THE STRING TENSION
IN TWO DIMENSIONAL GAUGE THEORIES

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We review and elaborate on properties of the string tension in two-dimensional gauge theories. The first model we consider is massive QED in the $m \ll e$ limit. We evaluate the leading string tension both in the fermionic and bosonized descriptions. We discuss the next to leading corrections in m/e. The next-to-leading terms in the long distance behavior of the quark-antiquark potential, are evaluated in a certain region of external versus dynamical charges. The finite temperature behavior is also determined. In QCD_2 we review the results for the string tension of quarks in cases with dynamical quarks in the fundamental, adjoint, symmetric and antisymmetric representations. The screening nature of SYM_2 is re-derived.

Contents

1 Introduction

Two-dimensional gauge theories serve as a theoretical laboratory for studying four-dimensional gauge theories. Non-perturbative issues, such as confinement and spectrum of models, can be addressed in these theories. For recent reviews see Refs. 1, 2.

In this framework, it is also possible to calculate the string tension σ of the confining part of the potential,

$$V = \sigma r \; . \tag{1}$$

The leading term of the string tension (in mass over charge parameter), in the massive Schwinger model ($U(1)$ gauge theory with massive matter, which we call electron), was calculated by using bosonization, long time ago[3]

$$\sigma_{QED} = m\,\mu \left(1 - \cos\left[2\pi \frac{q_{ext}}{q_{dyn}}\right]\right) , \tag{2}$$

where m is the electron mass, $\mu = e\,\mathrm{e}^{\gamma}/(2\pi^{3/2})$, e the gauge coupling, γ the Euler number and q_{ext}, q_{dyn} are the external and dynamical charges respectively (we measure charges in units of e, thus q_{ext} and q_{dyn} are dimensionless).

Note that the string tension vanishes whenever the external charge is an integer multiple of the dynamical charge, $q_{ext} = n\,q_{dyn}$. This is expected, since in this case dynamical charges, via electron-positron pairs, can screen the external source. Explicitly, n pairs are created, with n electrons screening the positive charge and the n positrons the negative one. Another important and somewhat unexpected result is that the string tension vanishes also when $m = 0$. This phenomenon can be explained in several ways. In the massless theory it is easy to produce pairs from the vacuum. Therefore, infinite amount of integer charges which are produced, may form a coherent state with a fractional charge and screen the fractional external charge. A second explanation is that, due to the chiral anomaly, the photon becomes massive resulting in a short range potential.

The expression in massive QCD_2 is[4]

$$\sigma_{QCD} = m\mu_R \sum_i \left(1 - \cos\left[4\pi\lambda_i \frac{k_{ext}}{k_{dyn}}\right]\right) , \tag{3}$$

where $\mu_R = e\mathrm{e}^{\gamma}/(2\pi^{3/2})$, λ_i are the isospin eigenvalues of the dynamical representation, k_{ext} and k_{dyn} are the affine current algebra levels of the external and dynamical representations, respectively. This expression was shown to hold for the fundamental and the adjoint representations. Other representations were also discussed in Ref. 4, with appropriate expressions for the string tension, as generalizations of Eq. (3).

Note that when $m = 0$ the string tension (3) vanishes, as in the Abelian case.[5] The explanation via acquiring mass is now more involved than in QED, as we are now in the gauge dependent sector.[6,7] Another explanation, which has no direct Abelian analogue, is related to the equivalence theorem of Kutasov and Schwimmer.[8] The massless adjoint fermions model is physically equivalent to the multi-flavor massless model with $N_f = N_c$ fermions in the fundamental representation. Therefore the original adjoint fermion can be expressed as a fundamental fermions which can screen the external source. The plan of the paper, which is an expanded version of Ref. 4, 9 but also with several new results, is as follows. In section 2 and 3 we calculate the string tension for the massive Schwinger model in both the fermionic and the bosonic languages. The bosonic language will be useful for the non-Abelian generalization and fermionic language will be useful when we will discuss supersymmetric theories.

Section 4 is devoted to quantum corrections to the string tension. Note that the expressions (2) and (3) are only the leading terms in mass perturbation theory and are valid when $m \ll e$. The next to leading order correction, in the Abelian case, was derived in Ref. 10 and it is reviewed briefly in this section.

In section 5 we discuss the short range corrections to the confining potential. We focus on the Abelian case, believing that the non-Abelian case is very similar. Our conclusion is that apart from the linear potential, a screening part, which arise from a massive component of the photon/gluon, is present.

In section 6 we comment on the behavior of the string tension when finite temperature is introduced. We follow Ref. 11 and conclude that confinement persists even at high temperatures. This is peculiar to two dimensions.

Sections 7 and 8 are devoted to the non-Abelian generalization. We compute the string tension for the cases of matter in the fundamental and adjoint representations (section 7) and symmetric and anti-symmetric representations (section 8). These sections are based on Ref. 4.

In section 9 we show that the string tension vanishes in supersymmetric gauge theories by showing that there is no $\langle tr\, \phi\bar{\lambda}\gamma_5\lambda\rangle = 0$ condensate in these models.

The appendix is devoted to a derivation of the quark anti-quark external current. It is shown that the relevant charge of the external source is the chiral anomaly (the affine Lie algebra level).

2 The Schwinger model

Let us review the derivation of the string tension in the massive Schwinger model, in the fermionic language. Consider the partition function of two dimensional massive QED

$$
\begin{aligned}
Z &= \int DA_\mu D\bar{\Psi} D\Psi \qquad (4)\\
&\times \exp\left(i \int d^2x \left[-\frac{1}{4e^2} F_{\mu\nu}^2 + \bar{\Psi}\, i \not{\partial}\, \Psi - m\bar{\Psi}\Psi - q_{dyn} A_\mu \bar{\Psi}\gamma^\mu \Psi \right] \right),
\end{aligned}
$$

where q_{dyn} is the charge of the dynamical fermions. Gauge fixing terms were not written explicitly. Let us add an external pair with charges $\pm q_{ext}$ at $\pm L$, namely $j_0^{ext} = q_{ext}\left[\delta(x+L) - \delta(x-L)\right]$, so that the change of $\mathcal{L}$ is $-j_\mu^{ext} A^\mu(x)$. Note that by choosing j_μ^{ext} which is conserved, $\partial^\mu j_\mu^{ext} = 0$, the action including the coupling to the external current is also gauge invariant.

