APPROXIMATION METHODS IN CLASSICAL AND QUANTUM THEORIES

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We show in this article how in certain cases Misha Marinov's ideas and advices produced strong influence, which lasts even after his passing away. In particular, his deep understanding of classical and quantum description of particles and fields enabled us to find simple approximate solutions to a few interesting problems, e.g. the Cerenkov effect in a gravitational field and a new approach to the periodic motion including the effects of spin in General Relativity.

Contents

1 Introduction **677**
1.1 Encounters with Misha Marinov 677
1.2 Simplicity as Art 678

2 Cherenkov radiation in gravitational background **680**

3 Radiating charges in the gravitational wave **684**
3.1 The gravitational wave background 684
3.2 Particle dynamics in the GW background 685

4 Epicycles in General Relativity **686**
4.1 World-line deviation equations 687
4.2 Application: the Coulomb-Reissner-Nordstrom field 688

5 Quasiclassical spinning bodies **692**

References **695**

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1 Introduction

1.1 Encounters with Misha Marinov

Although I had known some of Misha Marinov's work since long ago, especially his celebrated paper with Felix Berezin[1] and his original explanation of path integral methods in Quantum Theory,[2] I met him personally for the first time only in March 1993. It happened during the workshop *"Mathematical Physics towards the 21-st Century"* organized by R. Sen at the Ben Gurion University of Beersheva.

Before the end of the eighties it was quite difficult to travel to Russia privately, and international meetings held there were much scarcer than nowadays. And when came to visit Moscow in 1990 (and it was for me the first time since 1958) on the occasion of the 18-th International Colloquium on Group Theoretical Methods in Physics, Misha was already in Israel, where he could finally go with his family after eight years of enduring hardships imposed on them by Soviet system just because they expressed their will to leave for the only country in the world they could consider as their homeland.

Sharing Russian as mother tongue, we very quickly started to spend most of the time discussing and making comments on everything, physics first, of course, but also politics, history, Russian and Jewish literature and history. We listened to the radio news together, which I tried to understand with my quite rudimental knowledge of Hebrew, while Misha translated more difficult parts. I have identified the most frequently used words, "piguah" and "bitakhon", and in his usual ironical style Misha told me that these two words are fairly enough to understand the essence of the present situation.

Misha was only four years elder than myself, but his seniority was more than what these few years would suggest. He had the privilege to learn physics directly from Lev Landau and Evgeniy Lifshits, and to start his career under the guidance of Isaac Pomeranchuk. I was full of admiration for his deep and complete knowledge of theoretical physics; but what impressed me even more were his exceptional personal qualities, his integrity and firmness in opinions which he was ready to defend and stand up for, combined with a rare tolerance and openness towards the opinions of other people. He was always ready to listen and to discuss, no matter how far the opinions of his opponent could be from his own ones.

Very soon have met again at one of the Group Theoretical Conferences organized by H.D. Doebner in Goslar; then we have met in Poland, again in Germany, later on in Israel (where I visited him twice at the Technion, in 1996 and in 1999). He also visited me in Paris, as invited Professor to our Relativity and Cosmology Group (which I directed at that time) in February

1998. At that time we have become friends, also with his charming wife Lilia and his daughters Masha and Dina. Many memorable days and evenings were spent together, often with other friends invited to Marinov's home in Haifa, in particular Louis Michel and his wife Thérèse, and Julius Wess.

We have discussed a lot, and I have learned a lot, but we had no time to develop close collaboration e.g. writing papers together. We just discussed his last papers with his Ph.D. student E. Strahov, or my future projects, which ended up as articles published with other collaborators. Misha was one of the last students of Lev Landau, and my encounters with him did give me a better insight in the methods of thinking of Landau's school, above all the quest for simplicity and elegance. Several papers written since then are an attempt to achieve these standards.

1.2 Simplicity as Art

Very often the most powerful and long lasting ideas in physics are also the simplest ones. being aware that in the majority of cases we have to content ourselves with approximate solutions, the approximation methods become a crucial issue in physical problems we are dealing with.

When I visited Misha in Israel last time (early 1999), he showed me a book by the Academician V. Ginzburg with a beautiful personal dedication (the Academician Ginzburg has visited Misha in Haifa a few months before my short stay in the Technion). I have borrowed it and read almost all the articles it contained. In particular, I have discovered how beautiful was Ginzburg's explanation of the Cerenkov effect; then it gave me a few ideas, and with collaborators (one Russian, Sasha Balakin from Kazan, another Portuguese, Jose Lemos from Lisbon) we wrote a paper on the possibility of a Cerenkov-type effect from the electromagnetic waves interacting (very weakly, of course) via non-minimal terms of the type $R_{ijkl} F^{ij} F^{kl}$ with gravitation.[13] This paper has been published in "Classical and Quantum Gravity" in 2000. Another discussion with Misha helped me to better understand the solitons (see e.g. the paper[7] written with D.V. Gal'tsov in 2000).

Finally, Ginzburg's technique of developing everything in a Fourier series inspired our latest papers[16,14] on a new approach to the two-body problem in General Relativity – treating it as an infinite series of approximations (geodesic deviations from a circular orbit) – again, these results, in collaboration with Jan-Willem van Holten from Amsterdam, Sasha Balakin and my Ph.D. student Roberto Colistete appeared in "Classical and Quantum Gravity."

Another important influence can be traced back to Misha's visit to Paris as an Invited Professor in early 1998, and it also concerns an approximation technique. At that time Misha worked with his Ph.D. student Eugene Strahov on the adiabatic approximation for the description of spin in a variable magnetic field. The behavior of proton spin in varying magnetic field is an important problem which occurs in the analysis of nuclear magnetic resonance data. Its direct treatment turns out to be very difficult, but when the variations of the external field are not too rapid, the so-called *adiabatic approximation* can be successfully used. The results of this investigations have been published in Ref. 3.

The idea of separating slow and fast phenomena, or strong and weak ones, is inherent in the Fourier expansion and in some sense was always present in the description of motions of planets in our Solar system. Indeed, the Ptolemaic system was already based on such a separation – the first approximation given by the "secular" circular movement, then smaller perturbations, called "epicycles", being added to it. Such separation often becomes the only reliable tool when one is dealing with non-linear equations.

