# STRESS ENERGY TENSOR IN c = 0 LOGARITHMIC CONFORMAL FIELD THEORY

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We discuss the partners of the stress energy tensor and their structure in Logarithmic conformal field theories. In particular we draw attention to the fundamental differences between theories with zero and non-zero central charge. We analyze the OPE for $T$, $\bar{T}$ and the logarithmic partners $t$ and $\bar{t}$ for $c = 0$ theory.

## Contents

*This paper is dedicated to the memory of* ***Michael Marinov****. One of us (IIK) had a privilege to know and admire Misha Marinov. He was a man of principles – in all aspects of life, not only in theoretical physics. The decision to choose a subject about Logarithmic Conformal Field Theories was not accidental. These theories are at the boundary between theories with unitary and non-unitary evolution. Perhaps it is not a well known fact that Misha wrote pioneering papers in which he used non-unitary evolution to describe how a pure state can evolve into a mixed one*[1,2] *long time before this topic became popular through Hawking's famous paper.*[3] *In these early papers he discussed the phenomenology of* $K^0-\bar{K}^0$ *oscillations in the presence of quantum mixing. Unfortunately his very important papers were not widely known (see however Ref. 4 and M.B. Mensky paper*[5] *in this Volume) and did not receive the recognition they rightfully deserved. We hope this subject would have been to his liking.*

## 1 Introduction

The study of conformal invariance in two dimensions has been an extremely interesting and fruitful area of research for the last twenty years.[6]

During the last ten years an interesting class of conformal field theories (CFTs) has emerged called logarithmic conformal field theories (LCFTs). In Ref. 7 the concept of LCFT was introduced and the presence of logarithmic structure in the operator product expansion was explained by the indecomposable representations that can occur in the fusion of primary operators. These occur when there are fields with degenerate scaling dimensions having a Jordan block structure. It was shown that in any LCFT one of these degenerate fields becomes a zero norm state coupled to a logarithmic partner.[8] This together with another property – extra (hidden) symmetries,[8,9] coming from extra conserved currents in our theory, will be important for our analysis of the stress-energy tensor structure in LCFT. The above-mentioned hidden symmetry means that there are extra fields with integer conformal dimensions. One can even get extra states with zero dimension which means that we have a theory with a non-trivial vacuum. These operators play a prominent role in the Quantum-Hall effect,[10,11] In this case the descendents of this extra zero dimension operator may form logarithmic pairs with currents or higher dimension fields. It is therefore interesting to see what will happen in the case of the stress tensor itself – can it have logarithmic partners or not? – and will these partners be primary fields or descendents.

In our previous paper[12] we addressed this issue and suggested some kind of classification for LCFT based on the structure of the vacuum and the character of the degeneracy of the stress-energy tensor. In particular the structure of the partners to $T$ in LCFTs with non-zero central charge and LCFTs with zero central charge behave very differently. The second class, $c = 0$ theories, are a very special sub-class of LCFTs. They are of utmost importance for both disordered systems and critical strings. We showed that at $c = 0$ in order to get a non-trivial theory there must exist a state which is orthogonal to $T$ and is *not* a descendant of any other field. There could of course also be other states which are descendants but these are not required by general arguments. The appearance of such a state is characterised by at least two coefficients when we restricted ourselves to holomorphic sector only. In Ref. 12 we discussed the arguments presented in Refs. 14-17 concerning the existence of a logarithmic partner $t$ for the stress-energy tensor. In particular the emergence of logarithmic behaviour is not universal if we can decompose the theory into a sum of non-interacting sectors. It is the mixing between these sectors which makes the theory logarithmic. This issue was previously discussed in string theory with a ghost-matter mixing term.[20,21]

In Ref. 12 only the holomorphic sector was considered. Here we shall include the antiholomorphic sector for $c = 0$ and demonstrate how these become non-trivially mixed.

## 2 General properties of logarithmic operators

In LCFT there are logarithmic terms in some correlation functions but the theories are nonetheless compatible with conformal invariance. An LCFT appears when several operators, or their descendents[13], become degenerate. Here we shall discuss the simplest situation in which only two operators become degenerate and form a logarithmic pair, denoted by $C$ and $D$. The OPE of the stress-energy tensor T with the logarithmic operators $C$ and $D$ is non-trivial and involves mixing[7]

$$\begin{aligned} T(z)C(w,\bar{w}) &\sim \frac{h}{(z-w)^2}C(w,\bar{w}) + \frac{1}{(z-w)}\partial_z C + \dots\,, \qquad (1)\\ T(z)D(w,\bar{w}) &\sim \frac{h}{(z-w)^2}D(w,\bar{w}) + \frac{1}{(z-w)^2}C(w,\bar{w}) + \frac{1}{(z-w)}\partial_z D + \dots\,, \end{aligned}$$

where $h$ is the conformal dimension of the operators with respect to the holomorphic stress-energy tensor $T(z)$. The OPE with $\bar{T}$ has the same form but with $\bar{h}$ instead of $h$; as usual the scaling dimension is $h+\bar{h}$ and the spin of the field is $h-\bar{h}$.

