INVARIANT OPERATORS ON SUPERMANIFOLDS AND STANDARD MODELS

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Here we continue to list the differential operators invariant with respect to the 15 exceptional simple Lie superalgebras $\mathfrak{g}$ of polynomial vector fields. A part of the list (for operators acting on tensors with finite dimensional fibers) was earlier obtained in 2 of the 15 cases by Kochetkov and in one more instance by Kac and Rudakov. Broadhurst and Kac conjectured that some of these structures pertain to the Standard Models of elementary particles and the Grand Unified Theories. So, GUT, if any exists, will be formulated in terms of operators we found, or their r-nary analogs to be found. Calculations are performed with the aid of Grozman's *Mathematica*-based SuperLie package. When degeneracy conditions are violated (absence of singular vectors) the corresponding module of tensor fields is irreducible. We also verified some of the earlier findings.

Contents

To the memory of Misha Marinov (by D.L.)

Only one of us knew Misha. It was at the flat of one of my teachers, Felix Berezin, that we became acquainted. I was a student then and was bubbling with enthusiasm on account of fantastic visions, applications of supersymmetry, which at that time was not so clear to most. Misha shared the awe of the beauty of the mathematics related with SUSY. Besides this, he was a charming person. Moreover, he was writing a review on superstrings. So Misha fascinated me.

His review[41] illuminated my research for many years ahead (Ref. 16 is just a part of the story). Later Misha, being a man of passion, bitterly criticized the whole trend as useless waste of many talents of several generations. I did not take his vitriolic words at face value. I knew that however strong language he used, it was merely rhetoric. He similarly criticized me for not being able to quantize the "odd mechanic" (and Ref. 38 is a part of our answer to Misha) nor demonstrate the "usefulness" of what was rediscovered several years later by Batalin and Vilkovisky under the name antibracket together with the manifest demonstrations of its usefulness (cf. Refs. 31 and 10). When the antibracket was rediscovered Misha came to me and said, "Do you remember that stuff about the two types of mechanics, you told me some time ago? Recently Batalin talked a lot about what looks precisely like it, and he is one of the rare few who only speaks about what he knows. If I had not known him for a long time I'd think he had become crazy. He plans to do almost all gauge field theory by means of this new mechanic. So, let me get you acquainted, you two talk similar ..."

Later, before he applied for emigration and while waiting for permission (more likely for a refusal, at that time) Misha frequented an informal seminar I hosted at home being also ready to quit the regular ways. Some mathematicians having applied for emigration became *personae non grata* at official seminars; in my flat they could discuss new and old results with those who dared. Misha was our physics teacher, he tried to convey to us (J. Bernstein, A. Beilinson, me, and occasional "guests", from students to S. Gindikin) some ideas of modern physics.

Misha visited me at Stockholm recently. I took him to look at the frescos of Täby kyrkan, one of the most interesting churches in Stockholm. The most famous of the frescos is currently called "The Knight playing chess with Death". The original title, if it ever existed, is forgotten; the modern title is hardly correct: depicted are a chess-board (5×5, to simplify Death's task even further, I presume), Death, and the Knight, smiling his best (as his PR advisor, no doubt, taught him) politely inclined towards the ruling party. The Knight stands on the same side of the board as Death, so manifestly both are playing together against *us*. Misha, who was restless during the day being anxious not to miss his train, suddenly relaxed having seen this fresco. This was the last fresco we saw and on the way back we discussed something neutral. He said he enjoyed the frescoes. On our way back we got into a traffic jam and missed his train by a few minutes. Naturally, I paid the penalty. Misha wrote to me later insisting to repay or split the cost, so I answered that I did not change that much after emigration, and he should know better. He retorted as in the good old days when we all were younger saying, "I regret that Israel is so small, and so penalties for missing the train are low. Anyway, I hope to see you one day and promise to make you miss the train from Haifa to wherever." He also gave me a reprint of his superstring paper (somebody "borrowed" my copy of the reprint he gave me in Moscow) and wrote in Russian: "To DL, as a memory of our friendship that I hope not interrupt". The inscription looked very strange to me. Now I understand it: he already knew about his cancer.

I think all who knew Misha love him. One can not use past tense with this verb. And therefore it is infinitely sad not to be able, ever, to miss the train in Misha's company, this side of the Chess-Board we all try to describe before we meet the ultimate opponent.

1 Introduction

This is a part of an expanded transcript of two talks at the International Workshops "Supersymmetry and Quantum Symmetry": (1) at Dubna, July 22-26, 1997, we described the case of $\mathfrak{kas}$ and $\mathfrak{k}(1|6)$, sketched the case $\mathfrak{k}(1|n)$ and mentioned earlier results of Kochetkov on $\mathfrak{vle}(4|3)$ and $\mathfrak{mb}(4|5)$, (2) at Karpacz, September 23, 2001, we considered $\mathfrak{mb}(3|8)$ and $\mathfrak{ksle}(5|10)$.

Broadhurst and Kac observed[17] that some of the exceptional Lie superalgebras (listed in Refs. 50, 7) might pertain to a GUT or the Standard Model, their linear parts being isomorphic to $\mathfrak{sl}(5)$ or $\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathfrak{gl}(1)$. Kac demonstrated[18] that for the Standard Model with $\mathfrak{su}(3)\oplus\mathfrak{su}(2)\oplus\mathfrak{u}(1)$ as the gauge group a certain remarkable relation between $\mathfrak{vle}(3|6)$ and some of the known elementary particles does take place; it seems that for $\mathfrak{mb}(3|8)$ there is even better correspondence.

The total lack of enthusiasm from the physicists' community concerning these correspondences is occasioned, perhaps, by the fact that no real form of any of the simple Lie superalgebras of vector fields with polynomial coefficients has a unitary Lie algebra as its linear part. Undeterred by this, Kac and Rudakov calculated[19] some $\mathfrak{vle}(3|6)$-invariant differential operators. They calculated the operators for finite dimensional fibers only. This restriction makes calculations a sight easier but strikes out many operators. The amount of calculations for $\mathfrak{mb}(3|8)$ is too high to be performed by hands.

The problem we address — calculation of invariant differential operators acting in tensor fields on manifolds and supermanifolds with various structures — was a part of our Seminar on Supermanifold's agenda since mid-70's. Here we use Grozman's code SuperLie[14] to verify and correct earlier results and obtain new ones, especially when bare hands are inadequate. The usefulness of SuperLie was already demonstrated when we calculated the left-hand side of N-extended SUGRA equations for any N, cf. Ref. 15. We review the whole field with its open problems and recall interesting Kirillov's results and problems buried in the VINITI collection[22] which is not very accessible. Another nice (and accessible) review we can recommend in addition to Ref. 22 is Ref. 29.

What is done

(1) We list (degeneracy conditions) all differential operators, or rather the corresponding to them singular vectors, of degrees 1, and 2 and, in some cases, of all possible degrees (which often are ≤ 2), invariant with respect to several exceptional Lie superalgebras of vector fields. When degeneracy conditions are violated (absence of singular vectors) the corresponding induced

and coinduced modules are irreducible. For some exceptions EVERY module $I(V)$ has a singular vector. This is a totally new feature. For $\mathfrak{vle}(3|6)$ (and finite dimensional fibers) the answer coincides with Kac-Rudakov's one.

(2) We observe that the linear parts of two of the W-regradings of $\mathfrak{mb}$ are Lie superalgebras strictly greater than $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{gl}(1)$. They (or certain real forms of them) are natural candidates for the algebras of The *would be* Standard Models; modern "no-go" theorems do not preclude them.

1.1 Veblen's problem

The topology of differentiable manifolds has always been related with various geometric objects on them and, in particular, with operators invariant with respect to the group of diffeomorphisms of the manifold, operators which act in the spaces of sections of "natural bundles",[29] whose sections are tensor fields, or connections, etc. For example, an important invariant of the manifold, its *cohomology*, stems from the de Rham complex whose neighboring terms are bridged by an invariant differential operator — the exterior differential.

The role of invariance had been appreciated already in XIX century in relation with physics; indeed, differential operators invariant with respect to the group of diffeomorphisms preserving a geometric structure are essential both in formulation of Maxwell's laws of electricity and magnetism and in Einstein–Hilbert's formulation of relativity.

Simultaneously, invariance became a topic of conscious interest for mathematicians: the representation theory flourished in works of F. Klein, followed by Lie and É. Cartan, to name the most important contributors; it provided with the language and technique adequate in the study of geometric structures. Still, it was not until O. Veblen's talk in 1928 at the Mathematical Congress in Bologna[56] that invariant operators (such as, say, Lie derivative, the exterior differential, or integral) became the primary object of the study. In what follows we rule out the integral and other non-local operators; except in Kirillov's example, we only consider local operators.

Schouten and Nijenhuis tackled Veblen's problem: they reformulated it in terms of modern differential geometry and found several new bilinear invariant differential operators. Schouten conjectured that there is essentially one *un*ary invariant differential operator: the exterior differential of differential forms. This conjecture had been proved in particular cases by a number of people, and in full generality in 1977–78 by A. A. Kirillov and, independently, C. L. Terng (see Refs. 22, 55).