Now, one can eliminate this charge by performing a local, space-dependent left-handed rotation

$$
\Psi \to e^{i\alpha(x)\frac{1}{2}(1-\gamma_5)}\, \Psi \qquad (5)
$$

$$
\bar{\Psi} \to \bar{\Psi}\, e^{-i\alpha(x)\frac{1}{2}(1+\gamma_5)}\,, \qquad (6)
$$

where $\gamma^5 = \gamma^0\gamma^1$. We choose a left-handed rotation (or equally well a right-handed one) rather than an axial one, in analogy with the non-abelian case (see section 7), where it is simpler to do so.

The rotation introduce a change in the action, due to the chiral anomaly

$$
\delta S = \int d^2x\, \frac{\alpha(x) q_{dyn}}{2\pi} F\,, \qquad (7)
$$

where F is the dual of the electric field $F = \frac{1}{2}\epsilon^{\mu\nu} F_{\mu\nu}$.

The new action is

$$
\begin{aligned}
S &= \int d^2x \left(-\frac{1}{4e^2} F_{\mu\nu}^2 + \bar{\Psi} i \not{\partial} \Psi - \bar{\Psi}\partial_\mu \alpha(x) \gamma^\mu \frac{1}{2}(1-\gamma_5)\Psi - m\bar{\Psi} e^{-i\alpha(x)\gamma_5}\Psi \right.\\
&- \left. q_{dyn} A_\mu \bar{\Psi}\gamma^\mu\Psi - q_{ext}\left[\delta(x+L) - \delta(x-L)\right] A_0 + \frac{\alpha(x) q_{dyn}}{2\pi} F \right). \qquad (8)
\end{aligned}
$$

The external source and the anomaly term are similar, both being linear in the gauge potential. This is the reason why the θ-vacuum and electron-positron pair at the boundaries are the same in two dimensions.[3] In the following we assume $\theta = 0$, as otherwise we absorb it in α. Choosing the $A_1 = 0$ gauge and integrating by parts, the anomaly term looks like an external source

$$\frac{q_{dyn}}{2\pi} A_0 \partial_1 \alpha(x) \,. \tag{9}$$

This term can cancel the external source by the choice

$$\alpha(x) = 2\pi \frac{q_{ext}}{q_{dyn}} \left[\theta(x+L) - \theta(x-L)\right]. \tag{10}$$

Let us take the limit $L \to \infty$. The form of the action, in the region B of $-L < x < L$ is

$$S_B = \int_B d^2x \left(-\frac{F^2_{\mu\nu}}{4e^2} + \bar{\Psi} i \not{\partial} \Psi - m\bar{\Psi} e^{-i2\pi\gamma_5 q_{ext}/q_{dyn}} \Psi - q_{dyn} A_\mu \bar{\Psi}\gamma^\mu \Psi \right). \tag{11}$$

Thus, the total impact of the external electron-positron pair is a chiral rotation of the mass term. This term can be written as

$$\bar{\Psi} e^{-i2\pi\gamma_5 q_{ext}/q_{dyn}} \Psi = \cos\left[2\pi \frac{q_{ext}}{q_{dyn}}\right] \bar{\Psi}\Psi - i \sin\left[2\pi \frac{q_{ext}}{q_{dyn}}\right] \bar{\Psi}\gamma_5\Psi \,. \tag{12}$$

The string tension is the vacuum expectation value (v.e.v.) of the Hamiltonian density in the presence of the external source relative to the v.e.v. of the Hamiltonian density without the external source, in the $L \to \infty$ limit.

$$\sigma = \langle \mathcal{H} \rangle - \langle \mathcal{H}_0 \rangle_0 \,, \tag{13}$$

where $|0\rangle_0$ is the vacuum state with no external sources. The change in the vacuum energy is due to the mass term. The change in the kinetic term which appears in (8) does not contribute to the vacuum energy.[4] Thus

$$\sigma = m \cos\left[2\pi \frac{q_{ext}}{q_{dyn}}\right] \langle \bar{\Psi}\Psi \rangle - m \sin\left[2\pi \frac{q_{ext}}{q_{dyn}}\right] \langle \bar{\Psi} i\gamma_5 \Psi \rangle - m \langle \bar{\Psi}\Psi \rangle_0 \,. \tag{14}$$

Thus, the values of the condensates $\langle \bar{\Psi}\Psi \rangle$ and $\langle \bar{\Psi}\gamma_5\Psi \rangle$ are needed. The easiest way to compute these condensates is Bosonization, but it can also be computed directly in the fermionic language which at $m = 0$ is[12]

$$\langle \bar{\Psi}\Psi \rangle_{m=0} = -\frac{e}{2\pi^{3/2}} \mathrm{e}^{\gamma} \,, \tag{15}$$

$$\langle \bar{\Psi}\gamma_5\Psi \rangle_{m=0} = 0 \,. \tag{16}$$

Equation (16) is due to parity invariance (with our choice $\theta = 0$). The resulting string tension, to first order in m,

$$\sigma = m \frac{e}{2\pi^{3/2}} \mathrm{e}^{\gamma} \left(1 - \cos \left[2\pi \frac{q_{ext}}{q_{dyn}} \right] \right) . \tag{17}$$

Though this expression is only the leading term in a m/e expansion and might be corrected,[10] when q_{ext} is an integer multiple of q_{dyn} the string tension is *exactly* zero, since in this case the rotated action (11) is not changed from the original one (5).

3 The Schwinger model in Bosonic form

In their seminal paper[3] Coleman, Jackiw and Susskind used the bosonized version of the Schwinger model to calculate the string tension. We present here their calculation, for completeness.

The bosonized Lagrangian, in the gauge $A_1 = 0$, is the following

$$\mathcal{L} = \frac{1}{2e^2}(\partial_1 A_0)^2 + \frac{1}{2}(\partial_\mu \phi)^2 + M^2 \cos(2\sqrt{\pi}\phi) + \frac{q_{dyn}}{\sqrt{\pi}} A_0 \partial_1 \phi - A_0 j_{ext} \, , \tag{18}$$

where $M^2 = m\mu$, $\mu = \mathrm{e}^{\gamma} \mu_{(\phi)}/2\pi$, with $\mu_{(\phi)} = e \; q_{dyn}/\sqrt{\pi}$ the mass of the photon for $e \gg m$.