It is a consequence of Eq. (1) that under a conformal transformation $z \to$

$w = z + \epsilon(z)$ the logarithmic pair is transformed as

$$\begin{aligned} \delta C &= \partial_z \epsilon(z) h C + \epsilon(z) \partial_z C + \dots \,, \\ \delta D &= \partial_z \epsilon(z)(hD + C) + \epsilon(z) \partial_z D + \dots \,, \end{aligned} \tag{2}$$

which can be written globally as

$$\begin{pmatrix} C(z,\bar{z}) \\ D(z,\bar{z}) \end{pmatrix} = \left(\frac{\partial w}{\partial z}\right)^{\begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix}} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\begin{pmatrix} \bar{h} & 0 \\ 1 & \bar{h} \end{pmatrix}} \begin{pmatrix} C(w,\bar{w}) \\ D(w,\bar{w}) \end{pmatrix} . \tag{3}$$

One can see that even holomorphic (antiholomorphic) fields with dimensions $(h, 0)$ or $(0, \bar{h})$ are transformed under the action of *both* $T$ and $\bar{T}$, i.e. there is some sort of holomorphic anomaly for logarithmic pair.

From this conformal transformation one can derive the full two point functions for the logarithmic pair[7,8]

$$\begin{aligned} \langle C(x,\bar{x}) D(y,\bar{y}) \rangle &= \langle C(y,\bar{y}) D(x,\bar{x}) \rangle = \frac{\alpha_D}{(x-y)^{2h}(\bar{x}-\bar{y})^{2\bar{h}}} \,, \\ \langle D(x,\bar{x}) D(y,\bar{y}) \rangle &= \frac{1}{(x-y)^{2h}(\bar{x}-\bar{y})^{2\bar{h}}} \left(-2\alpha_D \ln|x-y| + \alpha'_D\right) , \\ \langle C(x,\bar{x}) C(y,\bar{y}) \rangle &= 0 \,, \end{aligned} \tag{4}$$

where the constant $\alpha_D$ is determined by the normalization of the $D$ operator and the constant $\alpha'_D$ can be changed by the redefinition $D \to D+C$. Note that Eq. (4) is absolutely universal and valid in any number of dimensions, because only the global properties of conformal symmetry were used to derive it. One can easily generalize these formulas to the case when there are $n$ degenerate fields and the Jordan cell is given by an $n \times n$ matrix, in which case the maximal power of the logarithm will be $\ln^{n-1} z$ .

## 3 Towards the classification of LCFT

LCFTs can be naturally divided into classes based on the dimension of the Jordan blocks involved. Here we shall concentrate on the case of rank 2 (one logarithmic partner) however it is obvious that our results will generalise to higher rank Jordan cells. It is perhaps still an interesting problem to understand if there can be a more complicated structure at higher rank.

These theories can be further grouped, as we shall show, into four distinct categories in which the stress tensor and its partners have different structures:

- $c = 0$ Theories

  - 0A: Non-degenerate vacua ($SU(2)_0$, Disordered Models)
  - 0B: Degenerate vacua ($OSp(2|2)_k$ for certain $k$)

- $c \neq 0$ Theories
  - IA: Non-degenerate vacua
  - IB: Degenerate vacua ($c_{p,1}$).

Throughout this paper we use the notation that non-degenerate and degenerate refer to the single vacuum and the vacuum with logarithmic pair respectively. There may also be other primaries at $h = 0$ with a trivial Jordan cell structure and we do not consider this possibility here.

Only in the case of the $c_{p,1}$ models and in particular the $c = -2$ triplet model has the structure of the theory been fully elucidated [18]. For the others some of the structure is known from explicit correlation functions. As far as we are aware there has been no examples of type IA in the literature. It is easy to see that a logarithmic partner for $T$ can only exist in cases $0A$, $0B$, $IB$ by the following simple arguments.