Curiously enough, this approximation technique has not been much used in the two-body problem in General Relativity. The most common approach was to start with the approximation given by the Newtonian theory, and then to add up progressive corrections which take into account various relativistic effects. A lot of important results have been obtained in, e.g., Refs. 22 and 23.

But if we start from the geodesic equation in General Relativity, describing the motion of a test particle with negligible mass, we can find an exact solution in the case of special symmetry. Circular orbits with constant four-velocity represent such exact solutions in the case of spherically symmetric gravitaional fields, especially in Schwarzschild and Reissner-Nordstrom metrics. Small perturbations around this solution satisfy the well known *Jacobi equation*, or the *geodesic deviation equation*. Higher-order generalizations of this equations are quite straightforward, and have been known since a long time.

When applied to the particular case of circular orbits in spherically symmetric gravitational field, these equations give rise to a series of linear systems whose solutions are nothing else but harmonic oscillators with frequencies given by the characteristic eigenvalues of the system. Next approximations give further corrections in the form of "epicycles", which are the consecutive terms in a Fourier series expansion of this quasi-periodic motion.

These results have been obtained quite recently in collaboration with A.Balakin, J.-W. van Holten and R. Colistete Jr.[16,14] Here I shall expose the main lines of this work, stressing the similarity with quasi-classical treatment of spin in the external field. This is my tribute to the direct and indirect influence Misha Marinov's deep understanding of Physics had on me during the last years of his life.

2 Cherenkov radiation in gravitational background

The radiation emission stimulated by a particle moving with a supraluminal speed in a dielectric medium, first observed by Cherenkov and Vavilov, and theoretically explained by Tamm and Frank, is one of the cornerstones of classical electrodynamics (see, e.g., Refs. 4 and 5). This phenomenon, called the Cherenkov radiation, is currently used in measuring devices of elementary particle physics, e.g. in high-energy collider detectors.

The Cherenkov radiation can be explained in simple terms in Minkowskian spacetime. Although it can be explained also in purely classical terms, the Quantum Oscillator approach introduced by V.L. Ginzburg is by far more elegant. It uses the fact that the two interacting entities, the electromagnetic field and the charged particle producing it, can be decomposed into an infinite sum of contributions having the form of quantum oscillators. The resonance terms present on the right-hand side of Maxwell equations (representing the source) give rise to spontaneous emission of quanta, which are interpreted as Cherenkov's waves.

Consider a charged particle moving uniformly with velocity $\mathbf{V}$ in a static isotropic dielectric medium with the refraction index n. This particle induces an electromagnetic field which can be represented as a Fourier integral as follows,

$$\mathbf{A} = \sum_l \left(q_l \mathbf{A}_l + q_l^* \mathbf{A}_l^* \right) \;\; \text{with } l = 1, 2, ...\infty, \;\; \text{and } \mathbf{A}_l = \frac{c}{n}\, \mathbf{e}_l \, e^{(i\mathbf{k}_l \cdot \mathbf{r})} \,. \tag{1}$$

Then, in Lorentz gauge $\partial_\mu A^\mu = 0$, and from

$$\frac{n^2}{c^2}\,\frac{\partial^2 \mathbf{A}}{\partial t^2} - \triangle \mathbf{A} = \frac{1}{c}\mathbf{j} \qquad \text{and} \qquad \frac{n^2}{c^2}\,\frac{\partial^2 \varphi}{\partial t^2} - \triangle \varphi = \frac{1}{c}\,\rho \,, \tag{2}$$

we get, in terms of Fourier components,

$$\frac{d^2 q_l}{dt^2} + \omega^2 q_l = \frac{1}{c} \int \mathbf{j} \cdot \mathbf{A}_l^* \, dV \,. \tag{3}$$

For a point particle with electric charge e moving with constant velocity $\mathbf{V}$

$$\mathbf{j} = e\mathbf{V}\, \delta \left(\mathbf{r} - \mathbf{V}t \right), \tag{4}$$

and we get the equation of an oscillator solicited by an external force:

$$\frac{d^2 q_l}{dt^2} + \omega^2 q_l = \frac{e}{n}\, \left(\mathbf{e}_l \cdot \mathbf{V} \right)\, e^{-i\,(\mathbf{k}_l \cdot \mathbf{V})\, t} \,, \tag{5}$$

in which each component l displays the frequency $\Omega_l = \mathbf{k}_l \cdot \mathbf{V} = k_l\, V \cos\theta$, where θ is the angle between the direction of the wave three-vector $\mathbf{k}_l$ and the three-velocity of the particle $\mathbf{V}$. When $\Omega_l = \omega$, a resonance occurs, making oscillator's amplitude grow linearly with t, which is interpreted as condition for radiation. In order for this radiation to represent a real electromagnetic wave, the relation between the frequency and the wave number should be $\omega = kc/n$. Since one must always have $k > k\cos\theta$, one finds that this condition is equivalent to

$$V > \frac{c}{n}\,, \tag{6}$$

i.e. the velocity of the particle should be greater than the velocity of light in the medium c/n. It is also clear that the maximal angle of radiation, the Cherenkov angle, is given by the condition

$$\cos\theta_0 = \frac{c}{nV}\,. \tag{7}$$

A possibility of the Cherenkov radiation emission by charged particles traveling with a quasi-luminal speed in vacuum where the strong gravitational field is present, is suggested by a sentence of the book "Classical Field Theory" of Landau and Lifshitz,[6] in the problem of paragraph 90, where it is stated that the gravitational field plays a role of a medium with electric and magnetic permeabilities equal to $1/\sqrt{h}$, where h is the determinant of the three-dimensional spatial induced metric. Then one may ask the following two questions: In which types of gravitational fields can the Cherenkov radiation be emitted? Is the Cherenkov effect possible in vacuum hosting a gravitational field?