Chiral rotation corresponds to a shift in the field ϕ. Upon the transformation

$$\phi = \tilde{\phi} + \sqrt{\pi} \frac{q_{ext}}{q_{dyn}} \left[\theta(x+L) - \theta(x-L) \right] . \tag{19}$$

The Lagrangian (18) takes, in the region B, the form

$$\mathcal{L}_B = \frac{1}{2e^2}(\partial_1 A_0)^2 + \frac{1}{2}(\partial_\mu \tilde{\phi})^2 + M^2 \cos \left[2\sqrt{\pi}\tilde{\phi} + 2\pi \frac{q_{ext}}{q_{dyn}} \right] + \frac{q_{dyn}}{\sqrt{\pi}} A_0 \partial_1 \tilde{\phi} \, . \tag{20}$$

Hence, similarly to the previous derivation, a local chiral rotation may be used to eliminate the external source. The calculation of the string tension is exactly the same as in the previous section.

The relevant part of the Hamiltonian density is

$$\mathcal{H} = -M^2 \cos \left[2\sqrt{\pi}\tilde{\phi} + 2\pi \frac{q_{ext}}{q_{dyn}} \right] . \tag{21}$$

To zeroth order in $(M/e)^2$, the vacuum is $\tilde{\phi} = 0$. Setting this choice in (21) and subtracting the v.e.v. of the free Hamiltonian, we arrive at (2).

4 Beyond the small mass Abelian string tension

The expression (2) contains only the leading m/e contribution to the Abelian string tension. This expression was computed in section 3, using a classical average. However, as we used the normal ordering scale μ_ϕ which is the photon mass for $e \gg m$, taking $\tilde{\phi} = 0$ actually gives the full quantum answer, as is evident by comparing with the fermionic calculation of section 2.

The full perturbative (in m) string tension can be written as[10]

$$\sigma_{QED} = m\mu \sum_{l=1}^{\infty} C_l \left(\frac{m}{eq_{dyn}}\right)^{l-1} \left(1 - \cos\left[2\pi l \frac{q_{ext}}{q_{dyn}}\right]\right) . \tag{22}$$

The first coefficient is $C_1 = 1$ and the next one is $C_2 = -8.91\mathrm{e}^{\gamma}/(8\pi^{1/2})$.[13] Higher coefficients have not been calculated yet.

Note that for finite m/e we have to minimize the potential

$$V = M^2 \left(1 - \cos\left[2\sqrt{\pi}\phi + 2\pi\frac{q_{ext}}{q_{dyn}}\right]\right) + \frac{\mu_\phi^2 \phi^2}{2} . \tag{23}$$

The minimum $\phi = \phi_m$ obeys

$$2\sqrt{\pi} M^2 \sin\left[2\sqrt{\pi}\phi_m + 2\pi\frac{q_{ext}}{q_{dyn}}\right] + \mu_\phi^2 \phi_m = 0 . \tag{24}$$

Thus, for the first order $m/(eq_{dyn})$ correction, we get a C_2 which is $-\sqrt{\pi}\mathrm{e}^{\gamma}/2$. This has the same sign, but a factor 1.41 larger, than the instanton contribution of Ref. 10.

Note that all above results for the string tension are symmetric under change of sign of the external charge, as expected on general grounds. However, when a θF term is introduced, we get odd terms as well, like $\sin(l\theta)\sin(2\pi l q_{ext}/q_{dyn})$.[10] The even terms are multiplied by $\cos(l\theta)$.

We expect that similar corrections as those in Eq. (22) will occur in the non-Abelian case. For the fundamental/adjoint case, the following expression may correct the leading term (3)

$$\sigma_{QCD} = m\mu_R \sum_{l=1}^{\infty} \tilde{C}_l \left[\frac{m}{ek_{dyn}}\right]^{l-1} \sum_j \left(1 - \cos\left[4\pi\lambda_j l \frac{k_{ext}}{k_{dyn}}\right]\right) . \tag{25}$$

Finally, let us remark that for very large m/e, the abelian case has a string tension which is $e^2 q_{ext}^2/2$.

5 Correction to the leading long distance Abelian potential

The potential (1) is the dominant long range term. However, there are, of course, corrections. In this section we present these corrections.

In the static case, the equations of motions, which follow from the bosonized Lagrangian (18), are

$$-\frac{1}{e^2}\,\partial_1^2 A_0 + \frac{q_{dyn}}{\sqrt{\pi}}\,\partial_1\phi - j_{ext} = 0\,, \tag{26}$$

$$-\,\partial_1^2\phi + 2\sqrt{\pi}M^2\sin\left(2\sqrt{\pi}\phi\right) + \frac{q_{dyn}}{\sqrt{\pi}}\,\partial_1 A_0 = 0\,. \tag{27}$$

In order to solve these equation, it is useful to eliminate the bosonized matter field ϕ. Using the approximation $\sin(2\sqrt{\pi}\phi) \sim 2\sqrt{\pi}\phi$, we arrive (in momentum space) at

$$A_0(k) = \frac{e^2(k^2+4\pi M^2)}{k^2(k^2+(4\pi M^2+e^2q_{dyn}^2/\pi))}\,j_{ext}(k)\,, \tag{28}$$

where k is the Fourier transform of the space coordinate. We will discuss the validity of our approximation for ϕ later in this section. The last equation can be rewritten as

$$A_0(k) = \left(\frac{m_1^2}{m_2^2}\frac{1}{k^2} + \left[1-\frac{m_1^2}{m_2^2}\right]\frac{1}{k^2+m_2^2}\right)e^2 j_{ext}(k)\,, \tag{29}$$

where

$$m_1^2 = 4\pi M^2\,, \tag{30}$$

$$m_2^2 = 4\pi M^2 + (e^2/\pi)\,q_{dyn}^2\,. \tag{31}$$

Note that the photon propagator has two poles: a massless pole, which reproduces the string tension, and a massive pole, which adds a screening term to the potential. Note that there is no *const*/L correction, which appears in higher dimensions,[14] since in the present case the string cannot fluctuate in transverse directions.

Note also that in the massless case, when $M^2 = 0$, only the second term survives and the photon has only one pole with mass square $e^2q_{dyn}^2/\pi$. This result is of course exact, independent of our approximation.

The resulting gauge field is

$$A_0(x) = \frac{2\pi^2M^2q_{ext}}{q_{dyn}^2}\Big(\mid x+L\mid - \mid x-L\mid\Big) \tag{32}$$

$$-\;\frac{e\sqrt{\pi}q_{ext}}{2q_{dyn}}\left(e^{-eq_{dyn}|x+L|/\sqrt{\pi}} - e^{-eq_{dyn}|x-L|/\sqrt{\pi}}\right),$$

where we took $M^2 \ll e^2$ for simplicity.