If $T$ has a logarithmic partner then $T$ itself must be a zero norm state[8]

$$\langle T(z)T(w)\rangle = 0\,. \tag{5}$$

Now consider the standard OPE for the stress tensor

$$T(z)T(w) \sim \frac{c\ I}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{6}$$

where $I$ is the identity operator. For consistency with (5) we see that we must have

$$c\,\langle I\rangle = c\,\langle 0|I|0\rangle = c\,\langle 0|0\rangle = 0\,. \tag{7}$$

For $c \neq 0$ this implies that the vacuum $|0\rangle$ must have zero norm and thus must be part of a logarithmic pair which excludes case IA. Thus partners to the stress tensor $T$ cannot occur in type IA theories. For this reason in the next section when discussing non-degenerate vacua we shall only discuss the case of $c = 0$.

## 4 Non-degenerate vacua and $\mathbf{c \to 0}$ limit

### *4.1 $c = 0$ catastrophe*

Here we review the construction given in Refs. 14, 15, 16.

For a primary field of conformal dimension $h$ we use the normalisation

$$\langle V(z_1,\bar{z}_1)V(z_2,\bar{z}_2)\rangle = \frac{A}{z_{12}^{2h}\bar{z}_{12}^{2\bar{h}}} \,. \tag{8}$$

Then we consider the correlator

$$\langle T(z)V(z_1,\bar{z}_1)V(z_2,\bar{z}_2)\rangle = \frac{A\ h}{(z-z_1)^2(z-z_2)^2 z_{12}^{2h-2}\bar{z}_{12}^{2\bar{h}}} \,. \tag{9}$$

The coefficient of the three point function is uniquely fixed by considering the limit $z \to z_1$ and using the property of a primary field,

$$T(z)V(w,\bar{w}) \sim \frac{hV(w,\bar{w})}{(z-w)^2} + \frac{\partial V(w,\bar{w})}{z-w} + \cdots \,. \tag{10}$$

Of course we have similar results following from insertions of $\bar{T}(\bar{z})$ in the correlator and we have taken $h = \bar{h}$ for simplicity. We now use

$$\langle T(z)T(0)\rangle = \frac{c}{2z^4}\,, \quad \langle 0|I|0\rangle = \frac{c}{2z^4}\,; \qquad \langle 0|0\rangle = 1\,, \tag{11}$$

and are explicitly using the fact that the identity field has non-zero norm. If $T$ is the only $h=2$ field present in our model we can deduce

$$V(z,\bar{z})V(0,0) \sim \frac{A(c)}{z^{2h}\bar{z}^{2h}}\left[1 + \frac{2h}{c}z^2T(0) + \frac{2\bar{h}}{c}\bar{z}^2\bar{T}(0) + \cdots\right]. \tag{12}$$

We have also assumed $c = \bar{c}$. Clearly for $c = 0$ if $A(0) \neq 0$ then the above OPE becomes ill-defined. However suppose that as $c$ approaches zero there is another spin 0 field $X$ with dimension $(2+\alpha,\alpha)$ and approaches $(2,0)$. For most of the following we shall concentrate on the operators with dimensions that converge to $(2,0)$ however a similar pattern occurs for the $(0,2)$ operators. Then for $c \neq 0$ we have

$$V(z,\bar{z})V(0,0) \sim \frac{A(c)}{z^{2h}}\left[1 + \frac{2h}{c}z^2T(0) + 2X(0,0)z^{2+\alpha(c)}\bar{z}^{\alpha(c)} + \cdots\right]. \tag{13}$$

Our starting point will be the two point functions for $c \neq 0$. Non-chiral fields $X(z,\bar{z})$, $\bar{X}(z,\bar{z})$ are of dimensions $(2+\alpha,\alpha)$ and $(\alpha,2+\alpha)$, respectively. The only non-trivial 2-point correlators (up to relation by conjugation) are

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{z_{12}^4}\,, \tag{14}$$

$$\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\rangle = \frac{1}{c}\frac{B(c)}{z_{12}^{4+2\alpha(c)}\bar{z}_{12}^{2\alpha(c)}}\,, \tag{15}$$

where we used the fact that $\langle T(z_1)X(z_2,\bar{z}_2)\rangle$ vanishes as they have different dimensions. We have exactly the same relations for the fields related by conjugation. We define the new fields $t$, $\bar{t}$ via

$$t = \frac{b}{c}T + \frac{b}{h}X, \qquad \bar{t} = \frac{b}{c}\,\bar{T} + \frac{b}{h}\,\bar{X}\,. \tag{16}$$

The parameter $b$ is defined through

$$b^{-1} \equiv -\lim_{c\to 0}\frac{\alpha(c)}{c} = -\alpha'(0). \tag{17}$$

We can now calculate the two point function,

$$\begin{aligned}\langle T(z_1)t(z_2,\bar{z}_2)\rangle &= \left\langle T(z_1)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_2,\bar{z}_2)\right\rangle = \frac{b}{c}\langle T(z_1)T(z_2)\rangle \\ &= \frac{b}{2}\frac{1}{z_{12}^4}.\end{aligned} \tag{18}$$