Thanks to the usual clarity and an enthusiastic way of Kirillov's presentation he drew new attention to this problem, at least, in Russia. Under the light of this attention it became clear (to J. Bernstein) that in 1973 A. Rudakov[43] also proved this conjecture by a *simple algebraic* method which reduces Veblen's problem for differential operators to a "computerizable" one.

Thus, a tough analytic problem reduces to a problem formally understandable by any first year undergraduate: a series of systems of linear equations in small dimensions plus induction. The only snag is the volume of calculations: to list all unary operators in the key cases requires a half page; for binary operators one needs about 50 pages and induction becomes rather nontrivial; for r-nary operators with $r > 2$ only some cases seem to be feasible.

Later Rudakov and, for the contact series, I. Kostrikin, classified unary differential operators in tensor fields on manifolds invariant with respect to each of the remaining three simple infinite dimensional $\mathbb{Z}$-graded Lie algebras $\mathfrak{L}$ of vector fields. In passing, the definition of the tensor field was generalized and primitive forms came to foreground.

1.2 Rudakov's breakthrough (following Bernstein[5])

Hereafter the ground field $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$. Without going into details which will be given later, observe that the spaces in which invariant operators act fall into two major cases: spaces of tensor fields (transformations depend on the 1-jet of diffeomorphism) and spaces depending on higher jets, called *HJ-tensors* for short. We will only study tensors here, not HJ-tensors.

1) Instead of considering $\mathrm{Diff}(U)$-invariant operators, where U is a local chart, let us consider $\mathfrak{vect}(U)$-invariant operators, where $\mathfrak{vect}(U)$ is the Lie algebra of vector fields on U with polynomial coefficients, or its formal completion. (*A posteriori* one proves that the global and the local problems are equivalent, cf. Ref. 5). Accordingly, instead of tensor fields with smooth coefficients, we consider their formal version: $T(V) = V \otimes \mathbb{K}[[x]]$, where $x = (x_1, \ldots, x_n)$ and $n = \dim U$.

2) We assume here that V is an *irreducible* $\mathfrak{gl}(n)$*-module with lowest weight*. Observe that while the requirement of lowest weight seems to be "obviously" reasonable, that on irreducibility is not, unless we confine ourselves to finite dimensional modules V. In super setting we are forced, in the absence of complete reducibility, to consider *indecomposable* representations even for finite dimensional modules. Irreducible modules is just the simplest first step.

3) Instead of the coinduced module, $T(V)$, consider the dual induced module, $I(V^*) = \mathbb{K}[\partial] \otimes V^*$, where $\partial_i = \frac{\partial}{\partial x_i}$. The reason: formulas for $\mathfrak{vect}(U)$-action are simpler for $I(V^*)$ than for $T(V)$. (The results, contrariwise, are more graphic in terms of tensor fields.)

Observe that each induced module is a "highest weight one" with respect to the whole $\mathfrak{g} = \mathfrak{vect}(U)$, i.e., the vector of the most highest weight with respect to the linear vector fields from $\mathfrak{g}_0 = \mathfrak{gl}(n)$ is annihilated by $\mathfrak{g}_+$, the subalgebra of $\mathfrak{g}$ consisting of all operators of degree > 0 relative the standard grading ($\deg x_i = 1$ for all i).

In what follows the vectors annihilated by $\mathfrak{g}_+$ will be called *singular* ones.

4) To every r-nary operator $D : T(V_1) \otimes \cdots \otimes T(V_r) \longrightarrow T(V)$ the dual operator corresponds

$$D^* : I(W) \longrightarrow I(W_1) \otimes \cdots \otimes I(W_r), \quad \text{where} \quad W = V^*,\ W_i = W_i^*,$$

and, since (for details see Ref. 43) each induced module is a highest weight one, *to list all the D's, is the same as to list all the $\mathfrak{g}_0$-highest singular vectors* $D^* \in I(W_1) \otimes \cdots \otimes I(W_r)$. In what follows r is called the *arity* of D.

5) In super setting, as well as for non-super but infinite dimensional one, the above statement is not true: the submodule generated by a singular vector does not have to be a maximal one; it could have another singular vector of the same degree due to the lack of complete reducibility.

For unary operators and Lie algebras this nuisance does not happen; this was one of the (unreasonable) psychological motivations to stick to the finite dimensional case even for Lie superalgebras,[6] cf. Ref. 54.

6) Rudakov's paper[43] contains two results:

(A) description of $\mathfrak{vect}(n)$-invariant operators in tensor fields (only the exterior differential exists) and (the main bulk of the paper)

(B) proof of the fact that between the spaces of HJ-tensors there are no unary invariant operators.

Problems 1) Describe r-nary invariant operators in the spaces of HJ-tensors for $r > 1$.

2) Describe r-nary invariant operators in the *super*spaces of HJ-tensors.

The dual operators Kirillov noticed[22] that by means of the invariant pairing (we consider fields on M with compact support and tensoring over the space $\mathcal{F}$ of functions)

$$B : T(V) \times (T(V^*) \otimes_{\mathcal{F}} Vol(M)) \longrightarrow \mathbb{R}, \quad (t, t^* \otimes vol) \mapsto \int (t, t^*) vol$$

one can define the duals of the known invariant operators. For the fields with formal coefficients we consider there is, of course, no pairing, but we consider a *would be* pairing induced by smooth fields with compact support. So the *formal dual* of $T(V)$ is not $T(V^*)$ because the pairing returns a function instead of a volume form to be integrated to get a number, and not $T(V)^*$ because $T(V)^*$ is a highest weight module while we need a lowest weight one. Answer: the formal dual of $T(V)$ is $T(V^*) \otimes_{\mathcal{F}} Vol(M)$.

Possibility to dualize, steeply diminishes the number of cases to consider in computations and helps to check the results. Indeed, with every invariant operator $D : T(V) \longrightarrow T(W)$ the dual operator $D^* : T(W^*) \otimes_{\mathcal{F}} Vol(M) \longrightarrow T(V^*) \otimes_{\mathcal{F}} Vol(M)$ is also invariant. For example, what is the dual of $d : \Omega^k \longrightarrow \Omega^{k+1}$? Clearly, it is the same d but in another incarnation: $d : \Omega^{n-k-1} \longrightarrow \Omega^{n-k}$. Though, roughly speaking, we only have one operator, d, the form of singular vectors corresponding to d differs with k and having found several "new" singular vectors we must verify that the corresponding operators are indeed distinct. This might be not easy.

Observe that these arguments do not work when we allow infinite dimensional fibers (dualization sends the highest module into a lowest weight one, so it is unclear if a highest weight module with a singular vector always correspond to this lowest weight one). Sometimes, being tired of calculations, or when the computer gave up, we formulated the description of singular vectors "up to dualization"; sometimes even the computer became "tired". We will mention such cases extra carefully; we intend to reconsider these cases on a more powerful computer.

1.3 Further ramifications of Veblen's problem

Rudakov's arguments[43] show that the fibers of HJ-tensors have to be of infinite dimension; the same holds for Lie superalgebras, though arguments are different. Traditionally, fibers of tensor bundles were only considered to be of finite dimension, though even in his first paper on the subject Rudakov[43] digressed from traditions.

1^o. In the study of invariant operators, one of the "reasons" for confining to tensors, moreover, the ones corresponding to finite dimensional fibers, is provided by two of Rudakov's results:[43] (1) there are no invariant operators between HJ-tensors, (2) starting with any highest weight modules $I(V)$, Rudakov unearthed singular vectors only for fundamental (hence, finite dimensional) representations. Though (1) only applies to *unary* operators, researchers were somewhat discouraged to consider HJ-tensors even speaking about binary operators.

Consider invariant operators of arity > 1 between the spaces of HJ-tensors (see Problem above). Is it true that in this case there are no invariant operators either?

2^o. Kirillov proved that (having fixed the dimension of the manifold and arity) *the degree of invariant (with respect to $\mathfrak{vect}(n)$) differential operators is bounded, even* dim *of the space of invariant operators is bounded.*

There seems to be no doubt that a similar statement holds on supermanifolds ... but Kochetkov's examples reproduced below and our own ones show that these expectations are false in some cases.

Problem Figure out the conditions when the dimension of the space of invariant operators is bounded. (We conjecture that this is true for all the series of simple vectorial Lie superalgebras in the standard grading.)

3^o. On the line, all tensors are λ-densities and every r-linear differential operator is of the form

$$L : (f_1 dx^{\lambda_1}, \dots, f_r dx^{\lambda_r}) \longrightarrow P_L(f_1, \dots, f_r) dx^{\lambda}.$$

Kirillov shows (with ease and elegance) that invariance of L is equivalent to the system

$$\sum_{s=1}^{r} \left[t_s \frac{\partial^{j+1}}{\partial t_s^{j+1}} + (j+1)\lambda_s \frac{\partial^j}{\partial t_s^j} \right] P_L(t) = \begin{cases} \lambda P_L(t) & \text{for } j = 0 \\ 0 & \text{for } j > 0 \end{cases} \qquad (*)$$

Clearly, differential operators correspond to polynomial solutions $P_L(t)$ and in this case $\lambda = \sum_{s=1}^{r} \lambda_s - \deg P_L$. Kirillov demonstrated that nonpolynomial solutions do exist: for $r = 2$ and $\lambda_1 = \lambda_2 = 0$ the function

$$P_L(t) = \frac{t_1 - t_2}{t_1 + t_2}$$

satisfies $(*)$ for $\lambda = 0$.