In order to calculate the potential we will use

$$V = \frac{1}{2}\int A_0(x) j_{ext}(x) dx\,. \tag{33}$$

Hence the potential is

$$V = 2\pi^2 M^2 \frac{q_{ext}^2}{q_{dyn}^2} \times 2L + \frac{e\sqrt{\pi}}{2}\frac{q_{ext}^2}{q_{dyn}}\left(1 - e^{-eq_{dyn}2L/\sqrt{\pi}}\right). \tag{34}$$

The first term is the confining potential which exists whenever the quark mass is non-zero. On top of it, there is always a screening potential.

The string tension which results from the above potential is

$$\sigma = m\mu \times 2\pi^2 \frac{q_{ext}^2}{q_{dyn}^2}\,, \tag{35}$$

which is exactly (2) in the approximation $2\pi q_{ext}/q_{dyn} \ll 1$. This turns out to be also the condition for $\sin(2\sqrt{\pi}\phi) \sim 2\sqrt{\pi}\phi$ that we assumed in the start of this section. To see that, we solve for ϕ from Eq. (26) as

$$\phi(k) = -ik\frac{q_{dyn}}{\sqrt{\pi}}\frac{e^2}{m_2^2}\left(\frac{1}{k^2} - \frac{1}{k^2 + m_2^2}\right) j_{ext}(k)\,. \tag{36}$$

Define $\phi = \phi_1 + \phi_2$, where ϕ_1 is the part with $1/k^2$, and ϕ_2 with $1/(k^2 + m_2^2)$. The ϕ_2 part goes to zero at long distances, i.e. $k \to 0$. As for the ϕ_1 part, its x-space form is

$$\phi_1(x) = \frac{e^2}{\sqrt{\pi}m_2^2} q_{dyn} q_{ext}\left[\theta(x+L) - \theta(x-L)\right], \tag{37}$$

which for small m/e reduced to

$$\phi_1(x) \sim \sqrt{\pi}\frac{q_{ext}}{q_{dyn}}\left[\theta(x+L) - \theta(x-L)\right]. \tag{38}$$

Thus, small $2\sqrt{\pi}\phi$ means

$$2\pi\frac{q_{ext}}{q_{dyn}} \ll 1\,, \tag{39}$$

the condition mentioned before.

Note that we could generalize the argument to values of $2\pi q_{ext}/q_{dyn}$ that are close to $2\pi n$, with integer n.

6 Finite temperature

In this section we would like to comment on the behavior of the string tension in the presence of finite temperature. It is interesting to check whether the string is torn due to high temperature and whether the system undergoes a phase transition from confinement to de-confinement.

The prescription for calculating quantities at finite temperature T is to formulate the theory on a circle in Euclidean time with circumference $\beta = T^{-1}$.

For the purpose of calculating the string tension, we can follow the same steps which were used in sections 2 and 3 leading to a modification of Eq. (17) as (a comprehensive discussion of this issue is given in Ref. 11)

$$\sigma = -m\langle\bar{\Psi}\Psi\rangle_T \left(1 - \cos\left[2\pi\frac{q_{ext}}{q_{dyn}}\right]\right). \tag{40}$$

It is enough to calculate $\langle\bar{\Psi}\Psi\rangle_T$, the condensate at finite temperature, in the massless Schwinger model.

Following Ref. 12, the chiral condensate behaves as

$$\langle\bar{\Psi}\Psi\rangle_{(T\to 0)} \to -\frac{e}{2\pi^{3/2}}\,\mathrm{e}^{\gamma} \tag{41}$$

and

$$\langle\bar{\Psi}\Psi\rangle_{(T\to\infty)} \to -2Te^{-\pi^{3/2}T/\mathrm{e}}. \tag{42}$$

The above result (42) indicates that the string is not torn even at very high temperatures. The explicit expression[11] shows that $\langle\bar{\Psi}\Psi\rangle_T$ is non-zero for all T. Thus, the system does not undergoes a phase transition. It is just energetically favorable to have the electron-positron pair confined.

7 Two-dimensional QCD

The action of bosonized QCD_2 with massive quarks in the fundamental representation of $SU(N)$ is [1]

$$\begin{aligned} S_{fundamental} &= \frac{1}{8\pi}\int_\Sigma d^2x\; tr(\partial_\mu g\partial^\mu g^\dagger) \\ &+ \frac{1}{12\pi}\int_B d^3y\,\epsilon^{ijk} tr(g^\dagger\partial_i g)(g^\dagger\partial_j g)(g^\dagger\partial_k g) \\ &+ \frac{1}{2}m\mu_{fund}\int d^2x\; tr(g+g^\dagger) - \int d^2x\;\frac{F^a_{\mu\nu}F^{a\mu\nu}}{4e^2} \\ &- \frac{1}{2\pi}\int d^2x\; tr\big(i\,g^\dagger\partial_+ g\,A_- + i\,g\partial_- g^\dagger A_+ + A_+ gA_- g^\dagger - A_+A_-\big)\,, \end{aligned} \tag{43}$$

where e is the gauge coupling, m is the quark mass, $\mu = e\,e^{\gamma}/(2\pi)^{3/2}$, g is an $N\times N$ unitary matrix, A_μ is the gauge field and the trace is over $U(N)$ indices. Note, however, that only the $SU(N)$ part of the matter field g is gauged.

When the quarks transform in the adjoint representation, the expression for the action is[15]

$$\begin{aligned} S_{adjoint} &= \frac{1}{16\pi}\int_\Sigma d^2x\; tr(\partial_\mu g \partial^\mu g^\dagger) + \frac{1}{24\pi}\int_B d^3y\, \epsilon^{ijk} tr(g^\dagger\partial_i g)(g^\dagger\partial_j g)(g^\dagger\partial_k g) \\ &+ \frac{1}{2} m\mu_{adj}\int d^2x\; tr(g+g^\dagger) - \int d^2x\; \frac{F^a_{\mu\nu}F^{a\mu\nu}}{4e^2} \\ &- \frac{1}{4\pi}\int d^2x\; tr\big(i\, g^\dagger\partial_+ g\, A_- + i\, g\partial_- g^\dagger A_+ + A_+ g A_- g^\dagger - A_+A_-\big)\,. \end{aligned} \tag{44}$$

A version which takes into account instanton effects is given elsewhere[16] but for our purposes it will not be needed.

The action (44) differs from (43) by a factor of one half in front of the WZW and interaction terms, because g is real and represents Majorana fermions. Another difference is that g now is an $(N^2-1)\times(N^2-1)$ orthogonal matrix. The two actions (43) and (44) can be schematically represented by one action

$$S \;=\; S_0 + \frac{m\mu_R}{2}\int d^2x\; tr(g+g^\dagger) - \frac{ik_{dyn}}{4\pi}\int d^2x\; (g\partial_- g^\dagger)^a A^a_+\,, \tag{45}$$

where $A_-=0$ gauge was used, S_0 stands for the WZW action and the kinetic action of the gauge field, k_{dyn} is the level (the chiral anomaly) of the dynamical charges ($k=1$ for the fundamental representation of $SU(N)$ and $k=N$ for the adjoint representation).