This problem was taken up in Ref. 8, where the Cherenkov radiation inside a material medium was considered. One may also consider the Cherenkov radiation in a gravitational wave background with vacuum interacting with curvature. More specifically, it was possible to show that a covariant analysis of this problem, based on exact solutions for the coupled system of a charged particle and an electromagnetic wave in a nonlinear gravitational wave background, makes it possible to answer these two questions.[12,13]

One should distinguish between pure vacuum and vacuum interacting with curvature.[9] The latter behaves as a medium with electric and magnetic properties, and can, therefore, be called a quasi-medium. The results of our investigation show that quasi-media and true media can display the Cherenkov effect, while the pure vacuum can not.

In order to adapt it to the curved space-time, the criterion for the Cherenkov radiation emitted by a charged particle must be reformulated in a covariant way. In a covariant formulation we cannot use the three-velocity vector $\mathbf{V}$, or the Cherenkov angle θ_0, because they are not Lorentz invariant quantities. There are two covariant vectors in this problem, the time-like momentum four-vector of the charged particle P^i, normalized according to

$P_i P^i = m^2 c^2 > 0$, and the wave four-vector k_i characterizing the electromagnetic plane wave which can, in principle, propagate inside the medium in a given space-time. We say therefore that the Cherenkov radiation can exist when the following equality is satisfied :

$$k_m P^m = 0\,, \tag{8}$$

where Latin indices are spacetime indices running from 0 to 3, and where we use the metric with signature $(+\;-\;-\;-)$.

One can easily recover the relations (6) and (7). In a standard three-dimensional context in Minkowskian space-time the relationship (8) can be rewritten in the usual form

$$k_m P^m = k_0 P^0 - \mathbf{k}\cdot\mathbf{P} = \frac{P^0}{c}\,(\omega - \mathbf{k}\cdot\mathbf{V}) = \frac{P^0\omega}{c}\Big(1 - \frac{1}{c}\,\mathbf{n}\cdot\mathbf{V}\Big) = 0\,. \tag{9}$$

The classical definition of frequency is $\omega = ck^0$, the vector refraction index is $\mathbf{n} = c\mathbf{k}/\omega$, and its square is $n^2 = \mathbf{n}^2$. We can now reproduce the criterion to have Cherenkov's radiation in flat space-time, i.e. we can reproduce Eqs. (6) and (7). For instance, using the definition of the angle θ, i.e. $\cos\theta = \mathbf{k}\cdot\mathbf{V}/kV = \mathbf{n}\cdot\mathbf{V}/nV$, as well as the condition $|\cos\theta| \leq 1$, one can see from (9) that $V \geq c/n$, i.e. particle's three-velocity should be bigger than speed of light in the medium, recovering (6). Equation (7) also follows in a straightforward way. Let us now proceed to find the necessary condition for the existence of Cherenkov's radiation. It can be given in three equivalent ways, the first involving the square $k_i k^i$ of the wave four-vector k^i, the second one involving the square n^2 of the scalar refraction index, and the third one involving the phase velocity of light in the medium, $v_{\rm ph}$. Note that this necessary condition does not depend on particle's momentum P^i.

It is well known that the frequency ω (considered as a scalar quantity), and the four-vector k_i^* (the spatial part of the wave four-vector) may be defined by the following relations (see, e.g., Ref. 10):

$$\frac{\omega}{c} \equiv k_i U^i \qquad \text{and} \qquad k_i^* \equiv k_l \Delta_i^l\,. \tag{10}$$

Here U^i is the four-velocity vector of the medium (or of the observer, if we consider propagation in pure vacuum), and Δ_i^l is the projector defined as

$$\Delta_j^i \equiv \delta_j^i - U^i U_j\,. \tag{11}$$

Following the standard definition (see p. 290 in Ref. 4), we introduce now the four-vector index of refraction n_i

$$n_i = \frac{c}{\omega}\,k_i^*\,. \tag{12}$$

The vector refraction index n_i is in general a spacelike four-vector, orthogonal to the four-velocity U_i, and only in an isotropic medium it reduces to a scalar. Its absolute value is referred to as the scalar index of refraction, (or refraction index). Its square is given by $n^2 \equiv -\, g^{ik} n_i n_k$.

We can now write the following useful identity satisfied by k^i,

$$k_i = U_i \,(k_l U^l) + k_i^* \,. \tag{13}$$

Now, in order to use the criterion (8) we note that P^i and k^i must be orthogonal. This happens only when k_i is spacelike, since the P^i four-vector is timelike. From the equation (13) we obtain the square of K_i,

$$k^2 \equiv -\, g^{im} k_i k_m = -\, k_m k^m = -\, (k^m U_m)^2 - k^l k^s \Delta_{ls} \,. \tag{14}$$

This equation can be written explicitly in two ways,

$$k^2 = \left(\frac{\omega}{c}\right)^2 (n^2 - 1) \qquad \text{or} \qquad k^2 = (k^*)^2 - \frac{\omega^2}{c^2} \,, \tag{15}$$

where

$$(k^*)^2 \equiv -\, k^l k^s \Delta_{ls} \,. \tag{16}$$

From the equation (16) we see that the sign of k^2 coincides with that of $(n^2{-}1)$. This means that k_i is spacelike for $n^2 > 1$. From the equation (15) we also see that k_i is spacelike when $\omega^2/c^2 < (k^*)^2$, i.e. when the phase velocity of light $v_{\mathrm{ph}} \equiv \omega/k^*$ obeys $v_{\mathrm{ph}} < c$.

Three equivalent invariant forms of the necessary condition for Cherenkov's radiation can be used, namely,

$$k_m k^m < 0 \,, \qquad \text{or} \qquad n^2 > 1 \,, \qquad \text{or} \quad v_{\mathrm{ph}} < c \,. \tag{17}$$

All of them require the knowledge of the four-vector k^i, which is obtained from the corresponding solution of Maxwell equations.

In order to use the criterion (8) in a curved background we have to resort now to a specific model. We have used as a background the pp-wave solution of Einstein equations in vacuum. First, the four-momentum P^i of a particle moving in this background is determined, then a specific solution of Maxwell equations is found, in the same gravitational background. After that, we can examine the criteria of existence of the Cherenkov radiation, and establish its spatial properties in this background. In a curved space-time different spatial directions are generally non-equivalent, which implies that one should know the evolution of particle's four-momentum P^i with arbitrary initial data in order to be able to use explicitly the criterion (8).

