturm-Liouville operators L_b and L_c in (2.9) (with $\sigma(x)$ and $\pi(x)$ relaxing fast to (2.2)).

Therefore, in deriving the asymptotic behavior of $G(x)$ and $\Delta(x)$ we may use as an input that $r_1(k)$ is analytic on the real k axis and that it decays at least as fast as $1/k$ as $k \to \infty$. Given these properties of $r_1(k)$, we are

[k] We tacitly assume that the potential wells in question tend to the same asymptotic value at $x = \pm\infty$ (as L_b does with $\sigma(x), \pi(x)$ satisfying (2.2)), and that they do not have any barriers above these asymptotic values.

allowed to expand $G(x)$ in powers of $1/x$ in the most natural way, namely, by repeatedly integrating by parts over k in (3.7).

Thus, for example, after three integrations we find

$$\begin{aligned}
2\pi G(x) &= -\frac{\sigma(\infty)r_1(0)}{2imx} + \frac{imr_1(0) + m\sigma(\infty)r_1'(0)}{(2imx)^2} \\
&- \frac{-\sigma(\infty)r_1(0) + 2im^2r_1'(0) + m^2\sigma(\infty)r_1''(0)}{(2imx)^3} \\
&- \int_0^\infty dk \frac{e^{2ikx}}{(2ix)^3}\frac{\partial^3}{\partial k^3}\left(\frac{ik+\sigma(\infty)}{\sqrt{k^2+m^2}}\, r_1(k)\right) ,
\end{aligned} \tag{3.8}$$

and so on and so forth. Clearly, the remaining integral in each step is subdominant by a power of $1/x$ relative to its predecessor, and thus, the expansion of $G(x)$ generated in this way is an asymptotic expansion.

From (3.8) (or by working out a few more terms in the asymptotic expansion if necessary) the following pattern emerges: the coefficient of $(1/2ix)^{2n+1}$ is a linear combination of the form

$$\sum_{j=0}^{2n} c_j i^j r_1^{(j)}(0)$$

with *real* coefficients c_j, and the coefficient of $(1/2ix)^{2n}$ is a linear combination of the form

$$i\sum_{j=0}^{2n-1} c_j i^j r_1^{(j)}(0)$$

with some other real coefficients c_j.

From the reflection property (A.2) $r_1(-k) = r_1^*(k)$ we immediately conclude that $r_1^{(n)}(0) = (-1)^n r_1^{(n)}(0)^*$: the even derivatives $r_1^{(2j)}(0)$ are real, and the odd derivatives $r_1^{(2j+1)}(0)$ are imaginary. Thus, the linear combinations $\sum c_j i^j r_1^{(j)}(0)$ are real, which makes each term on the right hand side of the asymptotic expansion (3.8) of $G(x)$ *pure imaginary.*

Using this result in (3.6) we conclude that all terms in the asymptotic expansion of $\Delta(x)$ in powers of $1/x$ vanish. Thus, $\Delta(x)$ vanishes faster than any power of $1/x$ as $x \to \infty$. This is consistent with our expectation that $\Delta(x)$ vanishes at least at an exponential rate when $x \to \infty$.

This concludes our discussion of the subtracted saddle point equation (3.3) for $\sigma(x)$ and its asymptotic behavior.

We can repeat the same story for the subtracted saddle point equation for $\pi(x)$

$$\frac{\pi\,(x) - \pi_{_{VAC}}}{iNg^2} = -\int \frac{d\omega}{2\pi}\,\left[(B(x) - B_{_{VAC}}) + (C(x) - C_{_{VAC}})\right]\,. \tag{3.9}$$

In a similar manner to our derivation of (3.5) and (3.6), we can show that

$$\frac{\pi(x) - \pi(\infty)}{Ng^2} \underset{x\to+\infty}{\longrightarrow} \text{exponentially small contribution of bound states}$$
$$-\mathrm{Re}\left[\int_0^\infty \frac{dk}{2\pi}\,\frac{(\sqrt{k^2+m^2} - \pi(\infty)) - (\sqrt{k^2+m^2} + \pi(\infty))\frac{\sigma(\infty)+ik}{\sigma(\infty)-ik}}{\sqrt{k^2+m^2}}\,r_1(k)e^{2ikx}\right]. \tag{3.10}$$

Due to the asymptotic boundary conditions on $\pi(x)$, the left hand side of (3.10) is exponentially small as $x \to \infty$, which bounds the dispersion integral on the right hand side of (3.10). As in our analysis of (3.7), we expand the integral in the square brackets in (3.10) in powers of $1/x$, and similarly to (3.8), we can show that all terms in that asymptotic series are pure imaginary. Thus, the right hand side of (3.10) vanishes faster than any power of $1/x$, consistent with the boundary conditions on $\pi(x)$.