Let us add an external charge to the action. We choose static (with respect to the light-cone coordinate x^+) charge and therefore we can omit its kinetic term from the action. Thus, an external charge coupled to the gauge field would be represented by

$$-\frac{ik_{ext}}{4\pi}\int d^2x\; (u\partial_- u^\dagger)^a A^a_+\,.$$

Suppose that we want to put a quark and an anti-quark at a very large separation. A convenient choice of the charges would be a direction in the algebra in which the generator has a diagonal form. The simplest choice is a generator of an $SU(2)$ subalgebra. Since a rotation in the algebra is always possible, the results are insensitive to this specific choice. As an example we write down the generator in the case of fundamental and adjoint representations:

$$T^3_{fund} = diag\,\big(1/2, -1/2, \underbrace{0, 0, \ldots, 0}_{N-2}\big),$$

$$T^3_{adj} = diag\left(1,\, 0, -1,\, \underbrace{1/2, -1/2,\, 1/2, -1/2,\, \ldots,\, 1/2, -1/2}_{2(N-2)\ \ doublets},\, \underbrace{0,\, 0,\, \ldots,\, 0}_{(N-2)^2}\right).$$

Generally, T^3 can be written as

$$T^3 = diag\ \left(\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_i,\, \ldots,\, 0,\, 0,\, \ldots\right),$$

where $\{\lambda_i\}$ are the 'isospin' components of the representation under the $SU(2)$ subgroup.

We take the $SU(N)$ part of u as (see Appendix)

$$u = \exp\left\{-i4\pi\left[\theta\left(x^- + L\right) - \theta\left(x^- - L\right)\right] T^3_{ext}\right\} \tag{46}$$

for $N > 2$ and similar expression with a 2π factor for $N = 2$. T^3_{ext} represents the '3' generator of the external charge and u is static with respect to the light-cone time coordinate x^+. The theta function is used as a limit of a smooth function which interpolates between 0 and 1 in a very short distance. In that limit $u = 1$ everywhere except at isolated points, where it is not well defined.

The form of the action (45) in the presence of an external source is

$$S = S_0 + \frac{m\mu_R}{2}$$
$$\times \int d^2x\ \left\{tr(g + g^\dagger) + \left[-\frac{ik_{dyn}}{4\pi}(g\partial_- g^\dagger)^a + k_{ext}\delta^{a3}\left[\delta(x^- + L) - \delta(x^- - L)\right]\right] A^a_+\right\}.$$

The external charge can be eliminated from the action by a transformation of the matter field. A new field $\tilde{g}$ can be defined as follows

$$-\frac{ik_{dyn}}{4\pi}\ (\tilde{g}\partial_-\tilde{g}^\dagger)^a = -\frac{ik_{dyn}}{4\pi}\ (g\partial_- g^\dagger)^a + k_{ext}\delta^{a3}\ \left[\delta(x^- + L) - \delta(x^- - L)\right].$$

This definition leads to the following equation for $\tilde{g}^\dagger$

$$\partial_-\tilde{g}^\dagger = \tilde{g}^\dagger\left(g\partial_- g^\dagger + i4\pi\frac{k_{ext}}{k_{dyn}}\left[\delta(x^- + L) - \delta(x^- - L)\right] T^3_{dyn}\right). \tag{47}$$

The solution of (47) is

$$\tilde{g}^\dagger = P\exp\left\{\int dx^-\left(g\partial_- g^\dagger + i4\pi\frac{k_{ext}}{k_{dyn}}\left[\,\delta\left(x^- + L\right) - \delta\left(x^- - L\right)\right] T^3_{dyn}\right)\right\}$$

$$= \exp\left[i4\pi\frac{k_{ext}}{k_{dyn}}\theta(x^- + L)T^3_{dyn}\right]\ g^\dagger\ \exp\left[-i4\pi\frac{k_{ext}}{k_{dyn}}\theta(x^- - L)T^3_{dyn}\right],$$

where P denotes path ordering and we assumed that T^3_{dyn} commutes with $g\partial_- g^\dagger$ for $x^- \geq L$ and with $g^\dagger$ for $x^- = -L$ (as we shall see, this assumption is self consistent with the vacuum configuration).

Let us take the limit $L \to \infty$. For $-L < x^- < L$, the above relation simply means that

$$g = \tilde{g}\ \exp\left[i4\pi \frac{k_{ext}}{k_{dyn}} T^3_{dyn}\right].$$

Since the Haar measure is invariant (and finite, unlike the fermionic case) with respect to unitary transformations, the form of the action in terms of the new variable $\tilde{g}$ reads

$$\begin{aligned} S \;=\; & S_{WZW}(\tilde{g}) + S_{kinetic}(A_\mu) - \frac{ik_{dyn}}{4\pi}\int d^2x\ (\tilde{g}\partial_-\tilde{g}^\dagger)^a A^a_+ \\ & + \frac{m\mu_R}{2}\int d^2x\ tr\left\{\tilde{g}\ \exp\left[i4\pi\frac{k_{ext}}{k_{dyn}}\,T^3_{dyn}\right] + \exp\left[-i4\pi\frac{k_{ext}}{k_{dyn}}\,T^3_{dyn}\right]\tilde{g}^\dagger\right\}, \end{aligned} \tag{48}$$

which is QCD$_2$ with a chiraly rotated mass term.

From Eq. (48), the string tension can be easily calculated.[3] It is simply the vacuum expectation value (v.e.v.) of the Hamiltonian density, relative to the v.e.v. of the Hamiltonian density of the theory without an external source,

$$\sigma = \langle H\rangle - \langle H_0\rangle .$$

The vacuum of the theory is given by $\tilde{g} = 1$. In terms of the variable g, this configuration points in the '3' direction and hence satisfies our assumptions while solving Eq. (47). The v.e.v. is

$$\begin{aligned} \langle H\rangle = & -\frac{m\mu_R}{2}\ tr\left\{\exp\left[i4\pi\frac{k_{ext}}{k_{dyn}}\,T^3_{dyn}\right] + \exp\left[-i4\pi\frac{k_{ext}}{k_{dyn}}\,T^3_{dyn}\right]\right\} \\ = & -m\mu_R\sum_i \cos\left[4\pi\lambda_i\frac{k_{ext}}{k_{dyn}}\right]. \end{aligned}$$

Therefore the string tension is

$$\sigma = m\mu_R\sum_i\left(1-\cos\left[4\pi\lambda_i\frac{k_{ext}}{k_{dyn}}\right]\right), \tag{49}$$

which is the desired result.