Problem What invariant operator corresponds to this solution? Describe all (any) of the nonpolynomial solutions of $(*)$ and the corresponding operators.

4^o. To select a reasonable type of r-nary operators is a good problem. Symmetric and skew-symmetric operators, as well as operators on λ-densities are the first choices but even in such simple cases there are few results. These results, though scanty, are rather interesting: quite unexpectedly, some of them are related to calculation of the N. Shapovalov determinant for the Virasoro algebra, cf. Ref. 9.

5^o. Since the real forms of simple vectorial Lie algebras are only trivial ones (in the natural polynomial basis replace all complex coefficients with reals), the results for $\mathbb{R}$ and $\mathbb{C}$ are identical. In super cases for nontrivial real forms some new operators might appear; we will discuss this in the detailed version of the text.

1.4 Arity > 1

Grozman added a new dimension to Rudakov's solution of Veblen's problem: in 1978 he described all *bi*nary invariant differential operators.[11,12] It turned out that there are plenty of them but not too many: modulo dualizations and permutations of arguments there are eight series of first order operators and several second and third order operators all of which are compositions of first order operators with one exception: the 3rd order irreducible Grozman operator on the line. There are no invariant bilinear operators of order > 3.

Miraculously, the 1st order differential operators determine, bar a few exceptions, a Lie superalgebra structure on their domain. (Here Lie superalgebras timidly indicated their usefulness in a seemingly nonsuper problem. Other examples, such as Quillen's proof of the index theorem, and several remarkable Witten's super observations followed soon.)

Limits of applicability of Rudakov's method Though fans of Rudakov's method,[a] let us point out that its application to simple finite dimensional subalgebras of the algebras of vector fields is extremely voluminous computational job; therefore, it is ill applicable, say, to isometries of a Riemannian manifold or the group preserving the Laplace operator.

Fortunately, when Rudakov's method fails, one can usually apply other methods (Laplace-Casimir operators, N. Shapovalov determinant, etc.).

1.5 Generalized tensors and primitive forms

Rudakov considered also operators invariant with respect to the Lie algebra of Hamiltonian vector fields on the symplectic manifold (M^{2n}, ω).[44] Thanks to nondegeneracy of ω we can identify Ω^i with Ω^{n-i}. So the operator $d^* : \Omega^{n-i-1} \longrightarrow \Omega^{n-i}$, dual to the exterior differential $d : \Omega^i \longrightarrow \Omega^{i+1}$ looks like a new operator, $\delta : \Omega^{i+1} \longrightarrow \Omega^i$, the co-differential. There are also (proportional to each other) compositions $\delta \circ \omega \circ \delta$ and $d \circ \omega^{-1} \circ d$, where ω^{-1} is the convolution with the bivector dual to ω.

A novel feature is provided by the fact that "tensors" now are sections of the representation of $\mathfrak{sp}(V)$, not $\mathfrak{gl}(V)$. Since various representations of $\mathfrak{sp}(V)$ can not be extended to representations of $\mathfrak{gl}(V)$ these "tensors" are, strictly speaking, new notions.

Another novel feature we encounter considering subalgebras $\mathfrak{g}$ of $\mathfrak{vect}$ are *primitive forms.* If the $\mathfrak{vect}$-module $I(V)$ contains a singular vector with respect to $\mathfrak{vect}$, so it does with respect to $\mathfrak{g}$. But the irreducible $\mathfrak{vect}_0$-module V does not have to remain irreducible with respect to submodule of the $\mathfrak{g}_0$. The $\mathfrak{g}_0$-irreducible component with the biggest highest weight in V is called the $\mathfrak{g}_0$-*primitive* (usually, just primitive) component. Examples: the primitive components appeared in symplectic geometry (we encounter their counterparts

[a] Interplay between restriction and induction functors goes back to Frobenius, but discovery of each instance deserves an acknowledgement, we presume.

in finite dimensional purely odd picture as spherical harmonics,[46,37]) contact analogues of primitive forms are described in Ref. 32. Spaces of primitive differential and integrable forms are just restrictions of the "usual" domain of the exterior differential; but other types of primitive tensors are domains of really new invariant operators.

Further examples A. Shapovalov and Shmelev considered the Lie *super*algebras of Hamiltonian vector fields and (following Bernstein[2] who considered the non-super case) their central extension, the Poisson Lie superlagebra, see Refs. 45, 46 and 52-54. Shmelev also considered the operators invariant with respect to the funny exceptional deformation $\mathfrak{h}_\lambda(2|2)$ of the Lie superalgebra $\mathfrak{h}(2|2)$ of Hamiltonian vector fields.[51] See also Ref. 38 for further description of $\mathfrak{h}_\lambda(2|2)$.

By that time I. Kostrikin described singular vectors for the contact Lie algebras and found a "new" 2nd order invariant operator. This operator was actually well-known in differential geometry as an *Euler operator* (for its description see Ref. 39; here we just briefly observe that it is not $\sum x_i\partial_i$, this is another Euler operator); it is needed for invariant formulation of Monge-Ampére equations, cf. Ref. 40. Leites[32] generalized I. Kostrikin's calculations to contact Lie *super*algebras and found out that there seem to be no analogue of Euler's operator in supersetting. This makes one contemplate on the following:

Problem What are superanalogs of Monge-Ampére equations, if any?

In 1977 "odd" analogs of the hamiltonian and contact series were discovered.[31,1] Batalin and Vilkovisky rediscovered the antibracket related to these series and showed its importance, cf. Ref. 10. Kochetkov[24−28] undertook the task of calculating the corresponding singular vectors. He digressed to consider two of the three known at that time Shchepochkina's exceptions[23] (and named after her with the first Cyrillic letter of her name), one of which was recently reconsidered in another realization in Ref. 19. At the last moment we have found out more singular vectors (= invariant operators) than in Ref. 23 , even with finite dimensional fibers; SuperLie is indeed indispensable. After the last moment we have considered ш_2; we reproduce Kochetkov's result for comparison. For the lack of resources we left out some possible cases of singular vectors, but we are sure they are improbable. Though for $\mathfrak{vle}(4|3)$ and $\mathfrak{mb}(4|5)$ all degrees can occure, we are sure induction à la Kochetkov (complete list of singular vectors) can be performed. Various regradings of $\mathfrak{ksle}$ seem to be a tougher problem.

1.6 Superization leads to new developments

The study of invariant differential operators on supermanifolds began in 1976 as a byproduct of attempts to construct an integration theory on supermanifolds similar to the integration theory of differential forms on manifolds. Bernstein and Leites became interested in Veblen's problem when they tried to construct an integration theory for supermanifolds containing an analog of the Stokes formula.[3,4] At that time there were only known the differential forms which are impossible to integrate and the volume forms of the highest degree. Unlike the situation on manifolds, no volume form coincides with any differential form and there was known no analogs of volume forms of lesser degrees.

Having discovered *integrable* forms[3,5] (i.e., the forms that can be integrated; Deligne calls them *integral* forms[8]) Bernstein and Leites wanted to be sure that there were no other tensor objects that can be integrated. Observe several points of this delicate question.

(1) The conventional Stokes formula on a manifold exists due to the fact that there is an invariant operator on the space of differential forms. The uniqueness of the integration theory with Stokes formula follows then from the above result by Rudakov and its superization due to Bernstein and Leites.

Since there are several superanalogs of the determinant, it follows that on supermanifolds, there are, perhaps, several analogs of integration theory, see Ref. 34, some of them without Stokes formula. Still, if we wish to construct an integration theory for supermanifolds containing an analog of the Stokes formula, and, moreover, coinciding with it when the supermanifold degenerates to a manifold, we have to describe all differential operators in tensor fields on supermanifolds.

(2) Bernstein and Leites confined themselves to *finite dimensional* representations ρ owing to tradition which says that *a tensor field is a section of a vector bundle with a finite dimensional fiber on which the general linear group acts*. Even in doing so Bernstein and Leites had to digress somewhat from the conventions and consider, since it was natural, ALL finite dimensional irreducible representations ρ of the general linear Lie superalgebra. Some of such representations can not be integrated to a representation of the general linear supergroup.

Inspired by Duflo, Leites used calculations of Ref. 5 to describe invariant differential operators acting in the superspaces of tensor fields with *infinite* dimensional fibers, see Ref. 36. These operators of order > 1 are totally new, though similar to fiberwise integration along the odd coordinates. The operators of order 1 are also not bad: though they are, actually, the good old exterior differential d, the new domain is that of semi-infinite forms, certain class of pseudodifferential forms. Observe that quite criminally (in V. I. Arnold's words) no example of the corresponding new type homology is calculated yet, except some preliminary (but important) results of Shander, see Ref. 33 v. 31, Ch. 4, 5.