3 Radiating charges in the gravitational wave

3.1 The gravitational wave background

Let us consider the space-time described by the exact pp-wave solution of Einstein's equations in vacuum.[11] The metric describing a gravitational wave propagating in the x^1 direction is supposed to take on the following form:

$$ds^2 = 2\,du\,dv - L^2 \left[e^{2\beta} {(dx^2)}^2 + e^{-2\beta} {(dx^3)}^2 \right], \tag{18}$$

where

$$u = \frac{ct - x^1}{\sqrt{2}} \qquad \text{and} \qquad v = \frac{ct + x^1}{\sqrt{2}} \tag{19}$$

are the retarded and the advanced times , respectively. The functions L and β depend only on the variable u, i.e. $L = L(u)$ and $\beta = \beta(u)$.

The pp-wave metric (18) is invariant under the G_5 symmetry group and admits the following set of five Killing vector fields $\xi_{(r)}$ (where the index (r) takes on the values (v), (2), (3), (4), (5) and characterizes each vector),

$$\xi^i_{(v)} = \delta^i_v\,, \qquad \xi^i_{(2)} = \delta^i_2\,, \qquad \xi^i_{(3)} = \delta^i_3\,,$$

$$\xi^i_{(4)} = x^2 \delta^i_v - \delta^i_2 \int g^{22}(u)\,du\,, \qquad \xi^i_{(5)} = x^3 \delta^i_v - \delta^i_3 \int g^{33}(u)\,du\,. \tag{20}$$

Here $g^{\alpha\beta}(u)$ $(\alpha, \beta = 2, 3)$ are the contravariant components of the metric tensor. The vector $\xi^i_{(v)}$ is isotropic, covariantly constant and orthogonal to the other four ones,

$$\nabla_k\, \xi^i_{(v)} = 0\,, \qquad g_{ik}\, \xi^i_{(v)}\, \xi^k_{(r)} = 0\,. \tag{21}$$

The three vectors $\xi^i_{(v)}$, $\xi^i_{(2)}$, and $\xi^i_{(3)}$ form the Abelian subgroup G_3 . The two functions $L(u)$ and $\beta(u)$ are coupled by the Einstein equation

$$L^{''} + L\ (\beta^{'})^2 = 0\,, \tag{22}$$

which is unique in this case. The function $\beta(u)$ can be chosen at will, and once given, one can solve the equation (22) for $L(u)$. The curvature tensor has two non-vanishing components

$$-\,R^2_{\cdot u2u} = R^3_{\cdot u3u} = L^{-2}\left[L^2\,\beta'\right]'. \tag{23}$$

Both the Ricci tensor R_{ik} and the curvature scalar R are equal to zero.

3.2 Particle dynamics in the GW background

The geodesic equation for a particle with mass m in the GW field (18) reads

$$\frac{DP_i}{D\tau} = 0\,, \qquad \text{with} \quad P^i = mc\,\frac{dx^i}{d\tau}\ . \tag{24}$$

Using the well-known property of the Killing vectors (20), we obtain the following expressions for the components of the momentum:

$$P_v \equiv P_i \xi^i_{(v)} = \text{const} \equiv C_v\,, \qquad P_\alpha \equiv P_i\, \xi^i_{(\alpha)} = \text{const} \equiv C_\alpha\,, \tag{25}$$

$$P_u = \frac{1}{2C_v}\left[m^2c^2 - g^{\alpha\beta} C_\alpha C_\beta\right], \tag{26}$$

where the last equation for the component P_u of the momentum followed from the normalization condition.

Since C_v and C_α are constants, they also represent the initial values of the corresponding momentum components at the initial surface defined by $u = 0$. These data determine the character of particle's motion. For example, when $C_\alpha = 0$, i.e. when the particle moves initially along the longitudinal direction (the direction of propagation of the GW x^1), then it will always move along that direction without acceleration. When $C_\alpha \neq 0$, the dynamical effects on the particle induced by the GW appear in its longitudinal motion (see the expression (26) for P_u). Thus, in the field of GW, the criterion (8) yields the following equation:

$$K_m P^m = K_u C_v + K_v\,\frac{1}{2C_v}\left[m^2c^2 - g^{\alpha\beta}(u)\, C_\alpha C_\beta\right] + g^{\alpha\beta}(u)\, K_\alpha C_\beta = 0\,. \tag{27}$$

It remains to represent explicitly the components of the wave four-vector K_i and we also have to check the condition (17), necessary for the emission of the Cherenkov radiation. The full solution of the combined field and particle motion equations has been found in Ref. 13 The main result is that although a plane gravitational wave modifies the dielectric properties of pure vacuum, it does not modify its scalar refraction index, and therefore there is no possibility of emission of the Cherenkov radiation in this particular case.

On the other hand, vacuum interacting with curvature, considered as a sort of quasi-medium, allows the possibility of the existence of the Cherenkov radiation. Cherenkov's radiation in vacuum interacting with curvature can propagate along a non-circular cone, the spatial structure of this cone being permanently modified with time. The Cherenkov radiation emission exists alternatively for each polarization of the electromagnetic wave.

Thus, we deal with a new phenomenon in Cherenkov's radiation, namely, polarized radiation with oscillating polarization direction. The Cherenkov angle is predetermined by the value of the Riemann tensor.

We have performed an exact treatment in two particular cases, with two privileged directions of particle motion, the longitudinal and the transversal ones. However, for arbitrary direction of particle motion, the structure and the inclination of the cone axis with respect to the GW front plane is given by more complicated modified expressions.

The dispersion phenomena for the Cherenkov radiation can not be ruled out, as the effective refraction index induced by the ambient curvature is also a function of the wavelength emitted. However, these effect seem to be rather of the academic interest, as their order of magnitude is apparently very small.

4 Epicycles in General Relativity

In General Relativity the problem of motion of planets, considered as test particles moving along geodesic lines in the metric of Schwarzschild's solution, has been solved in an approximate way by Einstein,[20] who found that the perihelion advance during one revolution is given in the near-Keplerian limit by the formula

$$\Delta\phi = \frac{6\pi\, GM}{a(1-e^2)} \ , \tag{28}$$

where G is Newton's gravitational constant, M the mass of the central body, a the greater half-axis of planet's orbit and e its eccentricity.