A few remarks should be made:

(i) The string tension (49) reduces to the abelian string tension (2) when abelian charges are considered. It follows that the non-abelian generalization is realized by replacing the charge q with the level k.

(ii) The string tension was calculated in the tree level of the bosonized action. Perturbation theory (with m as the coupling) may change Eq. (49), since the loop effects may add $O(m^2)$ contributions. However, we believe that it would not change its general character. In fact, one feature is that the string tension vanishes for any m when (k_{ext}/k_{dyn}) is an integer, as follows from Eq. (48), since the action does not depend then on k_{ext} at all.

(iii) When no dynamical mass is present, the theory exhibits screening. This is simply because non-abelian charges at the end of the world interval can be eliminated from the action by a chiral transformation of the matter field.

(iv) When the test charges are in the adjoint representation $k_{ext} = N$, equation (49) predicts screening by the fundamental charges (with $k_{dyn} = 1$).

(v) String tension appears when the test charges are in the fundamental representation and the dynamical charges are in the adjoint one.[17] The value of the string tension is

$$\sigma = m\mu_{adj} \Big\{ 2 \left[1 - \cos\left(4\pi/N\right) \right] + 4(N-2) \left[1 - \cos\left(2\pi/N\right) \right] \Big\} , \qquad (50)$$

as follows from Eq. (48) for this case.

The case of $SU(2)$ is special. The 4π which appears in Eq. (49) is replaced by 2π, since the bosonized form of the external $SU(2)$ fundamental matter differs by a factor of a half with respect to the other $SU(N)$ cases (see Appendix). Hence, the string tension in this case is $4m\mu_{adj}$.

(vi) We would like to add, that when computing the string tension in the pure YM case with external sources in representation R, the Wilson loop gives $e^2C_2(R)/2$, while our way of defining external source gives $e^2k_{ext}^2/2$. Thus, we need a factor $C_2(R)/k_{ext}^2$ to bring our result to the Wilson loop case. Analogous factors should be computed for the other cases, when dynamical matter is also present.

8 Symmetric and anti-symmetric representations

The generalization of (49) to arbitrary representations is not straightforward. However, we can comment about its nature (without rigorous proof).

Let us focus on the interesting case of the antisymmetric representation. One can show (in a way similar to Ref. 15) that the WZW action, with g taken to be $\frac{1}{2}N(N-1)\times\frac{1}{2}N(N-1)$ unitary matrices, is a bosonized version of QCD_2 with fermions in the antisymmetric representation.

The antisymmetric representation is described in the Young-tableaux notation by two vertical boxes. Its dimension is $N(N-1)/2$ and its diagonal $SU(2)$ generator is

$$T^3_{as} = diag\Big(\underbrace{\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},...,\frac{1}{2},-\frac{1}{2}}_{(N-2)\ doublets},0,0,...,0\Big) \tag{51}$$

and consequently, $k = N-2$. When the dynamical charges are in the fundamental and the external in the antisymmetric the string tension should vanish because tensor product of two fundamentals include the antisymmetric representation. Indeed, (49) predicts this result.

The more interesting case is when the dynamical charges are antisymmetric and the external are fundamentals. In this case, the value of the string tension depends on whether N is odd or even.[17] When N is odd the string tension should vanish because the anti-fundamental representation can be built by tensoring the antisymmetric representation with itself $(N-1)/2$ times. When N is even string tension must exist. Note that (49) predicts

$$\sigma = 2m\mu_{as}(N-2)\big(1-\cos\left[\,2\pi/(N-2)\right]\big), \tag{52}$$

which is not zero when N is odd.

The resolution of the puzzle seems to be the following. Non-Abelian charge can be static with respect to its spatial location. However, its representation may change in time due to emission or absorption of soft gluons (without cost of energy). Our semi-classical description of the external charge as a c-number is insensitive to this scenario. We need an extension of (46) which takes into account the possibilities of all various representations. One possible extension is

$$j^a_{ext} = \delta^{a3}k_{ext}(1+lN)\left[\,\delta\,(x^-+L)-\delta\,(x^- - L)\right], \tag{53}$$

where l is an arbitrary positive integer. This extension takes into account the cases which correspond to $1+lN$ charges multiplied in a symmetric way.
The resulting string tension is

$$\sigma = m\mu_R \sum_i \left(1-\cos\left[4\pi\lambda_i\frac{k_{ext}}{k_{dyn}}(1+lN)\right]\right), \tag{54}$$

which includes the arbitrary integer l. What is value of l that we should pick?

The dynamical charges are attracted to the external charges in such a way that the total energy of the configuration is minimal. Therefore the value of l which is needed, is the one that guarantees minimal string tension.

Thus, the extended expression for string tension is the following

$$\sigma = \min_{l} \left\{ m\mu_R \sum_i \left(1 - \cos \left[4\pi\lambda_i \frac{k_{ext}}{k_{dyn}} (1 + lN) \right] \right) \right\}. \tag{55}$$

In the case of dynamical antisymmetric charges and external fundamentals and odd N, $l = (N-3)/2$ gives zero string tension. When N is even the string tension is given by (52).

The expression (55) yields the right answer in some other cases also, like the case of dynamical charges in the symmetric representation. The bosonization for this case can be derived in a similar way to that of the antisymmetric representation, and T^3 is given by

$$T^3_{symm} = diag\Big(1, 0, -1, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}}_{(N-2)\ doublets}, 0, 0, ..., 0\Big) \tag{56}$$

and therefore $k = N + 2$. When the external charges transform in the fundamental representation and N is odd, Eq. (55) predicts zero string tension (as it should). When N is even the string tension is given by

$$\sigma = 2m\mu_{symm}\Big\{\Big(1 - \cos\big[4\pi/(N+2)\big]\Big) + (N-2)\Big(1 - \cos\big[2\pi/(N+2)\big]\Big)\Big\}.$$

We discussed only the cases of the fundamental, adjoint, anti-symmetric, and symmetric representations since we used bosonization techniques, which are applicable to a limited class of representations.[18]

9 Supersymmetric Yang-Mills

The same technique can be used to prove screening in SYM_2 . In this case, the action is[19]

$$S = \int d^2x\ tr \left(-\frac{1}{4e^2} F^2_{\mu\nu} + i\bar{\lambda}\, \not{D}\lambda + \frac{1}{2}(D_\mu\phi)^2 - 2ie\phi\bar{\lambda}\gamma_5\lambda \right), \tag{57}$$

where A_μ is the gluon field, λ the gluino (a Majorana fermion) and ϕ a pseudoscalar, are the components of the vector supermultiplet and transform as the adjoint representation of $SU(N_c)$. Also $D_\mu = \partial_\mu - i[A_\mu, \ldots]$.