Also,

$$\begin{aligned}\langle t(z_1,\bar{z}_1)t(z_2,\bar{z}_2)\rangle &= \left\langle\left[\frac{b}{c}T + \frac{b}{h}X\right](z_1,\bar{z}_1)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_2,\bar{z}_2)\right\rangle \\ &= \frac{b^2}{c^2}\langle T(z_1)T(z_2)\rangle + \frac{b^2}{h^2}\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)\rangle \\ &= \frac{b^2}{2c}\frac{1}{z_{12}^4} + \frac{b^2B(c)}{h^2c}\frac{1}{z_{12}^{4+2\alpha(c)}\bar{z}_{12}^{2\alpha(c)}} \\ &= \frac{b^2}{2c}\frac{1}{z_{12}^4} + \frac{b^2B(c)}{h^2c}\frac{1}{z_{12}^4}\left(1 - 2\alpha(c)\ln|z_{12}|^2 + \cdots\right) \\ &= \frac{1}{z_{12}^4}\left\{\left(\frac{b^2}{2c} + \frac{b^2B(c)}{h^2c}\right) - \frac{2b^2B(c)\alpha(c)}{h^2c}\ln|z_{12}|^2 + \cdots\right\}.\end{aligned} \tag{19}$$

As this is to be well defined we see that we must have $B(c) = -\frac{1}{2}h^2 + B_1c + O(c^2)$. Now using (17) we get the standard OPEs for a logarithmic pair,

$$\begin{aligned}\langle T(z_1)T(z_2)\rangle &= 0\,, \\ \langle T(z_1)t(z_2,\bar{z}_2)\rangle &= \frac{b}{2z_{12}^4}\,, \\ \langle t(z_1,\bar{z}_1)t(z_2,\bar{z}_2)\rangle &= \frac{B_1 - b\ln|z_{12}|^2}{z_{12}^4}\,.\end{aligned} \tag{20}$$

As discussed earlier $B_1$ can be removed by a redefinition of $t$ and we shall assume that this has been done. Note that although $t$ is a $(2,0)$ field it is not chiral as $\bar\partial t \neq 0$. We have similar expressions for the correlation functions related to these by conjugation. Also,

$$\langle T(z_1)\bar{t}(z_2,\bar{z}_2)\rangle = \left\langle T(z_1)\left[\frac{b}{c}+\frac{b}{h}X\right](z_2,\bar{z}_2)\right\rangle = 0\,, \tag{21}$$

$$\langle t(z_1,\bar{z}_1)\bar{t}(z_2,\bar{z}_2)\rangle = \left\langle\left[\frac{b}{c}+\frac{b}{h}X\right](z_1,\bar{z}_1)\left[\frac{b}{c}+\frac{b}{h}X\right](z_2,\bar{z}_2)\right\rangle = 0\,. \tag{22}$$

The OPE (13) now becomes,

$$V(z,\bar{z})V(0,0)\sim\frac{A(0)}{z^{2h}}\left[1+\frac{2h}{b}z^2\left(T\ln|z|+t\right)+\frac{2h}{b}\bar{z}^2\left(\bar{T}\ln|z|+\bar{t}\right)+\cdots\right], \tag{23}$$

which now only involves quantities that are perfectly well defined in the limit as $c\to 0$.

We can now continue and insist that $t$ is also well defined in the three point functions (See Appendix). Assuming now that the algebra closes these are then sufficient to determine the OPEs. These are,

$$T(z_1)t(z_2,\bar{z}_2)\sim\frac{b}{2z_{12}^4}+\frac{2t(z_2,\bar{z}_2)+T(z_2)}{z_{12}^2}+\frac{\partial t(z_2,\bar{z}_2)}{z_{12}}+\cdots\,, \tag{24}$$

$$T(z_1)\bar{t}(z_2,\bar{z}_2)\sim\frac{\bar{T}(\bar{z}_2)}{z_{12}^2}+\frac{\partial\bar{t}(z_2,\bar{z}_2)}{z_{12}}+\cdots\,, \tag{25}$$