This formula is deduced from the exact solution of the General Relativistic problem of motion of a test particle in the field of Schwarzschild metric, which leads to the expression of the angular variable φ as an elliptic integral, which is then evaluated after expansion of the integrand in terms of powers of the small quantity GM/r.

The formula has been successfully confronted with observation, and represents one of the best confirmations of Einstein's theory of gravitation. In the case of small eccentricities the formula (28) can be developed into a power series:

$$\Delta\phi = \frac{6\pi GM}{a}\,(1 + e^2 + e^4 + e^6 + \dots)\ . \tag{29}$$

One can note at this point that even for the case of Mercury ($e = 0.251$), the series truncated at the second term, i.e. taking into account only the factor $(1 + e^2)$ will lead to the result that differs only by 0.18% from the result predicted by relation (28), which is below the actual error bar.

This is why it is useful to present an alternative way of treating this problem, based on the use of geodesic deviation equations of first and higher orders. Instead of developing the exact formulae of motion in terms of powers of the parameter GM/r, we can start with an exact solution of a particularly simple form (i.e. a circular orbit with uniform angular velocity), and then generate approximate solutions as geodesics being close to this orbit.

One of the advantages of this method is the fact that it amounts to treating consecutively systems of linear equations with constant coefficients, all of them being of harmonic oscillator type, eventually with an extra right-hand side being a known periodic function of the proper time. The approximate solution obtained in this manner has the form of a Fourier series and represents the closed orbit as a superposition of epicycles with diminishing amplitude as their circular frequencies grow as multiples of the basic one. This approach is particularly well-suited for using numerical computations.

In this respect, it is similar to the treatment[3] of the spin precession in slowly varying magnetic field.

4.1 World-line deviation equations

According to the equivalence principle, structureless test bodies (sometimes referred to as point masses) in a gravitational field move on geodesics of space-time. Their world-line $x^\mu(\tau)$ is a solution of the geodesic equation

$$\frac{D^2 x^\mu}{D\tau^2} = \frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\lambda\nu}{}^{\mu}\, \frac{dx^\lambda}{d\tau}\, \frac{dx^\nu}{d\tau} = 0\,, \tag{30}$$

where the world-line parameter τ is the proper time. Introducing the four-velocity as the time-like tangent unit vector to the world-line: $u^\mu = dx^\mu/d\tau$, the equation can be written in geometrical language as

$$u \cdot \nabla u = 0\,, \qquad u^2 = -1\,. \tag{31}$$

with ∇ the covariant derivative. It is easily observed from Eq. (30), that the proper acceleration $a^\mu = d^2 x^\mu/d\tau^2$ is not a covariant object. In particular, its vanishing or non-vanishing has no observer-independent meaning.

In contrast, the *relative* acceleration between world lines is a covariant quantity, and its vanishing or non-vanishing does not depend on the frame of reference.[10] Consider a one-parameter congruence of geodesics $x^\mu(\tau;\lambda)$, where λ labels the geodesics and τ is the proper-time parameter along the geodesic. We suppose the parametrization to be smooth, hence we can construct the tangent vector fields $u^\mu = \partial x^\mu/\partial\tau$, and $n^\mu = \partial x^\mu/\partial\lambda$. It is straightforwardly established that

$$(u\cdot\nabla n)^{\mu} = \frac{\partial^2 x^{\mu}}{\partial\tau\,\partial\lambda} + \Gamma_{\lambda\nu}{}^{\mu}\,\frac{\partial x^{\lambda}}{\partial\tau}\,\frac{\partial x^{\nu}}{\partial\lambda} = (n\cdot\nabla u)^{\mu}. \tag{32}$$

As a corollary, we obtain

$$u\cdot\nabla(u\cdot\nabla n) = n\cdot\nabla(u\cdot\nabla u) + u^{\mu}n^{\nu}\,[\nabla_{\mu},\nabla_{\nu}]\,u = u^{\mu}n^{\nu}R_{\mu\nu}[u,\cdot\,]\,. \tag{33}$$

In component notation this reads

$$\frac{D^2 n^{\mu}}{D\tau^2} \;=\; R_{\kappa\lambda\nu}{}^{\mu}u^{\kappa}u^{\nu}n^{\lambda}\,. \tag{34}$$

If $x_0^{\mu}(\tau) = x^{\mu}(\tau;\lambda_0)$ is a solution of the geodesic equation (30), then (to first order) $x_1^{\mu} = x_0^{\mu} + n^{\mu}\Delta\lambda$ is a solution as well:

$$x^{\mu}(\tau;\lambda_1) = x^{\mu}(\tau;\lambda_0) + \Delta\lambda\,\frac{\partial x^{\mu}}{\partial\lambda}\,(\tau,\lambda_0) \approx x^{\mu}(\tau\,;\,\lambda_0+\Delta\lambda)\,. \tag{35}$$

It follows, that Eq. (34) describes the covariant relative acceleration between these world lines. Of course, n^{μ} is only a first approximation to the neighboring geodesic at $\lambda_1 = \lambda_0 + \Delta\lambda$. To increase the precision of the approximation, one has to compute higher-order derivatives w.r.t. λ, by solving higher-order versions of Eq. (34), involving not only the Riemann curvature tensor, but its derivatives as well. A systematic procedure of this type can be found in Ref. 14 .

4.2 Application: the Coulomb-Reissner-Nordstrom field

World-line deviation equations can be used to compute the relative motion between particles in given background fields, or to obtain an approximation to solutions for orbits close to a known one. We illustrate the general results with an application to the study of the motion of charged particles in a central gravitational and electric Coulomb-Reissner-Nordstrom field.

The vector potential and electric field strength for the Coulomb part of this solution of the Einstein-Maxwell equations are given by the one- and two-forms

$$A = -\,\frac{Q}{4\pi r}\;dt\,, \qquad F = dA = \frac{Q}{4\pi r^2}\;dr\wedge dt\,, \tag{36}$$

whilst the metric for the gravitational field can be taken as

$$-d\tau^2 = -B(r)\,dt^2 + \frac{1}{B(r)}\,dr^2 + r^2\big(d\theta^2 + \sin^2\theta\,d\varphi^2\big)\,, \tag{37}$$

where $B(r) = 1-(2M/r)+(Q^2/r^2)$, Q and M are the charge and the mass of the central body which is the source of the field.