(3) Even under all the restrictions Bernstein and Leites imposed, to say that "the only invariant differential operator is just the exterior differential" would be to disregard how drastically they expanded its domain (even though they ignored semi-infinite possibilities). It acts in the superspace of differential forms and in the space of *integrable forms*, which is natural, since the space of integrable forms is just the dual space to the superspace of differential forms. Though Bernstein and Leites did not find any new invariant differential operator (*this proves that an integration theory on supermanifolds containing an analog of the Stokes formula can only be constructed with integrable forms*), they enlarged the domain of the exterior differential to the superspace of pseudodifferential and pseudointegrable forms. These superspaces are not *tensor fields* on $\mathcal{M}^{m,n}$ unless $n = 1$, but they are always tensor fields on the supermanifold $\widehat{\mathcal{M}}$ whose structure sheaf $\mathcal{O}_{\widehat{\mathcal{M}}}$ is a completion of the sheaf of differential forms on $\mathcal{M}$; namely, the sections of $\mathcal{O}_{\widehat{\mathcal{M}}}$ are arbitrary functions of differentials, not only polynomial ones.

(4) Bernstein and Leites did not consider indecomposible representations ρ which are more natural in both the supersetting and for infinite dimensional fibers. The first to consider indecomposible cases was Shmelev;[54] his result was, however, "not interesting": there are no totally new operators, just compositions of the known ones with projections. For a review of indecomposible representations of simple Lie superalgebras see Ref. 35.

Integration and invariant differential operators for infinite dimensional fibers There are new operators invariant with respect to the already considered (super)groups of diffeomorphisms or, equivalently, their Lie superalgebras, if we let them act in the superspaces of sections of vector bundles with infinite dimensional fibers. These operators of high order have no counterparts on manifolds and are versions of the Berezin integral applied fiber-wise. (A year after the talk with these results[36] was delivered, I. Penkov and V. Serganova interpreted some of these new operators as acting in the superspaces of certain tensor fields on "curved" superflag and supergrassmann supervarieties.[42])

We hope to relate with some of these operators new topological invariants (or perhaps old, like cobordisms, but from a new viewpoint). Recall that since the de Rham cohomology of a supermanifold are the same as those of its underlying manifold, the "old type" operators are inadequate to study "topological" invariants of supermanifolds. The operators described here and related to vector bundles of infinite rank lead to new (co)homology theories (we prefix them with a "pseudo"). These pseudocohomologies provide us with invariants different from de Rham cohomology; regrettably, never computed yet.

The approach adopted here for the operators in the natural bundles with *infinite dimensional* fibers on supermanifolds prompts us to start looking for same on manifolds. From the explicit calculations in Grozman's thesis,[11] it is clear that there are some new bilinear operators acting in the spaces of sections of tensor fields with infinite dimensional fibers.

1.7 An infinitesimal version of Veblen's problem

Denote $\mathcal{F} = \mathbb{K}[[x]]$, where $x = (u_1, \dots, u_n, \xi_1, \dots, \xi_m)$ so that $p(u_i) = \bar{0}$ and $p(\xi_j) = \bar{1}$. Denote by (x) the maximal ideal in $\mathcal{F}$ generated by the x_i. Define a topology in $\mathcal{F}$ so that the ideals $(x)^r$, $r = 0, 1, 2, \dots$ are neighborhoods of zero, i.e., two series are r-*close* if they coincide up to order r. We see that $\mathcal{F}$ is complete with respect to this topology.

Denote by $\mathfrak{vect}(n|m)$ the Lie superalgebra of formal vector fields, i.e., of continuous derivations of $\mathbb{K}[[x]]$. By abuse of notations we denote $\mathfrak{der}\mathbb{K}[x]$, the Lie superalgebra of *polynomial* vector fields, also by $\mathfrak{vect}(n|m)$.

Define partial derivatives $\partial_i = \frac{\partial}{\partial x_i} \in \mathfrak{vect}(n|m)$ by setting $\partial_i(x_j) = \delta_{ij}$ with super-Leibniz rule. Clearly, $p(\partial_i) = p(x_i)$ and $[\partial_i, \partial_j] = 0$. Any element $D \in \mathfrak{vect}(n|m)$ is of the form $D = \sum f_i \partial_i$, where $f_i = D(x_i) \in \mathcal{F}$. We will denote $\mathfrak{vect}(n|m)$ by $\mathcal{L}$. In $\mathcal{L}$, define a filtration of the form $\mathcal{L} = \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$ setting

$$\mathcal{L}_r = \{D \in \mathfrak{vect}(n|m) \mid D(\mathcal{F}) \subset (x)^{r+1}\}.$$

This filtration defines a topology on $\mathcal{L}$, the superspaces $\mathcal{L}_r$ being the base of the topology, open neighborhoods of zero.

Denote by $L = \oplus L_r$, where $L_r = \mathcal{L}_r / \mathcal{L}_{r+1}$, the associated graded Lie superalgebra. Clearly, $L_0 \simeq \mathfrak{gl}(n|m)$ with $E_{ij} \longleftrightarrow x_j \partial_i$.

Let ρ be an irreducible representation of the Lie superalgebra $L_0 = \mathfrak{gl}(n|m)$ with lowest weight in a superspace V. Define a $\mathfrak{vect}(n|m)$-module $T(\rho)$ also denoted by $T(V)$ by setting $T(V) = \mathcal{F} \otimes_{\mathbb{K}} V$. The superspace $T(V)$ evidently inherits the topology of $\mathcal{F}$. To any vector field D, assign the operator $L_D : T(V) \longrightarrow T(V)$ — the Lie derivative — such that for $f \in \mathcal{F}$ and $v \in V$

$$L_D(fv) = D(f)v + (-1)^{p(D)p(f)} \sum D^{ij} \rho(E_{ij})(v), \tag{1.7.1}$$

where $D^{ij} = (-1)^{p(x_i)(p(f)+1)} \partial_i f_j$. We will usually write just D instead of L_D.

The elements $t \in T(V)$ will be called *tensor fields of type* V. The modules $T(V)$ are topological; their duals are spaces with discrete topology.

Observe that even if V is finite dimensional, the elements of $T(V)$ are generalized tensors as compared with the classical notion: the space V might not be realized in the tensor product of co- and contra-variant tensors, only as a subquotient of such; e.g., unlike the determinant (or trace, speaking on the Lie algebra level), the supertrace is not realized in tensors and we have to introduce new type of "tensors" — the λ-densities.

For any L_0-module V with highest weight and any L_0-module W with lowest weight set

$$I(V) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V; \quad T(W) = \mathrm{Hom}_{U(\mathcal{L}_0)}(U(\mathcal{L}), W), \tag{1.7.2}$$

where we have extended the action of L_0 to a $U(\mathcal{L}_0)$-action by setting $\mathcal{L}_1 V = 0$ and $\mathcal{L}_1 W = 0$. Clearly,

a) $I(V)$ is an $\mathcal{L}$-module with discrete topology.

b) $I(V)^* \cong T(V^*)$

c) definition of the tensor fields with $\mathcal{L}$-action (1.7.1) is equivalent to the one given by (1.7.2).

Thus, instead of studying invariant maps $T(W_1) \longrightarrow T(W_2)$ (or $T(W_1) \otimes T(W_2) \longrightarrow T(W_3)$, etc.) we may study submodules — or, equivalently, singular vectors — of $I(V)$ (resp. of $I(V_1) \otimes I(V_2)$, etc.). They are much easier to describe.

Further generalization of tensors. The highest weight theorem Let

$$\mathcal{L} = \mathcal{L}_{-d} \supset \cdots \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \dots$$

be a Lie superalgebra of vector fields with formal or polynomial coefficients and endowed with a Weisfeiler filtration/grading described in what follows (for the time being consider a "most natural" grading, like that in $\mathfrak{vect}$ above). We define the space of *generalized* tensor fields and its dual by the same formula (1.7.2) as for the usual tensor fields given any L_0-module V with highest weight and any L_0-module W with lowest weight such that $\mathcal{L}_1 V = 0$ and $\mathcal{L}_1 W = 0$.

Observe that for the Lie algebra of divergence-free vector fields the spaces $T(W)$ are the same as for $\mathfrak{vect}$. For some other Lie superalgebras the notion of tensors we give is different because there are representations of L_0 distinct from tensor powers of the identity one. For example, for the Lie superalgebra $\mathcal{L}$ of Hamiltonian vector fields $\mathfrak{h}(2n|m)$ such is the spinor representation (for $n = 0$); if we consider infinite dimensional fibers such is the oscillator representation (for $m = 0$), and in the general case such is the spinor-oscillator representation, cf. Ref. 37.

Thus, the first step in the study of $\mathcal{L}$-invariant operators is a description of irreducible L_0-modules, at least in terms of the highest/lowest weight. For the majority of the L_0's this is not a big deal, but the catch is that for some L_0's there is no easy highest/lowest weight theorem, even for finite dimensional modules. We will encounter this phenomenon with $\mathfrak{as}$, the linear part of $\mathfrak{vas}$.

An aside remark: being interested not only in representations of vectorial algebras (with polynomial coefficients) but in their stringy analogs (with Laurent coefficients), too, observe that vacuum over L_0 can be degenerate.

Problem For all Weisfeiler gradings of simple vectorial superalgebras $\mathcal{L}$ describe conditions for the highest (lowest) weight under which the irreducible quotient of the Verma module over L_0 is finite dimensional and describe the corresponding module (say, in terms of a character formula, cf. Ref. 42).