$$\begin{aligned} t(z_1,\bar{z}_1)t(z_2,\bar{z}_2)\sim{}&\frac{-b\ln|z_{12}|^2}{z_{12}^4}+\frac{1}{z_{12}^2}\Bigg[\left(1-4\ln|z_{12}|^2\right)t(z_2,\bar{z}_2)\\ &+\left(\frac{2a}{b}-\ln|z_{12}|^2-2\ln^2|z_{12}|^2\right)T(z_2)\Bigg]\\ &+\frac{\bar{z}_{12}^2}{z_{12}^4}\left[\bar{t}(z_2,\bar{z}_2)+\left(\frac{2f}{b}+\ln|z_{12}|^2\right)\bar{T}(\bar{z}_2)\right]+\cdots\,, \end{aligned} \tag{26}$$

$$\begin{aligned} t(z_1,\bar{z}_1)\bar{t}(z_2,\bar{z}_2)\sim{}&\frac{1}{\bar{z}_{12}^2}\left[\left(\frac{2f}{b}-\ln|z_{12}|^2\right)T(z_2)+\cdots\right]\\ &+\frac{1}{z_{12}^2}\left[\left(\frac{2f}{b}-\ln|z_{12}|^2\right)\bar{T}(\bar{z}_2)+\cdots\right]. \end{aligned} \tag{27}$$

The appearance of a state $|t\rangle = t\,|0\rangle$ in this way is equivalent to postulating a logarithmic partner for the null vector $T$. This prevents $T$ from decoupling despite the fact that it is a zero-norm state. Note that once one fixes

$$L_0\, |t\rangle \quad = \quad 2\, |t\rangle + |T\rangle\ , \tag{28}$$

$$\bar{L}_0\, |t\rangle \quad = \quad |T\rangle \tag{29}$$

then the parameter $b$ cannot be removed by rescaling and thus different values of $b$ correspond to inequivalent representations.

Let us note that in our notation parameter $b$ is different by a factor of 2 from a definition given in Ref. 15. We also want to stress that the $t(z)t(0)$ OPE is determined by several parameters, not by only $b$ as in Ref. 15. The constant terms $\frac{2a}{b}$, $\frac{2f}{b}$ cannot be removed by scale transformation – one can absorb it in $\ln|z|$ term, but not into $\ln^2|z|$ . One of the important open problems is to find if the classification of all $c = 0$ theories of this type can be reduced to the classification of all possible triplets $(a, b, f)$. The parameter $a$ cannot be determined from singular terms in (23), but only from the full 3-point functions of $(T, t)$ pair (see Appendix).

### *4.2 $c = 0$ and separability*

There is a third rather trivial way out of the paradox at $c = 0$. It is simply that the full theory is constructed from two parts $T = T_1 \oplus T_2, c = c_1 + c_2 = 0$ both having $c_i \neq 0$. Then in the OPE of two fields from one part we will only see the stress tensor for that part rather than the full one. Operators in the full theory are just the direct product $V = V_1 \otimes V_2$. Then (writing only the holomorphic part)

$$\begin{aligned} &V(z)V(0) = V_1(z)V_1(0) \;\; V_2(z)V_2(0) \\ &\sim \frac{1}{z^{2h_1}}\left(1 + z^2\frac{2h_1}{c_1}T_1(0) + \cdots\right)\frac{1}{z^{2h_1}}\left(1 + z^2\frac{2h_2}{c_2}T_2(0) + \cdots\right) \\ &\sim \frac{1}{z^{2h}}\left[1 + z^2\left(\frac{2h_1T_1}{c_1} + \frac{2h_2T_2}{c_2}\right) + \cdots\right]. \end{aligned} \tag{30}$$

This expression is now perfectly well defined as $c_1, c_2 \neq 0$. Of course this is as expected as the two decoupled theories are perfectly regular.

In critical string theory the ghost and matter sectors are normally assumed to be non-interacting. However this may not be the most general if we wish to allow not just positive but also zero norm states in our final theory.[20]

## 5 Conclusions

We studied here the stress tensor and its partners in LCFT. In particular in non-trivial $c = 0$ theories we have demonstrated that it is necessary for there to be a primary field of dimension 2 orthogonal to the stress tensor. The indecomposable representations are characterised by at least three parameters: $a$, $b$ and $f$. The third parameter $f$ emerges when we include antiholomorphic sector. In a logarithmic theory one can not avoid mixing between holomorphic and antiholomorphic sectors.


### Acknowledgements

We would like to thank J. Cardy and V. Gurarie for interesting comments. A.N. is funded by the Martin Senior Scholarship, Worcester College, Oxford. I.I.K. is partly supported by PPARC rolling grant PPA/G/O/1998/00567 and EC TMR grant HPRN-CT-1999-00161.