The orbits of particles with mass m and charge q in this background can be computed in closed form in terms of elliptic integrals.[16] More precisely, the orbits are given by

$$r(\varphi) = \frac{r_0}{1 + e\cos y(\varphi)}\ , \tag{38}$$

where $y(\varphi)$ is the solution of the differential equation

$$\frac{dy}{d\varphi} = \sqrt{A + B\cos y + C\cos^2 y}\ , \tag{39}$$

with coefficients given by

$$\begin{aligned} A &= 1 + \frac{Q^2}{\ell^2}\left(1 - \left(\frac{q}{4\pi m}\right)^2\right) - \frac{6M}{r_0} + \frac{Q^2}{r_0^2}\,(6 + e^2)\,, \\ B &= -\,\frac{2e}{r_0}\left(M - \frac{2Q^2}{r_0}\right), \qquad C = \frac{e^2 Q^2}{r_0^2}\,. \end{aligned} \tag{40}$$

Here ℓ is the constant angular momentum per unit of mass. As the periastra of the orbit are at the points $y(\varphi) = 2\pi n$, one can now compute the angular distance $\Delta\varphi$ between successive periastra. From $\Delta\varphi = 2\pi + \delta\varphi$, it follows that the periastron shift per orbit is

$$\delta\,\varphi = 2\pi\left(\frac{3M}{r_0} - \frac{Q^2}{2Mr_0}\right) + \ \dots\ , \tag{41}$$

the dots denoting terms of higher order in e, M/r_0 or Q/r_0.

Eqs. (38) and (39) describe a general orbit in the exterior region of the central body. However, they do not provide all information about the orbit. In particular, as the time coordinate has been eliminated from these equations, the solution does not tell us where in its orbit the test particle is at any moment. Such information can be relevant for some important applications, e.g. to compute estimates of the amount of electro-magnetic and gravitational radiation emitted by the system. The method of world-line deviations is useful to obtain parametrized expressions of orbits $(r(t), \varphi(t))$.

As the reference orbit, the zeroth order approximation to the real orbit, we take a circular one with constant radial coordinate R. Constants of motion on all orbits are the angular momentum per unit of mass, $\ell = \omega R^2$, with the angular velocity $\omega = \dot{\varphi}$, and the energy ε per unit of mass defined by

$$\frac{dt}{d\tau} = \frac{\varepsilon -\ q\,Q/4\pi mR}{1 - 2M/R + Q^2/R^2}\ . \tag{42}$$

Then, on circular orbits the constants R, ℓ and ε are related by

$$\left(\varepsilon - \frac{qQ}{4\pi m R}\right)^2 = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)\left(1 + \frac{\ell^2}{R^2}\right), \tag{43}$$

and

$$\left[\frac{\ell^2}{R} - M\left(1 + \frac{3\ell^2}{R^2}\right) + \frac{Q^2}{R}\left(1 + \frac{2\ell^2}{R^2}\right)\right]^2 \tag{44}$$

$$= \left(\frac{qQ}{4\pi m}\right)^2\left(1 + \frac{\ell^2}{R^2}\right)\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right).$$

As all orbits are planar, we can always choose the orientation of the coordinate system such that $\theta = \pi/2$ for the reference orbit. For orbits tilted w.r.t. this one, we then find from Eq. (63) that

$$\ddot{n}^\theta + \omega^2 n^\theta = 0\,, \tag{45}$$

from which it follows, as the physics dictates, that the distance perpendicular to the plane of the reference orbit oscillates with the period of the circular orbit $T = 2\pi/\omega = 2\pi R^2/\ell$.

Considering orbits in the plane of the reference orbit, the world-line deviation equations (63) for the other components $n^i = (n^t, n^r, n^\varphi)$ become

$$\ddot{n}^i + \gamma^i{}_j \dot{n}^j + m^i{}_j n^j = 0\,, \tag{46}$$

where the coefficient matrices take the form

$$\gamma = \begin{pmatrix} 0 & \gamma^t{}_r & 0 \\ \gamma^r{}_t & 0 & \gamma^r{}_\varphi \\ 0 & \gamma^\varphi{}_r & 0 \end{pmatrix}, \qquad m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m^r{}_r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

This represents a system of coupled linear oscillators, which has solutions

$$n^t(\tau) = n^t_0 \sin\omega_1\tau\,, \qquad n^r(\tau) = n^r_0 \cos\omega_1\tau\,, \qquad n^\varphi(\tau) = n^\varphi_0 \sin\omega_1\tau\,, \tag{48}$$

where ω_1 is the solution of the characteristic equation for (46). The detailed form of this equation, using explicit expressions for the elements of the matrices γ and m were given in Ref. 16. The resulting expression for the characteristic frequency is

$$\omega_1 = \omega\left(1 - \frac{3M}{R} + \frac{Q^2}{2MR} + \ldots\right), \tag{49}$$

where the dots represent terms of higher order in M/R, Q/M or q/m. We also observe, that the amplitudes n^i_0 are not all independent: as $u^2 = -1$ both on the original orbit and on the displaced world-line, it follows that n is space-like and $u\cdot n = 0$. In the present case this general result translates to the constraint

$$\left(\varepsilon - \frac{q\,Q}{4\pi mR}\right)n_0^t - \frac{q\,Q}{4\pi m\,\omega_1 R^2}\,u^t n_0^r - \ell\,n_0^\varphi = 0\,. \tag{50}$$