Examples of generalized tensor fields Clearly, for $\mathcal{L} = \mathfrak{vect}(n|m)$ we have $\mathcal{L} \equiv T(\mathrm{id})$, where $\mathrm{id} = Span(\partial_i \mid 1 \leq i \leq n+m)$ is the (space V of the) identity representation of $L_0 = \mathfrak{gl}(V) = \mathfrak{gl}(n|m)$. The spaces $T(E^i(\mathrm{id}^*))$ are denoted by Ω^i; their elements are called *differential i-forms* and the right dual elements to ∂_i are denoted by $\widehat{x_i} = dx_i$, where $p(\widehat{x_i}) = p(x_i) + \bar{1}$. In particular, let $\mathcal{F} = \Omega^0$ be the algebra of functions.

The algebra $\widehat{\Omega}$ of arbitrary, not only polynomial, functions in $\widehat{x_i} = dx_i$ is called the algebra of *pseudodifferential forms*. An important, as Shander showed in Ref. 33 v. 31, Ch. 5, subspace $\widehat{\Omega}^{(\lambda)}$ of homogeneous *pseudodifferential forms of homogeneity degree* $\lambda \in \mathbb{K}$ is naturally defined as functions of homogeneity degree λ with respect to the hatted indeterminates.

Define the space of volume forms *Vol* to be $T(\mathrm{str})$; denote the volume element by $vol(x)$ or $vol(u|\theta)$. (Observe again that it is a bad habit to denote, as many people still do, *vol* by $d^n u d^m \theta$: their transformation rules are totally different, see, e.g., Refs. 5, 8.)

The space of *integrable* i-forms is $\Sigma_i = \mathrm{Hom}_{\mathcal{F}}(\Omega^i, Vol)$. In other words, integrable forms are *Vol*-valued polyvector fields. Pseudointegrable forms are defined as elements of $\widehat{\Sigma} = \mathrm{Hom}_{\mathcal{F}}(\widehat{\Omega}, Vol)$; the subspace $\widehat{\Sigma}_{(\lambda)} = \mathrm{Hom}_{\mathcal{F}}(\widehat{\Omega}^{(\lambda)}, Vol)$ of homogeneous forms is also important.

Particular cases: a) $m = 0$. We see that $\Omega^i = 0$ for $i > n$ and $\Sigma_i = 0$ for $i < 0$. In addition, the mapping $vol \mapsto \hat{x}_1 \cdots \hat{x}_n$ defines an isomorphism of Ω^i with Σ_i preserving all structures.

b) $n = 0$. In this case there is an even $\mathcal{L}$-module morphism $\int : \Sigma_{-m} \longrightarrow \mathbb{K}$ called the *Berezin integral*. It is defined by the formula

$$\int \xi_1 \cdots \xi_m \, vol = 1, \text{and} \int \xi_1^{\nu_1} \cdots \xi_m^{\nu_m} \, vol = 0 \quad \text{if} \quad \prod \nu_i = 0.$$

We will also denote by $\int$ the composition $\int : \Sigma_{-m} \to \mathbb{K} \hookrightarrow \Omega^0$ of the Berezin integral and the natural embedding.

c) $m = 1$. We generalize Ω^i and Σ_j to the spaces Φ^λ of pseudodifferentional and pseudointegral forms containing Ω^i and Σ_j, where $\lambda \in \mathbb{K}$. Let $x = (u_1, \cdots, u_n, \xi)$. Consider a $\mathbb{K}$-graded Ω-module $\Phi = \oplus\Phi^\lambda$ (we assume that $\deg \hat{x}_i = 1 \in \mathbb{K}$) generated by $\hat{\xi}^\lambda$, where $\deg \hat{\xi}^\lambda = \lambda$ and $p(\hat{\xi}^\lambda) = \bar{0}$, with relations $\hat{\xi} \cdot \hat{\xi}^\lambda = \hat{\xi}^{\lambda+1}$. Define the action of partial derivatives ∂_i and $\hat{\partial}_j$ for $1 \leq i, j \leq n+1$ via $\hat{\partial}_j(x_i) = 0$, $\partial_i(\hat{\xi}^\lambda) = 0$, $\partial_{\hat{u}_i}(\hat{\xi}^\lambda) = 0$ and $\partial_{\hat{\xi}}(\hat{\xi}^\lambda) = \lambda\hat{\xi}^{\lambda-1}$.

On Φ, the derivations d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$ consistent with the exterior derivation d, the inner product $i_{\mathfrak{D}}$ and the Lie derivative $L_{\mathfrak{D}}$ on Ω are naturally defined.

It is easy to see that $\Phi = \oplus\Phi^\lambda$ is a supercommutative superalgebra.

Clearly, Φ is a superspace of tensor fields and for $\Phi^{\mathbb{Z}} = \oplus_{r\in\mathbb{Z}}\Phi^r$ we have a sequence

$$0 \longrightarrow \Omega \xrightarrow{\alpha} \Phi^{\mathbb{Z}} \xrightarrow{\beta} \Sigma \longrightarrow 0 \qquad (*)$$

where the maps α and β are defined by

$$\alpha(\omega) = \omega\hat{\xi}^0, \quad \beta(\hat{u}_1 \cdots \hat{u}_n\hat{\xi}^{-1}) = vol.$$

Clearly, the homomorphisms α and β are consistent with the Ω-module structure and the operators d, $i_{\mathfrak{D}}$ and $L_{\mathfrak{D}}$. The explicit form of the $\mathcal{F}$-basis in Ω, Σ and Φ easily implies that $(*)$ is exact.

1.8 Operators invariant with respect to nonstandard realizations

At the moment the $\mathfrak{L}$-invariant differential operators are described for all but one series of simple vectorial Lie superalgebras in the standard realization. Contrariwise, about operators invariant with respect to same in nonstandard realizations almost nothing is known, except for $\mathfrak{vect}(m|n; 1)$, see Ref. 36.

For series, the standard realization is the one for which $\dim \mathfrak{L}/\mathfrak{L}_0$ is minimal; for exceptional algebras the notion of the standard realization is more elusive, and since there are 1 to 4 realizations, it is reasonable and feasible to consider all of them. It is also natural to consider $\mathfrak{h}_\lambda(2|2)$ and $\mathfrak{h}_\lambda(2|2; 1)$ as exceptional algebras, especially at exceptional values of λ.

1.9 On description of irreducible $\mathfrak{L}$-modules

Having described $\mathfrak{vect}(n|m)$-invariant differential operators in tensor fields with finite dimensional fibers (answer: only d, and $\int$ if $n = 0$, $m \neq 0$), we consider the quotients of $T(V)$ modulo the image of the invariant operator. It could be that the quotient also contains a submodule. In the general case there are no such submodules (Poincaré lemma), in other cases anything can happen, see Refs. 27 and 26,28.

Observe that to describe irreducible $\mathfrak{L}$-modules, it does not always suffice to consider only one realization of $\mathfrak{L}$. It is like considering generalized Verma modules induced or co-induced from distinct parabolic subalgebras. Similarly, the description of invariant operators must be performed from scratch in each realization.

Here we do not specifically consider the irreducible $\mathfrak{L}$-modules; so far, the answers are known for tensors with finite dimensional fibers and in two cases only: Ref. 20 ($\mathfrak{vle}(3|6)$) and Ref. 32 ($\mathfrak{k}(1|n)$; weights of singular vectors are corrected below).

2 Brief description of the exceptional algebras

To save space, we do not reproduce the details of definitions of exceptional superalgebras, see Ref. 50. Thus,[43,26,5] suffice to grasp the details of the theory;[50] to catch on with the list of exceptions we consider.

Some of the exceptional algebras $\mathfrak{g} = \sum_{i\geq -d} \mathfrak{g}_i$ are isomorphic as abstract ones; there are five abstract families altogether. We realize them as Lie superalgebras of polynomial vector fields with a particular, Weisfeiler (shortly W-), grading or filtration. In any W-grading (a) the sum of the terms of positive degree is a maximal subalgebra of finite codimension and (b) the linear part, $\mathfrak{g}_0$ irreducibly acts on $\mathfrak{g}_{-1}$. If depth $d = 1$, then each $\mathfrak{g}$ is constructed as the Cartan prolong (or its generalization) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. To construct $\mathfrak{kas}$, still another generalization of the Cartan prolongation is applied. For these generalizations, first described in Refs. 47 and 48, see Ref. 50. In Ref. 7 there is given a 15-th regrading we missed in Ref. 50, and a nice interpretation of the exceptional algebras in terms of $\mathfrak{g}_{\bar{0}}$ and the $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}_{\bar{1}}$. This interpretation is convenient in some problems, but in interpretations of our results the realization of $\mathfrak{g}$ as (generalized) Cartan prolong is often more useful.

We only recall that the classical Cartan prolong $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is defined inductively, as a subalgebra of vector fields in $\dim \mathfrak{g}_{-1}$ indeterminates with given $\mathfrak{g}_{-1}$ and the linear part $\mathfrak{g}_0$, and where for $i > 0$

$$\mathfrak{g}_{i+1} = \{D \in \mathfrak{vect}(\dim \mathfrak{g}_{-1}) \mid [D, \mathfrak{g}_{-1}] \subset \mathfrak{g}_i.$$

Hereafter $\mathfrak{vect}(m|n)$ is the general vectorial Lie superalgebra on m even and n odd indeterminates; $\mathfrak{svect}(m|n)$ is its divergence-free or special subalgebra; $\mathfrak{k}(2m+1|n)$ the contact algebra that preserves the Pfaffian equation $\alpha = 0$, where

$$\alpha_1 = dt + \sum_{1\leq i\leq m} (p_i dq_i - q_i dp_i) + \sum_{j\leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + \sum_{k\geq n-2r} \theta_k d\theta_k.$$

For $f \in \mathbb{C}[t, p, q, \Theta]$, where $\Theta = (\xi, \eta, \theta)$, define the contact field K_f by setting:

$$K_f = (2-E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y_i are all the coordinates except t), and H_f is the hamiltonian field with Hamiltonian f that preserves $d\alpha_1$:

$$H_f = \sum_{i \leq n} \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)} \left(\sum_{j \leq m} \frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j} \right).$$

Replacement of some of the θ's in the above formula with ξ, η leads to obvious modifications.