We conclude that the asymptotic behavior of the static saddle point equations (3.3) and (3.9) is consistent with the asymptotic boundary conditions (2.2) on $\sigma(x)$ and $\pi(x)$ for *any* reflection amplitude $r_1(k)$: all terms in the asymptotic expansions of the dispersion integral in the square brackets in (3.10) and also of $G(x)$ in (3.7) in powers of $1/x$ are imaginary.

Contrary to the argument we made in Ref. 21 the reflectionless property of the solutions $\sigma(x)$ and $\pi(x)$ of (3.1) does not emerge as a necessary condition from consistency of the asymptotic behavior of (3.5) and (3.10) and the boundary conditions on the background fields.

Acknowledgments

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Appendix: Asymptotics of the Dirac resolvent

In this Appendix we discuss the spatial asymptotic behavior of the diagonal resolvent of the Dirac operator (2.25).

According to our discussion in Sec. 2.2, given, for example, $B(x)$, we may determine $D(x)$, $A(x)$ and $C(x)$ using (2.26) first, then (2.27) and finally, (2.29).

Thus, it is enough to determine the asymptotic behavior of $B(x)$. According to (2.26), $B(x) = b_1(x)b_2(x)/W_b$, so we need to determine the asymptotic behavior of $b_1(x), b_2(x)$, the fundamental solutions of $L_b b(x) = 0$.

By definition, according to (2.12), and with the standard choice (2.18) of the coefficients $A_b^{(1)} = A_b^{(2)} = 1$, these functions satisfy

$$b_1(x) \underset{x\to-\infty}{\longrightarrow} e^{-ikx}\,, \qquad b_2(x) \underset{x\to+\infty}{\longrightarrow} e^{ikx}$$

at the two opposite sides of the one dimensional world. Thus, what remains is to determine the asymptotic behavior of each of these functions on the other side of the world. Since σ and π relax to the vacuum manifold as $x \to \pm\infty$ (Eq. (2.2)), the operators L_b and L_c in (2.9) degenerate into free particle operators, and thus we must have

$$\begin{aligned} b_1(x) \quad &\underset{x\to+\infty}{\longrightarrow} \quad \frac{1}{t_1(k)}e^{-ikx} + \frac{r_1(k)}{t_1(k)}e^{ikx} \\ b_2(x) \quad &\underset{x\to-\infty}{\longrightarrow} \quad \frac{1}{t_2(k)}e^{ikx} + \frac{r_2(k)}{t_2(k)}e^{-ikx}\,, \end{aligned} \tag{A.1}$$

where $t_{1,2}(k), r_{1,2}(k)$ are the appropriate transmission and reflection amplitudes, respectively.

It is enough to consider $k \geq 0$, since the operators L_b and L_c in (2.9) are real and thus $b_{1,2}(x,-k) = b^*_{1,2}(x,k)$, leading to

$$t_{1,2}(-k) = t^*_{1,2}(k) \quad \text{and} \quad r_{1,2}(-k) = r^*_{1,2}(k)\,. \tag{A.2}$$

Thus, $b_1(x)$ corresponds to a setting with a source at $x = +\infty$ which emits to the left, and $b_2(x)$ describes a source at $x = -\infty$ which emits to the right.

The wronskian of $b_1(x)$ and $b_2(x)$ (Eq. (2.14))

$$W_b(\omega) = \frac{b_2(x)b_1^{'}(x) - b_1(x)b_2^{'}(x)}{\omega + \pi(x)}$$

is independent of x. Thus, evaluating it at $x \to \pm\infty$ we find

$$W_b(\omega) = \frac{-2ik}{t_2(k)(\omega + \pi(-\infty))} = \frac{-2ik}{t_1(k)(\omega + \pi(+\infty))}\,, \tag{A.3}$$

and thus,

$$t_1(k)(\omega+\pi(+\infty)) = t_2(k)(\omega+\pi(-\infty))\,. \tag{A.4}$$

Like the wronskian of $\{b_1(x), b_2(x)\}$, the wronkians of the pairs of independent solutions $\{b_1(x), b_1^*(x)\}$ and $\{b_2(x), b_2^*(x)\}$ (here we assume that k is real) are also independent of x. In fact, these wronskians are proportional to the Schrödinger probability currents carried by $b_1(x)$ and $b_2(x)$, respectively. Thus, using (2.12) and (A.1) to evaluate each of these wronskians at $x=\pm\infty$ and then equating the results, we obtain

$$\begin{aligned}
W_b[b_1,b_1^*] &= \frac{b_1^*(x)b_1'(x) - b_1(x)b_1^{*'}(x)}{\omega+\pi(x)} \\
&= \frac{-2ik}{\omega+\pi(-\infty)} = \frac{-2ik}{\omega+\pi(+\infty)}\frac{1-|r_1|^2}{|t_1|^2}\,; \\
W_b[b_2,b_2^*] &= \frac{b_2^*(x)b_2'(x) - b_2(x)b_2^{*'}(x)}{\omega+\pi(x)} \\
&= \frac{2ik}{\omega+\pi(+\infty)} = \frac{2ik}{\omega+\pi(-\infty)}\frac{1-|r_2|^2}{|t_2|^2}\,.
\end{aligned} \tag{A.5}$$