Although the components n^μ define the direction of the deviation, they do not determine the actual distance between neighboring world lines; this is given by equation (61) as $\Delta x^\mu = n^\mu \Delta\lambda$. Therefore, for any particular orbit specified by the circular reference orbit (zeroth order approximation) and a world-line deviation vector n (first order approximation), we must determine in addition the scale factor $\Delta\lambda$ to be applied. This can be done as follows. Comparing the approximate solution (46) with the exact solution (38), we observe that

$$r(\varphi) = R + \Delta r \approx R - e\,R\cos y(\varphi)\,. \tag{51}$$

Hence at the periastron, one has

$$\Delta r = -\,\Delta\lambda\,n_0^r = -\,e\,R\,. \tag{52}$$

Thus the scale is set by the eccentricity of the orbit. Finally, we can determine the shift in angular coordinate between successive periastra, i.e. the advance of the periastron per orbital period. First observe, that the periastron occurs at the minima of $n^r(\tau)$, i.e. for $\tau_n = (2n+1)\pi/\omega_1$. Thus the amount of proper time elapsing between periastra is $\Delta\tau = 2\pi/\omega_1$; the corresponding period of observer time is

$$T = \int_0^{2\pi/\omega_1} d\tau\,\frac{dt}{d\tau} = \int_0^{2\pi/\omega_1} d\tau\,(u^t + \dot{n}^t\,\Delta\lambda) = \frac{2\pi}{\omega_1}\,u^t. \tag{53}$$

Here u^t is the rate of change of t per unit of proper time along the circular reference orbit. Next we observe, that at the proper times τ_n the angular coordinates at the reference orbit and the true orbit coincide: $n^\varphi(\tau_n) = 0$. Hence the change in angular coordinate φ between successive periastra is the same as the change of this coordinate along the circular reference orbit after time T. This we can easily compute. Defining

$$\delta\varphi = \varphi\,(t_0 + T) - \varphi\,(t_0) - 2\pi\,, \tag{54}$$

and using the expression $d\varphi/dt = \dot{\varphi}\,d\tau/dt = \omega/u^t$ for the angular velocity, we find

$$\delta\varphi = \frac{\omega\,T}{u^t} - 2\pi = 2\pi\left(\frac{\omega}{\omega_1} - 1\right) \approx 2\pi\left(\frac{3M}{R} - \frac{Q^2}{2MR}\right). \tag{55}$$

This is in perfect agreement with the expression (41) obtained from the analytical form of the orbit.

It is of interest to consider the geometrical interpretation of the approximation scheme we have used in a little more detail. The zeroth order approximation to the orbit we have constructed is a fully relativistic circular solution of the Einstein-Lorentz equation in a Coulomb-Reissner-Nordstrom field, with period $T_0 = 2\pi/\omega_0$. Included in this result is of course the simpler case of a circular geodesic in a Schwarzschild field. The first-order correction is a geodesic deviation which oscillates in all its components in the same plane with period $T_1 = 2\pi/\omega_1$. Geometrically this represents another circular movement on the background of the zeroth-order solution, i.e. an *epicycle*, with period slightly different from the zeroth-order approximation. This has two immediate consequences: the orbit becomes eccentric, and the period between extrema of the orbit differs from the period of the average (zeroth order) circular motion. This is in contrast with Newtonian gravity, where the periods are equal. Thus the extrema of the orbit (periastron and apastron) are shifted compared to the Newtonian approximation, by the amount predicted by the analytic description of the orbit.

It can be easily shown,[14] that higher-order world-line deviations all satisfy linear harmonic-oscillator type equations. Thus, computing higher-order corrections to our result amounts to the construction of higher-order epicycles. For the case of orbits in a central field, the method of world-line deviations then becomes a fully relativistic version of the Ptolemaean scheme,[21] which differs genuinely from the standard post-Newtonian approximation scheme because it uses the eccentricity of the orbit and the quantities M/R, Q/M as expansion parameters, rather than v/c. As such this scheme offers an alternative to post-Newtonian calculations of binary systems in a different physical regime, e.g. in the calculation of radiative effects.

5 Quasiclassical spinning bodies

The generalized geodesic deviation equations can be extended to the case of the particles carrying an electric charge and/or quasi-classical spin. In these cases particles do not move on geodesics, but on more general world lines.[15] For the case of charged particles in a combined electro-magnetic and gravitational field, the resulting world line deviation equation was derived in Ref. 16.

To this end, it is useful to consider the Lagrangian formulation of the geodesic deviation equations.We first observe, that this equation is linear and homogeneous in n^{μ}. It is therefore not very difficult to construct an action from which it can be derived. The Lagrangian of interest reads

$$\mathcal{L}\,(n) = \frac{1}{2}\, g_{\mu\nu}\, \frac{Dn^{\mu}}{D\tau}\, \frac{Dn^{\nu}}{D\tau} + \frac{1}{2}\, R_{\mu\rho\nu\lambda}\, u^{\rho}\, u^{\lambda}\, n^{\mu}\, n^{\nu}\,. \tag{56}$$

In this Lagrangian the metric, connection and curvature are those on the given reference geodesic $x_0^\mu(\tau)$, with $u^\mu(\tau) = \dot{x}_0^\mu$ representing the four-velocity. These quantities will be treated as background variables. Only the $n^\mu(\tau)$ should be considered as independent Lagrangian coordinates which are to be varied in the action. The action (56) can be derived independently by starting from the geodesic Lagrangian

$$\mathcal{L}\,(x) = \frac{1}{2}\, g_{\mu\nu}\,(x)\, \frac{dx^{\,\mu}}{d\tau}\, \frac{dx^\nu}{d\tau}\ , \tag{57}$$

and expanding $x^{\,\mu}(\tau)$ near the given background geodesic solution in the form $x^{\,\mu} = x_0^\mu + n^\mu \Delta\lambda$. The term independent of $\Delta\lambda$ does not contain n^μ, and contributes, after integration, only a constant term to the action. Next all terms linear in $\Delta\lambda$ drop out of the result because x_0 is a solution of the geodesic equation. Finally, the terms quadratic in $\Delta\lambda$ reproduce the expression (56), up to a total proper-time derivative and terms which vanish because of the geodesic equation for $x_0^\mu(\tau)$. Thus the Lagrangian (56) represents the lowest-order non-trivial term in a systematic expansion of our action integral:

$$S\,[x] = m\int d\tau L(x_0) + m\,(\Delta\lambda)^2 \int d\tau L(n) + \mathcal{O}\left[(\Delta\lambda)^3\right]. \tag{58}$$

The higher-order approximations can also be derived in this way.