The "odd" analog of the contact structure is given by the *even* form

$$\alpha_0 = d\tau + \sum_{1 \leq i \leq n} (\xi_i dq_i + q_i d\xi_i)$$

and formula for the vector field M_f — the analogs of K_f — generated by the function of τ, ξ, q are similar. The fields M_f span the "odd" contact Lie superalgebra, $\mathfrak{m}(n)$.

The regradings are given after the dimension of the supermanifold in the standard realization ($r = 0$, optional, marked by $(*)$) after semicolon. Observe that the codimension of $\mathfrak{L}_0$ attains its minimum in the standard realization.

Lie superalgebra	its $\mathbb{Z}$-grading
$\mathfrak{vect}(n\|m; r)$, $0 \leq r \leq m$	$\deg u_i = \deg \xi_j = 1$ for any i, j $(*)$
	$\deg \xi_j = 0$ for $1 \leq j \leq r$; $\deg u_i = \deg \xi_{r+s} = 1$ for any i, s
$\mathfrak{m}(n; r)$, $0 \leq r \leq n$, $r \neq n-1$	$\deg \tau = 2$, $\deg q_i = \deg \xi_i = 1$ for any i $(*)$
	$\deg \tau = \deg q_i = 2$, $\deg \xi_i = 0$ for $1 \leq i \leq r < n$; $\deg q_{r+j} = \deg \xi_{r+j} = 1$ for any j
$\mathfrak{m}(1\|n; n)$	$\deg \tau = \deg q_i = 1$, $\deg \xi_i = 0$ for $1 \leq i \leq n$
$\mathfrak{k}(2n+1\|m; r)$, $0 \leq r \leq [\frac{m}{2}]$, $r \neq k-1$ for $m = 2k$ and $n = 0$	$\deg t = 2$, $\deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1$ for any i, j, k $(*)$
	$\deg t = \deg \xi_i = 2$, $\deg \eta_i = 0$ for $1 \leq i \leq r \leq [\frac{m}{2}]$; $\deg p_i = \deg q_i = \deg \theta_j = 1$ for $j \geq 1$ and all i
$\mathfrak{k}(1\|2m; m)$	$\deg t = \deg \xi_i = 1$, $\deg \eta_i = 0$ for $1 \leq i \leq m$

Now the last notations: $\mathfrak{hei}(2n|m)$ is the *Heisenberg algebra*, it is isomorphic to $\mathfrak{k}(2n+1|m)_-$, the negative part of $\mathfrak{k}(2n+1|m)$; similarly, $\mathfrak{ab}(m)$ is the *antibracket algebra*, isomorphic to $\mathfrak{m}(m)_-$.

A. Sergeev central extension, $\mathfrak{as}$, of $\mathfrak{spe}(4)$: we represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$ and z is the central element. We define the bracket in $\mathfrak{as}$ in its matrix realization in the spinor representation:

$$\left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z\,, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z\right] = \left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix}\right] + \operatorname{tr} c\tilde{c}' \cdot z\,,$$

where $\tilde{}$ is extended via linearity from matrices $c_{ij} = E_{ij} - E_{ji}$ on which $\tilde{c}_{ij} = c_{kl}$ for any even permutation $(1234) \mapsto (ijkl)$.

The five types of exceptional W-graded Lie superalgebras are given above in their realizations as Cartan's prolongs $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ or generalized Cartan's prolongs $(\mathfrak{g}_-, \mathfrak{g}_0)_*^{mk}$ for $\mathfrak{g}_- = \mathop{\oplus}\limits_{-d\leq i\leq -1} \mathfrak{g}_i$ expressed for $d = 2$ as $(\mathfrak{g}_{-2}, \mathfrak{g}_{-1}, \mathfrak{g}_0)_*^{mk}$ together with one of the serial Lie superalgebras as an ambient which contains the exceptional one as a maximal subalgebra. The regradings $R(r)$ of the ambients given below are sometimes not of Weisfeiler type.

$\mathfrak{vle}(4\vert 3; r) = (\Pi(\Lambda(3))/\mathbb{C}\cdot 1, \mathfrak{cvect}(0\vert 3))_* \subset \mathfrak{vect}(4\vert 3; R), \quad r = 0, 1, K$
$\mathfrak{vas}(4\vert 4) = (\mathrm{spin}, \mathfrak{as})_* \subset \mathfrak{vect}(4\vert 4)$
$\mathfrak{kas}^\xi(1\vert 6; r) \subset \mathfrak{k}(1\vert 6; r), \quad r = 0, 1\xi, 3\xi$ $\mathfrak{kas}^\xi(1\vert 6; 3\eta) = (\mathit{Vol}_0(0\vert 3), \mathfrak{c}(\mathfrak{vect}(0\vert 3)))_* \subset \mathfrak{svect}(4\vert 3)$
$\mathfrak{mb}(4\vert 5; r) = (\mathfrak{ab}(4), \mathfrak{cvect}(0\vert 3))_*^m \subset \mathfrak{m}(4\vert 5; R), \quad r = 0, 1, K$
$\mathfrak{ksle}(9\vert 6; r) = (\mathfrak{hei}(8\vert 6), \mathfrak{svect}(0\vert 4)_{3,4})_*^k \subset \mathfrak{k}(9\vert 6; r), \quad r = 0, 2,$ CK $\mathfrak{ksle}(9\vert 6; K) = (\mathrm{id}_{\mathfrak{sl}(5)}, \Lambda^2(\mathrm{id}^*_{\mathfrak{sl}(5)}), \mathfrak{sl}(5))_*^k \subset \mathfrak{svect}(5\vert 10; R)$

Certain regradings $R(r)$ of the ambients are so highly nonstandard that even the homogeneous fibers are of infinite dimension:

1) $\mathfrak{vle}(4|3; r) = (\Pi(\Lambda(3)/\mathbb{C}\cdot 1), \mathfrak{cvect}(0|3))_* \subset \mathfrak{vect}(4|3), \quad r = 0, 1, K;$
 <u>$r = 0$</u>: $\deg y = \deg u_i = \deg \xi_i = 1$;
 <u>$r = 1$</u>: $\deg y = \deg \xi_1 = 0$, $\deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1$, $\deg u_1 = 2$;
 <u>$r = K$</u>: $\deg y = 0$, $\deg u_i = 2$; $\deg \xi_i = 1$.
2) $\mathfrak{vas}(4|4) = (\mathrm{spin}, \mathfrak{as})_* \subset \mathfrak{vect}(4|4)$;
3) $\mathfrak{kas} \subset \mathfrak{k}(1|6; r), \ r = 0, 1, 3\xi; \quad \mathfrak{kas}(1|6; 3\eta) \subset \mathfrak{svect}(4|3)$;
 <u>$r = 0$</u>: $\deg t = 2, \quad \deg \eta_i = 1; \quad \deg \xi_i = 1$;
 <u>$r = 1$</u>: $\deg \xi_1 = 0, \ \deg \eta_1 = \deg t = 2, \ \deg \xi_2 = \deg \xi_3 = \deg \eta_2 = \deg \eta_3 = 1$;
 <u>$r = 3\xi$</u>: $\deg \xi_i = 0, \quad \deg \eta_i = \deg t = 1$;
 <u>$r = 3\eta$</u>: $\deg \eta_i = 0$, $\deg \xi_i = \deg t = 1$.
4) $\mathfrak{mb}(4|5; r) = (\mathfrak{ab}(4), \mathfrak{cvect}(0|3))_*^m \subset \mathfrak{m}(4), \quad r = 0, 1, K$;
 <u>$r = 0$</u>: $\deg \tau = 2, \quad \deg u_i = \deg \xi_i = 1$ for $i = 0, 1, 2, 3$;
 <u>$r = 1$</u>: $\deg \tau = \deg \xi_0 = \deg u_1 = 2, \quad \deg u_2 = \deg u_3 = \deg \xi_2 = \deg \xi_3 = 1$;
 $\deg \xi_1 = \deg u_0 = 0$;
 <u>$r = K$</u>: $\deg \tau = \deg \xi_0 = 3, \quad \deg u_0 = 0, \quad \deg u_i = 2; \quad \deg \xi_i = 1$ for $i > 0$.
5) $\mathfrak{ksle}(9|6; r) = (\mathfrak{hei}(8|6), \mathfrak{svect}_{3,4}(4))_*^k \subset \mathfrak{k}(9|6), \quad r = 0, 2$, K, CK;
 <u>$r = 0$</u>: $\deg t = 2, \quad \deg p_i = \deg q_i = \deg \xi_i = \deg \eta_i = 1$;
 <u>$r = 2$</u>: $\deg t = \deg q_3 = \deg q_4 = \deg \eta_1 = 2, \ \deg q_1 = \deg q_2 = \deg p_1 = \deg p_2$

$=\deg\eta_2=\deg\eta_3=\deg\zeta_2=\deg\zeta_3=1;\quad \deg p_3=\deg p_4=\deg\zeta_1=0;$
$\underline{r=K}$: $\deg t=\deg q_i=2,\ \deg p_i=0;\quad \deg\zeta_i=\deg\eta_i=1;$
$\underline{r=CK}$: $\deg t=\deg q_1=3, \deg p_1=0; \deg q_2=\deg q_3=\deg q_4=\deg\zeta_1$
$=\deg\zeta_2=\deg\zeta_3=2; \deg p_2=\deg p_3=\deg p_4=\deg\eta_1=\deg\eta_2=\deg\eta_3=1.$

To determine the minimal ambient is important for our problem: every operator invariant with respect to an algebra is, of course, invariant with respect to any its subalgebra.