The action (57) is invariant under SUSY

$$\begin{aligned} \delta A_\mu &= -\, ie\,\bar{\epsilon}\,\gamma_5\gamma_\mu\sqrt{2}\,\lambda\,, \\ \delta\phi &= -\,\bar{\epsilon}\,\sqrt{2}\,\lambda\,, \\ \delta\lambda &= \frac{1}{2\sqrt{2}\,e}\,\epsilon\,\epsilon^{\mu\nu}F_{\mu\nu} + \frac{i}{\sqrt{2}}\,\gamma^\mu\epsilon D_\mu\phi\,. \end{aligned}$$

We now introduce an external current. The external source breaks explicitly supersymmetry. However, this breaking does not affect our derivation. We assume a semi-classical quark anti-quark pair which points in some direction in the algebra. Without loss of generality this direction can be chosen as the '3' direction ('isospin'). The additional part in the Lagrangian is $-tr\ j^{ext}_{\mu} A^{\mu}$, where $j_0^{a\ ext} = \big[C(R_{ext})\big]\,\delta^{a3}\big[\,\delta(x+L) - \delta(x-L)\,\big]$ and $\big[C(R_{ext})\big]$ is a c-number which depends on the representation of the external source, in analogy with k_{ext} of Chapter 7. The interaction term can be eliminated by a left-handed rotation of the gluino field in the '3' direction (we are using a spherical basis, and so we can perform appropriate complex transformation also for real fermions)

$$\lambda \to \tilde{\lambda} \;=\exp\biggl(i\alpha(x)\,\frac{1}{2}\,(1-\gamma_5)\,T^3\biggr)\,,\lambda \tag{58}$$

$$\bar{\lambda} \to \tilde{\bar{\lambda}} \;= \bar{\lambda}\,\exp\biggl(-\,i\alpha(x)\,\frac{1}{2}\,(1+\gamma_5)\,T^3\biggr)\,. \tag{59}$$

T^3 is in the 3 direction of the adjoint representation

$$\begin{aligned} T^3 &= diag\,\big(\mu_1,\mu_2,...,\mu_{N_c^2-1}\big)\\ &= diag\,\big(\,1,\,0,-1,\,\underbrace{1/2,-1/2,\,1/2,-1/2,\,...,\,1/2,-1/2}_{2(N_c-2)\ \ doublets},\,\underbrace{0,\,0,\,...\,,\,0}_{(N_c-2)^2}\big)\,. \end{aligned}$$

The chiral rotation introduces an anomaly term $tr\big\{\,[\alpha(x)T^3/4\pi]\ F\big\}$, which is used to cancel the external charges.

The choice

$$\alpha(x) = 2\pi\frac{C(R_{ext})}{N_c}\big[\,\theta\,(x+L) - \theta\,(x-L)\big]$$

leads to an action which is similar to the original (57), but has a chiral rotated term. The information of the external source is now transformed into a rotation angle.

The terms which are relevant to the computation of the string tension are those which appear in the interaction Lagrangian. In this case, it is the gluino pseudo-scalar term

$$tr\Big[2i\phi\bar{\lambda}\gamma_5\lambda\Big] \to tr\Big[2i\phi\tilde{\bar{\lambda}}\gamma_5\tilde{\lambda}\Big]\,. \tag{60}$$

Let us see how this change influences the Hamiltonian vacuum energy. In the original theory, without the external source, the Hamiltonian H_0 has no v.e.v., since the theory is supersymmetric and $H_0 \sim Q^2$ (where Q is the supercharge). In particular it means that there is no $\langle tr\ \phi\bar{\lambda}\gamma_5\lambda\rangle$ condensate.[a] The reason is the following. The light-cone Hamiltonian density of the system is given by

$$\mathcal{H} = tr\Big[e^2\Big(\frac{1}{\partial_-}j^+\Big)^2\Big] + tr\Big[2ie\phi\bar{\lambda}\gamma_5\lambda\Big], \tag{61}$$

where j^+ denotes the total, scalar and gluino, current which couples to the gauge field. SUSY implies that $\langle\mathcal{H}\rangle = 0$. In addition we may use the Feynman-Helman theorem

$$0 = \Big\langle\frac{\partial\mathcal{H}}{\partial e}\Big\rangle = tr\Big[2e\Big(\frac{1}{\partial_-}j^+\Big)^2\Big] + tr\Big[2i\phi\bar{\lambda}\gamma_5\lambda\Big]. \tag{62}$$

Thus, there are no non-trivial condensates:

$$\Big\langle tr\Big[e^2\Big(\frac{1}{\partial_-}j^+\Big)^2\Big]\Big\rangle = \Big\langle tr\ F^2\Big\rangle = 0\,, \tag{63}$$

$$\Big\langle tr\Big[2ie\phi\bar{\lambda}\gamma_5\lambda\Big]\Big\rangle = 0\,. \tag{64}$$

Note that we assumed that SUSY is not broken dynamically. The numerical analysis[20] indicates that this is indeed the case.

Let us compute the Hamiltonian density of the rotated theory. In the regime $-L < x < L$

$$\langle\mathcal{H}\rangle = 2ie\Big\langle tr\Big[\phi\tilde{\bar{\lambda}}\gamma_5\tilde{\lambda}\Big]\Big\rangle\,. \tag{65}$$

By using the fact that T^3 is diagonal, and the vacuum state is color symmetric, we get

$$\Big\langle tr\Big[\phi\tilde{\bar{\lambda}}\gamma_5\tilde{\lambda}\Big]\Big\rangle = \frac{1}{N_c^2-1}\Big\{\sum_a \cos(\alpha\mu_a)\langle tr\ \phi\bar{\lambda}\gamma_5\lambda\rangle - i\sum_a \sin(\alpha\mu_a)\langle tr\ \phi\bar{\lambda}\lambda\rangle\Big\}, \tag{66}$$

where $\alpha = \lim_{L\to\infty}\alpha(x)$. The first term on the right hand side vanishes since as argued before $\langle\, tr\,[\phi\bar{\lambda}\gamma_5\lambda]\,\rangle = 0$, and the second term vanishes since the isospin eigenvalues μ_a come in pairs of opposite signs.