### Appendix: Three point functions

We now consider the three point functions. They are the following:

$$\langle T(z_1)T(z_2)T(z_3)\rangle = \frac{c}{z_{12}^2 z_{13}^2 z_{23}^2} \,, \tag{31}$$

$$\langle T(z_1)X(z_2,\bar z_2)X(z_3,\bar z_3)\rangle = \frac{1}{c}\,\frac{C(c)}{z_{12}^2 z_{13}^2 z_{23}^{2+2\alpha(c)}\bar z_{23}^{2\alpha(c)}} \,,$$

$$\langle X(z_1,\bar z_1)X(z_2,\bar z_2)X(z_3,\bar z_3)\rangle = \frac{1}{c^2}\,\frac{D(c)}{z_{12}^{2+\alpha(c)} z_{13}^{2+\alpha(c)} z_{23}^{2+\alpha(c)}\bar z_{12}^{\alpha(c)}\bar z_{13}^{\alpha(c)}\bar z_{23}^{\alpha(c)}} \,,$$

$$\left\langle T(z_1)\bar X(z_2,\bar z_2)\bar X(z_3,\bar z_3)\right\rangle = \frac{E(c)}{z_{12}^2 z_{13}^2 z_{23}^{2\alpha(c)-2}\bar z_{23}^{4+2\alpha(c)}} \,,$$

$$\left\langle X(z_1,\bar z_1)X(z_2,\bar z_2)\bar X(z_3,\bar z_3)\right\rangle = \frac{1}{c}\,\frac{F(c)}{z_{12}^{4+\alpha(c)} z_{13}^{\alpha(c)} z_{23}^{\alpha(c)}\bar z_{12}^{\alpha(c)-2}\bar z_{13}^{2+\alpha(c)}\bar z_{23}^{2+\alpha(c)}} \,.$$

We note that all correlators are single-valued for any $\alpha(c)$ and therefore must also be at the critical point. This is important as logarithmic terms should only emerge in the form $\ln|z|$.

We have already fixed the normalisation of the two point functions (14). Then by expanding the three point functions we see that

$$C(c) \quad = \quad (2+\alpha(c))B(c) = (2+\alpha(c))\left(-\frac{1}{2}h^2 + B_2c^2 + \cdots\right) \,, \tag{32}$$

$$E(c) \quad = \quad \frac{\alpha(c)}{c}B(c) = \frac{\alpha(c)}{c}(-\frac{1}{2}h^2 + B_2c^2 + \cdots) \, . \tag{33}$$

As we wish to have well defined operators $T,\ t,\ \bar{T},\ \bar{t}$ they must in particular have regular 3-point functions. This will be enough to determine the leading behaviour of the functions above. Consider

$$\langle T(z_1)T(z_2)t(z_3,\bar{z}_3)\rangle = \left\langle T(z_1)T(z_2)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_3,\bar{z}_3)\right\rangle = \frac{b}{z_{12}^2 z_{13}^2 z_{23}^2} \, . \tag{34}$$

$$\begin{aligned}\langle T(z_1)t(z_2,\bar{z}_2)t(z_3,\bar{z}_3)\rangle \quad &= \quad \left\langle T(z_1)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_2,\bar{z}_2)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_3,\bar{z}_3)\right\rangle \\ &= \quad \frac{b^2}{c^2}\frac{c}{z_{12}^2 z_{13}^2 z_{23}^2} + \frac{b^2}{h^2 c}\frac{C(c)}{z_{12}^2 z_{13}^2 z_{23}^{2+2\alpha(c)}\bar{z}_{23}^{2\alpha(c)}} \\ &= \quad \frac{b^2}{z_{12}^2 z_{13}^2 z_{23}^2}\left[\frac{1}{c} + \frac{C(c)}{h^2 c}\left(1 - 2\alpha(c)\ln|z_{23}|^2 + \cdots\right)\right] .\end{aligned} \tag{35}$$

Now using the form of $C(c)$ (32) we get

$$\langle T(z_1)t(z_2,\bar{z}_2)t(z_3,\bar{z}_3)\rangle = \frac{-2b\ln|z_{23}|^2 + \frac{b}{2}}{z_{12}^2 z_{13}^2 z_{23}^2} \, , \tag{36}$$