This derivation of the deviation equations can be also applied to the case of charged particles. One should start from the action

$$S_q[x] = \int d\tau \left[\frac{m}{2}\, g_{\mu\nu}(x)\, \dot{x}^{\,\mu}\dot{x}^\nu + q\, A_\mu(x)\, \dot{x}^{\,\mu}\right], \tag{59}$$

with the overdot being the usual short-hand for proper-time derivatives. The world-lines generated by this action are solutions of the Einstein-Lorentz equation

$$\frac{D^2 x^{\,\mu}}{D\tau^2} = \frac{q}{m}\, F^\mu_{\ \ \nu}\, \frac{dx^\nu}{d\tau}\ . \tag{60}$$

Now given a solution $x_0^\mu(\tau)$ of this equation, and expanding the path in $S_q[x]$ as

$$x^{\,\mu}(\tau) = x_0^\mu(\tau) + \Delta\lambda\, n^\mu(\tau)\,, \tag{61}$$

one can expand to second order in $\Delta\lambda$:

$$\begin{aligned} S_q[x] \;&=\; S_q[x_0] + \frac{(\Delta\lambda)^2}{2} \int d\tau \Bigg[m\bigg(g_{\mu\nu}\, \frac{Dn^\mu}{D\tau}\, \frac{Dn^\nu}{D\tau} + R_{\mu\rho\nu\sigma} u^\rho u^\sigma n^\mu n^\nu \bigg) \\ &+\; q\bigg(F_{\mu\nu}\, n^\mu\, \frac{Dn^\nu}{D\tau} + \nabla_\mu F_{\nu\rho}\, u^\rho n^\mu n^\nu \bigg)\Bigg] + \mathcal{O}\left[(\Delta\lambda)^3\right]. \end{aligned} \tag{62}$$

To this order we then find that other (close to $x_0^{\mu}(\tau)$) solutions of the world-line equation (60) are given by (61), with n^{μ} being the solution of the world line deviation equation[16,17]

$$\frac{D^2 n^{\mu}}{D\tau^2} = R_{\rho\nu\sigma}{}^{\mu} u^{\rho} u^{\sigma} n^{\nu} + \frac{q}{m} F^{\mu}{}_{\nu} \frac{Dn^{\nu}}{D\tau} + \frac{q}{m} \nabla_{\rho} F^{\mu}{}_{\nu} u^{\nu} n^{\rho} . \tag{63}$$

The alternative interpretation of n^{μ}, as parametrizing the distance between two particles on neighboring world lines, holds in this case as well, provided the particles have the same charge-to-mass ratio q/m.

The same result can be obtained (63) from reduction of the geodesic equation and geodesic deviation equation in five-dimensional space-time, as particles with different q/m ratio in four dimensions correspond to particles with different momenta in five-dimensional space-time.[17]

Similarly, pseudo-classical spinning particles can be described by the supersymmetric Lagrangian[18,19]

$$\mathcal{L}_{spin}(x, \psi) = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} + \frac{i}{2} \psi_a \frac{D\psi^a}{D\tau} , \tag{64}$$

with ψ^a an anti-commuting tangent-space vector[a] such that the pseudo-classical spin is described by $S^{ab} = -i\, \psi^a \psi^b$. The corresponding equations of motion for spinning particles can be written as

$$\begin{aligned} \frac{D^2 x^{\mu}}{D\tau^2} &= \frac{1}{2} S^{ab} R^{\mu}{}_{\nu ab} u^{\nu} , \\ \frac{DS^{ab}}{D\tau} &= 0 . \end{aligned} \tag{65}$$

Starting from a one-parameter congruence of solutions $(x^{\mu}(\tau; \lambda), \psi^a(\tau; \lambda))$ we define the deviation vectors

$$n^{\mu} = \frac{\partial x^{\mu}}{\partial \lambda} , \quad \xi^a = \frac{D\psi^a}{D\lambda} = \frac{\partial \psi^a}{\partial \lambda} - n^{\mu} \omega_{\mu}{}^{a}{}_{b} \psi^b , \tag{66}$$

where $\omega_{\mu}{}^{a}{}_{b}$ is the spin connection. Then, the covariant change in the spin-tensor is

$$J^{ab} = \frac{DS^{ab}}{D\lambda} = -i[\psi^a \xi^b + \xi^a \psi^b] . \tag{67}$$

[a] Note that the transition between base-space and tangent-space vectors is made, as usual, by the vierbein $e_{\mu}{}^{a}$ and its inverse.

These vectors satisfy the world line deviation equations

$$
\begin{aligned}
\frac{D^2 n^\mu}{D\tau^2} &= R_{\sigma\nu\rho}{}^{\mu}\, u^\sigma u^\rho n^\nu + \frac{1}{2}\, S^{ab} R^{\mu}{}_{\nu ab}\, \frac{Dn^\nu}{D\tau} \\
&\quad + \frac{1}{2}\left[S^{ab}\, \nabla_\rho R^{\mu}_{\nu ab}\, u^\nu n^\rho + J^{ab} R^{\mu}_{\nu ab}\, u^\nu\right], \qquad (68) \\
\frac{DJ^{ab}}{D\tau} &= [S, R_{\mu\nu}]^{ab} u^\mu n^\nu .
\end{aligned}
$$

They define the stationary points of the quadratic deviation action

$$
\begin{aligned}
\mathcal{L}_{spin}(n,\xi) = \frac{1}{2}\, g_{\mu\nu} \frac{Dn^\mu}{D\tau}\, \frac{Dn^\nu}{D\tau} + \frac{i}{2}\, \xi_a\, \frac{D\xi^a}{D\tau} + \frac{1}{2}\, R_{\mu\rho\nu\sigma}\, u^\rho u^\sigma n^\mu n^\nu \qquad (69) \\
- \frac{i}{4}\, \psi^a \psi^b \left[R_{\mu\nu ab}\, n^\mu \frac{Dn^\nu}{D\tau} + \nabla_\mu R_{\nu\sigma ab}\, u^\sigma n^\mu n^\nu\right] - i\, R_{\mu\nu ab}\, n^\mu u^\nu \xi^a \psi^b .
\end{aligned}
$$

This set of coupled equations is of interest for the analysis of fine gyroscopic effects, namely in the field of Kerr's metric.

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