Thus, $\langle\mathcal{H}\rangle = 0$ and the string tension is zero.

Note that though we used the classical expression for the external current and the effective Hamiltonian may include other terms, these terms cannot change the value of the string tension. It is so because this theory contains only one dimension-full parameter, the gauge coupling e, and therefore the string tension is some number times e^2. We showed that this number is zero

[a]We are grateful to D.J. Gross for a discussion about this issue.

and higher terms in e which may appear in the effective Hamiltonian cannot affect the string tension.

The meaning of the last result is that a quark anti-quark pair located at $x = \pm\infty$ generate a linear potential. Physically, it is a consequence of infinitely many adjoint fermions and scalars which are produced from the vacuum, as there is no mass gap, that are attracted to the external source, form a soliton in the fundamental representation and result in screening it. A complementary argument in Refs. 6 and 21 is that due to loop effects, the intermediate gauge boson acquires a mass $M^2 \sim e^2 N_c$, which leads to a Yukawa potential between the external quark anti-quark pair.

The above result can be generalized to theories with extended supersymmetry and additional massive or massless matter content.

We argue that any supersymmetric gauge theory in two dimensions is screening. Technically, the reason is that the gluino is coupled to other fields in such a way that $\langle\mathcal{H}\rangle = 0$ (guaranteed if SUSY is not broken dynamically) and therefore there are no non-trivial chiral condensates. However, since the string tension is proportional to chiral condensates, SUSY leads to zero string tension. Physically, it follows from the fact that the gluino is an adjoint *massless* fermion. Since it does not acquire mass, external sources are screened, as in the non-supersymmetric massless model. Recently the spectrum of various supersymmetric models was derived. The shape of the spectrum confirms our prediction.[20,22]

In fact, the essential requirement for a screening nature of the type argued above, is to have among the charged particles at least one massless particle whose masslessness is protected by an unbroken symmetry. The symmetry can be gauge symmetry combined with supersymmetry or chiral symmetry.

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Appendix – The external field

We give here a detailed derivation of the external quark anti-quark field (46) for $N > 2$ and with 2π in exponent for $N = 2$.

For the case of external charges in a real representation the u field can chosen to point in some special direction in the $SU(N)$ algebra which we take to be '3', namely $u = \exp(-i\,T^3\phi)$. For external charges in a complex representation one has to dress this ansatz with a baryon number part, namely $u = \exp(-i\chi)\exp(-i\,T^3\phi)$. Let us view the external source as the limit of a dynamical variable with very large mass. Let us choose the gauge $A_-{=}0$. Then we can take $A_+{=}a_+T^3$, as the other directions do not couple. The Lagrangian for the real case takes the form

$$\begin{aligned}\mathcal{L} = {} & \frac{k}{8\pi}\,(\partial_-\phi)(\partial_+\phi) + \frac{1}{2e^2}\,(\partial_-a_+)^2 \\ & + M^2\sum_i \cos\left(\lambda_i\phi\right) + \frac{k}{4\pi}\,\partial_-(\phi a_+)\,,\end{aligned} \tag{67}$$

where k is the level and λ_i the isospin entries of the diagonal sub $SU(2)$ generator T^3.

The equations of motion for the matter and gauge fields are

$$\frac{k}{4\pi}\,\partial_-\partial_+\phi + M^2\sum_i \lambda_i\sin\left(\lambda_i\phi\right) + \frac{k}{4\pi}\,\partial_-a_+ = 0\,, \tag{68}$$

$$\partial_-^2\,a_+ = e^2\,\frac{k}{4\pi}\,\partial_-\phi\,. \tag{69}$$

Integrating (69) with zero boundary conditions and substituting in (68) we obtain

$$\frac{k}{4\pi}\,\partial_-\partial_+\phi + M^2\sum_i \lambda_i\sin\left(\lambda_i\phi\right) + e^2\left(\frac{k}{4\pi}\right)^2\phi = 0\,. \tag{70}$$

Let us assume a solution for ϕ which describes an infinitely heavy light-cone static quark anti-quark system

$$\phi = \alpha\left[\,\theta\,(x^-\!+L) - \theta\,(x^-\!-L)\,\right], \tag{71}$$

where α is a yet unknown coefficient.

For the region $-L<x^-<L$ we obtain

$$M^2\sum_i \lambda_i\sin\left(\lambda_i\alpha\right) + e^2\left(\frac{k}{4\pi}\right)^2\alpha = 0\,. \tag{72}$$

When $M^2 \gg e^2$ the solution for α is of the form

$$\alpha{=}4\pi n{+}\epsilon\,, \tag{73}$$

where n is integer (we will pick the minimal $n=1$ possibility) and the small parameter ϵ is determined by the substitution in (72):

$$M^2\sum_i \lambda_i^2\epsilon + e^2\left(\frac{k}{4\pi}\right)^2 4\pi \approx 0\,. \tag{74}$$

Thus, α is given by

$$\alpha = 4\pi - \frac{e^2}{M^2}\left(\frac{k}{4\pi}\right)^2 \frac{4\pi}{\sum_i \lambda_i^2} + O\Big(\big(e^2/M^2\big)^2\Big)\,. \tag{75}$$

In the limit $M^2 \to \infty$, u is

$$u = \exp\big\{-i4\pi\big[\,\theta\,(x^- + L) \;-\; \theta\,(x^- - L)\big]\,T^3\big\}\,. \tag{76}$$

When u is in a complex representation $u = \exp(-i\chi)\exp(-iT^3\phi)$, we find the following expression by repeating the above derivation:

$$\begin{aligned} u &= \exp\big\{-i2\pi\big[\,\theta\,(x^- + L) \;-\; \theta\,(x^- - L)\big]\big\} && (77)\\ &\quad\times \exp\big\{-i4\pi\big[\,\theta\,(x^- + L) \;-\; \theta\,(x^- - L)\big]\,T^3\big\} \qquad && \text{for } U(N > 2) \end{aligned}$$

and

$$\begin{aligned} u &= \exp\big\{-i\pi\big[\,\theta\,(x^- + L) \;-\; \theta\,(x^- - L)\big]\big\} && (78)\\ &\quad\times \exp\big\{-i2\pi\big[\,\theta\,(x^- + L) \;-\; \theta\,(x^- - L)\big]\,T^3\big\} \qquad && \text{for } U(2). \end{aligned}$$

Note that the $SU(2)$ part has a 2π prefactor. The reason is that for $SU(2)$, it is the only case where the adjoint does not contain isospin 1/2.

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