and

$$\begin{aligned}&\langle t(z_1,\bar{z}_1)t(z_2,\bar{z}_2)t(z_3,\bar{z}_3)\rangle \\ &= \left\langle \left[\frac{b}{c}T + \frac{b}{h}X\right](z_1,\bar{z}_1)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_2,\bar{z}_2)\left[\frac{b}{c}T + \frac{b}{h}X\right](z_3,\bar{z}_3)\right\rangle \\ &= \frac{b^3}{c^3}\langle T(z_1)T(z_2)T(z_3)\rangle + \frac{b^3}{h^2 c}\left(\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)T(z_3)\rangle\right. \\ &+ \left.\langle X(z_1,\bar{z}_1)T(z_2)X(z_3,\bar{z}_3)\rangle + \langle T(z_1)X(z_2,\bar{z}_2)X(z_3,\bar{z}_3)\rangle\right) \\ &+ \frac{b^3}{h^3}\langle X(z_1,\bar{z}_1)X(z_2,\bar{z}_2)X(z_3,\bar{z}_3)\rangle \\ &= \frac{b^3}{c^2}\frac{1}{z_{12}^2 z_{13}^2 z_{23}^2} + \frac{b^3}{h^3 c^2}\frac{D(c)}{z_{12}^2 z_{13}^2 z_{23}^2} z_{12}^{-\alpha(c)} z_{13}^{-\alpha(c)} z_{23}^{-\alpha(c)} \\ &+ \frac{b^3}{h^2 c^2}\frac{C(c)}{z_{12}^2 z_{13}^2 z_{23}^2}\left[z_{12}^{-2\alpha(c)}\bar{z}_{12}^{-2\alpha(c)} + z_{13}^{-2\alpha(c)}\bar{z}_{13}^{-2\alpha(c)} + z_{23}^{-2\alpha(c)}\bar{z}_{23}^{-2\alpha(c)}\right] .\end{aligned} \tag{37}$$

Now expanding this and using (17):

$$\langle t(z_1)t(z_2)t(z_3)\rangle = \frac{b^3}{h^3 c^2}\left(-2h^3 + D_0\right) \tag{38}$$

$$+\frac{b^2}{h^3c}\left[\left(D_0-2h^3\right)\left(\ln|z_{12}|^2+\ln|z_{13}|^2+\ln|z_{23}|^2\right)+bD_1+\frac{3}{2}h^3\right]+O(1).$$

Thus if this is to be regular in the limit we must have

$$D_0=2h^3,\qquad D_1=-\frac{3h^3}{2b}. \tag{39}$$

Then from the $O(1)$ terms we get

$$\begin{aligned}\langle t(z_1,\bar z_1)t(z_2,\bar z_2)t(z_3,\bar z_3)\rangle &= \frac{1}{z_{12}^2z_{13}^2z_{23}^2}\Big\{-b\left(\ln^2|z_{12}|^2+\ln^2|z_{13}|^2+\ln^2|z_{23}|^2\right)\\ &+2b\left(\ln|z_{12}|^2\ln|z_{13}|^2+\ln|z_{12}|^2\ln|z_{23}|^2+\ln|z_{13}|^2\ln|z_{23}|^2\right)\\ &-\frac{b}{2}\left(\ln|z_{12}|^2+\ln|z_{13}|^2+\ln|z_{23}|^2\right)+a\Big\},\end{aligned} \tag{40}$$

where we have defined the constant $a$ by

$$a\equiv-\frac{b^3}{2h^3}\left(-2D_2-12hB_2+\frac{3}{2}h^3\alpha''(0)\right). \tag{41}$$

Now consider correlators involving the $\bar T,\bar X$ fields as well. For instance

$$\langle T(z_1)T(z_2)\bar t(z_3,\bar z_3)\rangle=\left\langle T(z_1)T(z_2)\left[\frac{b}{c}\bar T+\frac{b}{h}\bar X\right](z_3,\bar z_3)\right\rangle=0, \tag{42}$$

$$\langle T(z_1)\bar T(\bar z_2)t(z_3,\bar z_3)\rangle=\left\langle T(z_1)\bar T(\bar z_2)\left[\frac{b}{c}T+\frac{b}{h}X\right](z_3,\bar z_3)\right\rangle=0,$$

$$\langle T(z_1)t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle=\left\langle T(z_1)\left[\frac{b}{c}T+\frac{b}{h}X\right](z_2,\bar z_2)\left[\frac{b}{c}\bar T+\frac{b}{h}\bar X\right](z_3,\bar z_3)\right\rangle=0\,.$$

More non-trivially

$$\begin{aligned}\langle T(z_1)\bar t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle &= \left\langle T(z_1)\left[\frac{b}{c}\bar T+\frac{b}{h}\bar X\right](z_2,\bar z_2)\left[\frac{b}{c}\bar T+\frac{b}{h}\bar X\right](z_3,\bar z_3)\right\rangle\\ &=\frac{b^2}{h^2}\,\frac{E(c)}{z_{12}^2z_{13}^2z_{23}^{2\alpha(c)-2}\bar z_{23}^{4+2\alpha(c)}}\,.\end{aligned} \tag{43}$$