Thus, we deduce the generalized unitarity conditions

$$\frac{1-|r_1|^2}{|t_1|^2} = \frac{|t_2|^2}{1-|r_2|^2} = \frac{\omega+\pi(+\infty)}{\omega+\pi(-\infty)}\,. \tag{A.6}$$

The usual unitarity condition $|t|^2+|r|^2=1$ holds for backgrounds in which $\pi(+\infty)=\pi(-\infty)$.

Putting everything together, we finally learn that

$$\begin{aligned}
B(x) \quad &\underset{x\to-\infty}{\longrightarrow} \quad \frac{1+r_2(k)e^{-2ikx}}{-2ik}(\omega+\pi(-\infty)) \\
B(x) \quad &\underset{x\to+\infty}{\longrightarrow} \quad \frac{1+r_1(k)e^{2ikx}}{-2ik}(\omega+\pi(+\infty))\,.
\end{aligned} \tag{A.7}$$

Recall from (2.34) that

$$B_{VAC}(\pm\infty) = \frac{\omega+\pi(\pm\infty)}{2\sqrt{m^2-\omega^2}} = \frac{\omega+\pi(\pm\infty)}{-2ik} \tag{A.8}$$

corresponds to the appropriate vacuum configurations $(\sigma(\pm\infty), \pi(\pm\infty))$. Thus, we can rewrite (A.7) as

$$B(x) \quad \underset{x\to-\infty}{\longrightarrow} \quad B_{VAC}(-\infty)\left(1+r_2(k)e^{-2ikx}\right)$$

$$B(x) \underset{x\to+\infty}{\longrightarrow} B_{VAC}(+\infty)\left(1 + r_1(k)e^{2ikx}\right) . \tag{A.9}$$

Using (2.27), (2.28) and (2.29) we then find the asymptotic behaviors of the remaining entries of the diagonal resolvent (2.25):

$$\begin{aligned} A(x) = D(x) &\underset{x\to-\infty}{\longrightarrow} D_{VAC}(-\infty) + \frac{ik - \sigma(-\infty)}{2k} r_2(k)e^{-2ikx} \\ A(x) = D(x) &\underset{x\to+\infty}{\longrightarrow} D_{VAC}(+\infty) - \frac{ik + \sigma(+\infty)}{2k} r_1(k)e^{2ikx} , \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} C(x) &\underset{x\to-\infty}{\longrightarrow} C_{VAC}(-\infty)\left[1 + \frac{\sigma(-\infty) - ik}{\sigma(-\infty) + ik} r_2(k)e^{-2ikx}\right] \\ C(x) &\underset{x\to+\infty}{\longrightarrow} C_{VAC}(+\infty)\left[1 + \frac{\sigma(+\infty) + ik}{\sigma(+\infty) - ik} r_1(k)e^{2ikx}\right] , \end{aligned} \tag{A.11}$$

with

$$C_{VAC}(\pm\infty) = \frac{-\omega + \pi(\pm\infty)}{2\sqrt{m^2 - \omega^2}} = \frac{\omega - \pi(\pm\infty)}{2ik} \tag{A.12}$$

from (2.34).

Finally, we can write these results more compactly as

$$\begin{aligned} &\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \underset{x\to+\infty}{\longrightarrow} \begin{pmatrix} A(+\infty) & B(+\infty) \\ C(+\infty) & D(+\infty) \end{pmatrix}_{VAC} + \\ &\begin{pmatrix} \frac{ik+\sigma(+\infty)}{-2k} & B_{VAC}(+\infty) \\ C_{VAC}(+\infty)\frac{\sigma(+\infty)+ik}{\sigma(+\infty)-ik} & \frac{ik+\sigma(+\infty)}{-2k} \end{pmatrix} r_1(k)e^{2ikx} , \\ &\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \underset{x\to-\infty}{\longrightarrow} \begin{pmatrix} A(-\infty) & B(-\infty) \\ C(-\infty) & D(-\infty) \end{pmatrix}_{VAC} + \\ &\begin{pmatrix} \frac{ik-\sigma(-\infty)}{2k} & B_{VAC}(-\infty) \\ C_{VAC}(-\infty)\frac{\sigma(-\infty)+ik}{\sigma(-\infty)-ik} & \frac{ik-\sigma(-\infty)}{2k} \end{pmatrix} r_2(k)e^{-2ikx} . \end{aligned} \tag{A.13}$$

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