Inserting the known expression for of $E(c)$ we get

$$\langle T(z_1)\bar t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle=\frac{\frac{b}{2}}{z_{12}^2z_{13}^2z_{23}^{-2}\bar z_{23}^4}\,. \tag{44}$$

The last correlator we have to consider is the following:

$$\begin{aligned}
&\langle t(z_1,\bar z_1)t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle \qquad (45)\\
&= \left\langle \left[\frac{b}{c}T+\frac{b}{h}X\right](z_1,\bar z_1)\left[\frac{b}{c}T+\frac{b}{h}X\right](z_2,\bar z_2)\left[\frac{b}{c}\bar T+\frac{b}{h}\bar X\right](z_3,\bar z_3)\right\rangle\\
&= \frac{b^3}{ch^2}\left\langle X(z_1,\bar z_1)X(z_2,\bar z_2)\bar T(z_3,\bar z_3)\right\rangle + \frac{b^3}{h^3}\left\langle X(z_1,\bar z_1)X(z_2,\bar z_2)\bar X(z_3,\bar z_3)\right\rangle\\
&= \frac{b^3E(c)/ch^2}{z_{12}^{4+2\alpha(c)}\bar z_{12}^{2\alpha(c)-2}\bar z_{13}^2\bar z_{23}^2} + \frac{b^3F(c)/ch^3}{z_{12}^{4+\alpha(c)}z_{13}^{\alpha(c)}z_{23}^{\alpha(c)}\bar z_{12}^{\alpha(c)-2}\bar z_{13}^{2+\alpha(c)}\bar z_{23}^{2+\alpha(c)}}\ .
\end{aligned}$$

Thus we find

$$F(c) = -\frac{h^3}{2b} + F_1c + O(c^2). \qquad (46)$$

Finally we get

$$\langle t(z_1,\bar z_1)t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle = \frac{\frac{b}{2}\left(\ln|z_{12}|^2-\ln|z_{13}|^2-\ln|z_{23}|^2\right)+f}{z_{12}^4\bar z_{12}^{-2}\bar z_{13}^2\bar z_{23}^2}\ , \qquad (47)$$

where the coefficient $f = -b^3(-\frac{1}{2}h^3\alpha''(0)-2F_1)/(2h^3)$.

In summary we have found the following correlators which yield the OPEs given in the text:

$$\langle T(z_1)t(z_2,\bar z_2)\rangle = \frac{b}{2z_{12}^4}\ , \qquad (48)$$

$$\langle t(z_1,\bar z_1)t(z_2,\bar z_2)\rangle = \frac{-b\ln|z_{12}|^2}{z_{12}^4}\ , \qquad (49)$$

$$\langle T(z_1)T(z_2)t(z_3,\bar z_3)\rangle = \frac{b}{z_{12}^2z_{13}^2z_{23}^2}\ , \qquad (50)$$

$$\langle T(z_1)t(z_2,\bar z_2)t(z_3,\bar z_3)\rangle = \frac{-2b\ln|z_{23}|^2+\frac{b}{2}}{z_{12}^2z_{13}^2z_{23}^2}\ , \qquad (51)$$

$$\begin{aligned}
\langle t(z_1,\bar z_1)t(z_2,\bar z_2)t(z_3,\bar z_3)\rangle = \frac{1}{z_{12}^2z_{13}^2z_{23}^2}\Big\{&-b\left(\ln^2|z_{12}|^2+\ln^2|z_{13}|^2+\ln^2|z_{23}|^2\right)\\
&+2b\left(\ln|z_{12}|^2\ln|z_{13}|^2+\ln|z_{12}|^2\ln|z_{23}|^2+\ln|z_{13}|^2\ln|z_{23}|^2\right)\\
&-\frac{b}{2}\left(\ln|z_{12}|^2+\ln|z_{13}|^2+\ln|z_{23}|^2\right)+a\Big\}, \qquad (52)
\end{aligned}$$

$$\langle T(z_1)\bar t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle = \frac{\frac{b}{2}}{z_{12}^2z_{13}^2z_{23}^{-2}\bar z_{23}^4}\ , \qquad (53)$$

$$\langle t(z_1,\bar z_1)t(z_2,\bar z_2)\bar t(z_3,\bar z_3)\rangle = \frac{\frac{b}{2}\left(\ln|z_{12}|^2 - \ln|z_{13}|^2 - \ln|z_{23}|^2\right) + f}{z_{12}^4 \bar z_{12}^{-2} \bar z_{13}^2 \bar z_{23}^2}\,. \tag{54}$$

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