Here is the list of nonpositive terms of the exceptional algebras. Notations: $\mathfrak{c}(\mathfrak{g})$ denotes the trivial central extension with the 1-dimensional even center generated by z; $\mathbb{C}[k]$ is the trivial $\mathfrak{g}_0$-module on which the central element z from $\mathfrak{g}_0$ chosen so that $z|_{\mathfrak{g}_i}=i\cdot\mathrm{id}_{\mathfrak{g}_i}$ acts as multiplcation by k; $\mathfrak{a}\subplus\mathfrak{b}$ denotes the semidirect product in which $\mathfrak{a}$ is the ideal; let d determine the $\mathbb{Z}$-grading of $\mathfrak{g}$ and not belong to $\mathfrak{g}$, we shorthand $\mathfrak{g}\supplus\mathbb{C}(az+bd)$ to $\mathfrak{g}_{a,b}$; $Vol(n|m)$ is the space of volume forms (densities) on the superspace of superdimension indicated, subscript 0 singles the subspace with integral 0; $T^{1/2}$ is the representation in the space of half-densities and T^0_0 is the quotient of Vol_0 modulo constants (over divergence-free algebra).

Observe that none of the simple W-graded vectorial Lie superalgebras is of depth >3 and only two algebras are of depth 3: $\mathfrak{mb}(4|5;K)$, for which we have

$$\mathfrak{mb}(4|5;K)_{-3}\cong\Pi(\mathrm{id}_{\mathfrak{sl}(2)}),$$

and another one, $\mathfrak{ksle}(9|6;CK)=\mathfrak{ck}(11|9)$, for which we have :

$$\mathfrak{ck}(11|9)_{-3}\simeq\Pi(\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-3]).$$

Here are the other terms $\mathfrak{g}_i$ for $i\leq 0$ of the 15 exceptional W-graded algebras.

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	r
$\mathfrak{vle}(4\|3)$	$-$	$\Pi(\Lambda(3)/\mathbb{C}\cdot 1)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	0
$\mathfrak{vle}(5\|4)$	$\mathbb{C}[-2]$	$\mathrm{id}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\subplus T^{1/2}(\mathfrak{vect}(0\|2))$	1
$\mathfrak{vle}(3\|6)$	$\mathrm{id}_{\mathfrak{sl}(3)}$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-1]$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	K
$\mathfrak{vas}(4\|4)$	$-$	spin	$\mathfrak{as}$	0
$\mathfrak{kas}(1\|6)$	$\mathbb{C}[-2]$	$\Pi(\mathrm{id})$	$\mathfrak{co}(6)$	0
$\mathfrak{kas}(5\|5)$	$\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathrm{id}_{\mathfrak{gl}(2)}\otimes\Lambda(1)$	$(\mathfrak{sl}(2)\oplus\mathfrak{gl}(2)\otimes\Lambda(1))\subplus\mathfrak{vect}(0\|1)$	1ξ
$\mathfrak{kas}(4\|4)$	$-$	$\Lambda(3)$	$\Lambda(3)\oplus\mathfrak{sl}(1\|3)$	3ξ
$\mathfrak{kas}(4\|3)$	$-$	$Vol_0(0\|3)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	3η
$\mathfrak{mb}(4\|5)$	$\Pi(\mathbb{C}[-2])$	$Vol(0\|3)$	$\mathfrak{c}(\mathfrak{vect}(0\|3))$	0
$\mathfrak{mb}(5\|6)$	$\Lambda(2)/\mathbb{C}1$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\Lambda(2)$	$\mathfrak{c}(\mathfrak{sl}(2)\otimes\Lambda(2)\subplus T^{1/2}(\mathfrak{vect}(0\|2))$	1
$\mathfrak{mb}(3\|8)$	$\mathrm{id}_{\mathfrak{sl}(3)}$	$\Pi(\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\mathrm{id}_{\mathfrak{sl}(2)}\otimes\mathbb{C}[-1])$	$\mathfrak{sl}(3)\oplus\mathfrak{sl}(2)\oplus\mathbb{C}z$	K
$\mathfrak{ksle}(9\|6)$	$\mathbb{C}[-2]$	$\Pi(T^0_0)$	$\mathfrak{svect}(0\|4)_{3,4}$	0
$\mathfrak{ksle}(11\|9)$	$\Pi(\mathrm{id}_{\mathfrak{sl}(1\|3)})$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\Lambda(3)$	$(\mathfrak{sl}(2)\otimes\Lambda(3))\subplus\mathfrak{sl}(1\|3)$	2
$\mathfrak{ksle}(5\|10)$	id	$\Pi(\Lambda^2(\mathrm{id}^*))$	$\mathfrak{sl}(5)$	K
$\mathfrak{ksle}(9\|11)$	$\mathrm{id}^*_{\mathfrak{sl}(3)}\otimes\Lambda(1)$	$\mathrm{id}_{\mathfrak{sl}(2)}\otimes\left(\mathrm{id}_{\mathfrak{sl}(3)}\otimes\Lambda(1)\right)$	$\mathfrak{sl}(2)\oplus(\mathfrak{sl}(3)\otimes\Lambda(1)\subplus\mathfrak{vect}(0\|1))$	CK

In what follows, we write just f instead of M_f or K_f, and I instead of M_1. So, $f \cdot g$ denotes $M_f M_g$ or $K_f K_g$, not M_{fg} or K_{fg}. We shorthand $D : I(\chi) \longrightarrow I(\psi)$ to $\chi = (a, b, c, d) \longrightarrow \psi = (e, f, g, h)$. Having selected a basis of Cartan subalgebra, we use it for every regrading.

The negated degree of the singular vector, i.e, the degree of the corresponding operator, is denoted by d). We are sure that in some cases there are singular vectors (operators) of degree higher than listed ones; in such cases we write "in degrees indicated the singular vectors are ..." rather than "the following are all possible singular vectors". In what follows m_1 is the nonzero highest weight vector of the $\mathfrak{g}_0$-module V.

3 Singular vectors for $\mathfrak{g} = \mathfrak{vle}(3|6)$

We denote the indeterminates by x (even) and ξ (odd); the corresponding partial derivatives by ∂ and δ. The Cartan subalgebra is spanned by

$$\begin{array}{ll} h_1 = -2\,x_4 \otimes \partial_4 - \xi_1 \otimes \delta_1 - \xi_2 \otimes \delta_2 - \xi_3 \otimes \delta_3, & h_2 = -x_2 \otimes \partial_2 - x_3 \otimes \partial_3 - \xi_1 \otimes \delta_1, \\ h_3 = -x_1 \otimes \partial_1 - x_3 \otimes \partial_3 - \xi_2 \otimes \delta_2, & h_4 = -x_1 \otimes \partial_1 - x_2 \otimes \partial_2 - \xi_3 \otimes \delta_3 \end{array}$$

We consider the following negative operators from $\mathfrak{g}_0$:

$$a_1 = \partial_4, \quad a_{12} = -x_4{}^2\partial_4 - x_4\xi_1\delta_1 - x_4\xi_2\delta_2 - x_4\xi_3\delta_3 + \xi_1\xi_2\partial_3 - \xi_1\xi_3\partial_2 + \xi_2\xi_3\partial_1$$

$$\begin{array}{ll} a_2 = -x_1\partial_1 - x_2\partial_2 - x_3\partial_3 + x_4\partial_4, & a_3 = x_2\partial_2 + x_3\partial_3 + \xi_1\delta_1 \\ a_4 = -x_2\partial_1 + \xi_1\delta_2, & a_5 = -x_3\partial_1 + \xi_1\delta_3 \\ a_6 = -x_1\partial_2 + \xi_2\delta_1, & a_7 = x_1\partial_1 + x_3\partial_3 + \xi_2\delta_2 \\ a_8 = -x_3\partial_2 + \xi_2\delta_3, & a_9 = -x_1\partial_3 + \xi_3\delta_1 \\ a_{10} = -x_2\partial_3 + \xi_3\delta_2, & a_{11} = x_1\partial_1 + x_2\partial_2 + \xi_3\delta_3 \end{array}$$

and the operators from $\mathfrak{g}_-$:

$$\begin{array}{lll} n_1 = \delta_1, & n_4 = x_4\delta_3 - \xi_1\partial_2 + \xi_2\partial_1, & n_7 = \partial_1 \\ n_2 = \delta_2, & n_5 = x_4\delta_2 - \xi_1\partial_3 + \xi_3\partial_1, & n_8 = \partial_2 \\ n_3 = \delta_3, & n_6 = x_4\delta_1 - \xi_2\partial_3 + \xi_3\partial_2, & n_9 = \partial_3 \end{array}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{ll} m_1 \text{ is the highest weight vector} & m_9 = a_4 \cdot a_8 \cdot m_1 \\ m_2 = a_{12} \cdot m_1 & m_{10} = a_8 \cdot a_8 \cdot m_1 \\ m_3 = a_4 \cdot m_1 & m_{11} = a_5 \cdot m_1 \\ m_4 = a_8 \cdot m_1 & m_{12} = a_{12} \cdot a_{12} \cdot a_{12} \cdot m_1 \\ m_5 = a_{12} \cdot a_{12} \cdot m_1 & m_{13} = a_{12} \cdot a_{12} \cdot a_4 \cdot m_1 \\ m_6 = a_{12} \cdot a_4 \cdot m_1 & m_{14} = a_{12} \cdot a_{12} \cdot a_8 \cdot m_1 \\ m_7 = a_{12} \cdot a_8 \cdot m_1 & m_{15} = a_{12} \cdot a_4 \cdot a_4 \cdot m_1 \\ m_8 = a_4 \cdot a_4 \cdot m_1 & m_{16} = a_{12} \cdot a_4 \cdot a_8 \cdot m_1 \end{array}$$

Theorem In $I(V)$, there are only the following singular vectors of degree d):

1a) $(k, k, l, l) \longrightarrow (k + 1, k + 1, l, l)$: $n_1 \otimes m_1$;

1b) $(k, k, k - 1, k - 1) \longrightarrow (k + 1, k, k, k - 1)$: $n_2 \otimes m_1 + n_1 \otimes m_3$;

1c) $(k-3,k,k,k-1) \longrightarrow (k-2,k,k,k)$: $n_3 \otimes m_1 + n_2 \otimes m_4 + n_1 \otimes m_9$;
1d) $(-k,k-2,l,1) \longrightarrow (-k-1,k-1,l,l)$, where $k \neq 0$

$$k\, n_6 \otimes m_1 + n_1 \otimes m_2;$$

1e) $(-k,k-2,k-1,k-1) \longrightarrow (-k-1,k-2,k,k)$, where $k \neq 0,-1$

$$k\,(n_5 \otimes m_1 + n_6 \otimes m_3) - n_2 \otimes m_2 + n_1 \otimes m_6;$$

1f) $(-k,k+1,k+1,k) \longrightarrow (-k-1,k+1,k+1,k+1)$, where $k \neq 0$

$$k\,(n_4 \otimes m_1 - n_5 \otimes m_4 + n_6 \otimes m_9) + n_3 \otimes m_2 + n_2 \otimes m_7 + n_1 \otimes m_{16};$$

2a) $(0,-2,k,k) \longrightarrow (0,0,k-1,k-1)$: $(n_6 {\cdot} n_1) \otimes m_1$;

2b) $(0,-2,0,0) \longrightarrow (0,-1,1,0)$:

$$(n_5{\cdot}n_1 - n_6{\cdot}n_2) \otimes m_1 + (n_6{\cdot}n_1) \otimes m_3$$

2c) $(1,0,0,-1) \longrightarrow (1,1,0,0)$:

$$-(n_8+n_4{\cdot}n_1)\otimes m_1+(n_3{\cdot}n_1)\otimes m_2+(n_9+n_5{\cdot}n_1)\otimes m_4+(n_2{\cdot}n_1)\otimes m_7-(n_6{\cdot}n_1)\otimes m_9$$

2d) $(-3,0,0,-1) \longrightarrow (-3,1,0,0)$

$$\begin{aligned}&(n_8 + n_4{\cdot}n_1 - 2\ (n_6{\cdot}n_3)) \otimes m_1 + (n_3{\cdot}n_1) \otimes m_2\\ &- (n_9 + n_5{\cdot}n_1 + 2\ (n_6{\cdot}n_2)) \otimes m_4 + (n_2{\cdot}n_1) \otimes m_7 - (n_6{\cdot}n_1) \otimes m_9\end{aligned}$$

2e) $(0,k,-1,k+1) \longrightarrow (0,k,1,k+1)$ (The dual cases were not calculated.)

$$-k\ (1+k)\ (n_5{\cdot}n_2) \otimes m_1 - k\ (n_5{\cdot}n_1 - k\, n_6{\cdot}n_2) \otimes m_3 + (n_6{\cdot}n_1) \otimes m_8$$

3a) $(k-2,k,k,k) \longrightarrow (k+1,k+1,k+1,k+1)$: $(n_3{\cdot}n_2{\cdot}n_1) \otimes m_1$;
3b) $(-3,-1,-1,-1) \longrightarrow (-2,0,0,0)$ (The dual cases were not calculated.)

$$(n_7{\cdot}n_1+n_8{\cdot}n_2+n_9{\cdot}n_3-n_4{\cdot}n_2{\cdot}n_1-n_5{\cdot}n_3{\cdot}n_1-n_6{\cdot}n_3{\cdot}n_2) \otimes m_1 - (n_3{\cdot}n_2{\cdot}n_1) \otimes m_2$$

4 Singular vectors for $\mathfrak{g} = \mathfrak{vle}(4|3)$

Here $\mathfrak{g} = \mathfrak{vle}(4|3)$, former ш_1. In $\mathfrak{g}_0 = \mathfrak{c}(\mathfrak{vect}(0|3))$ considered in the standard grading, we take the usual basis of Cartan subalgebra, $\xi_i \frac{\partial}{\partial_i}$, and z we identify the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ with $\Pi(\Lambda(\xi)/\mathbb{C} \cdot 1)$, by setting

$$\begin{aligned}&\partial_i = \Pi(\xi_i); \quad \partial_0 = \Pi(\xi_1\xi_2\xi_3),\\ &\delta_i = \operatorname{sign}(ijk)\Pi(\xi_j\xi_k) \text{ for } (i,j,k) \in S_3.\end{aligned}$$

We consider the following negative operators from $\mathfrak{g}_0$:

$$\begin{array}{ll} y_1 = -x_1\partial_2 + \xi_2\delta_1 & y_2 = -x_2\partial_3 + \xi_3\delta_2 \\ y_3 = -x_0\delta_3 - \xi_1\partial_2 + \xi_2\partial_1 & y_4 = x_1\partial_3 - \xi_3\delta_1 \\ y_5 = x_0\delta_2 - \xi_1\partial_3 + \xi_3\partial_1 & y_6 = x_0\delta_1 + \xi_2\partial_3 - \xi_3\partial_2 \end{array}$$

and the basis of Cartan subalgebra:

$$\begin{array}{l} h_0 = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_0\partial_0 + \xi_1\delta_1 + \xi_2\delta_2 + \xi_3\delta_3 \\ h_1 = x_2\partial_2 + x_3\partial_3 + \xi_1\delta_1 \\ h_2 = x_1\partial_1 + x_3\partial_3 + \xi_2\delta_2 \\ h_3 = x_1\partial_1 + x_2\partial_2 + \xi_3\delta_3 \end{array}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{lll} m_2 = y_1\, m_1 & m_{17} = y_5\, m_1 & m_{54} = y_2\, y_3\, y_4\, m_1 \\ m_3 = y_2\, m_1 & m_{26} = y_1\, y_5\, m_1 & m_{57} = y_3\, y_5\, m_1 \\ m_7 = y_3\, m_1 & m_{31} = y_3\, y_4\, m_1 & m_{82} = y_1\, y_3\, y_5\, m_1 \\ m_8 = y_4\, m_1 & m_{32} = (y_4)^2\, m_1 & m_{94} = y_3\, y_6\, m_1 \\ m_{12} = y_1\, y_3\, m_1 & m_{33} = y_6\, m_1 & m_{148} = y_3\, y_4\, y_5\, m_1 \\ m_{15} = y_2\, y_3\, m_1 & m_{46} = y_1\, y_2\, y_5\, m_1 & m_{150} = y_5\, y_6\, m_1 \end{array}$$

and $m_{320} = y_3\, y_5\, y_6\, m_1$. Observe that our choice of ordering obscures the fact that the vectors m_{129}, m_{148}, and m_{150} are proportional.

Theorem In $I(V)$ in degrees indicated, there are only the following singular vectors:

1a) $\lambda \longrightarrow \lambda + (-1,1,1,1)$, where $2\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$: $\quad \partial_0\, m_1$

1b) $(k,l,l,k-l+1) \longrightarrow (k-1,l+1,l+1,k-l+1)$: $\quad \partial_0\, m_7 + (k-l)\delta_3)\, m_1$

1c) $(k,k-1,1,k) \longrightarrow (k-1,k,1,k+1)$:

$$\partial_0\,(m_{15} + (k-2)m_{17}) - (k-1)\delta_2\, m_1 + \delta_3\, m_3$$

1d) $(k,l,k-l,k-l) \longrightarrow (k-1,l,k-l+1,k-l+1)$:

$$\partial_0\,(m_{26}+m_{31}+(1-k+2l)m_{33})+((k-2l)(1+l))\delta_1\, m_1-(l+1)\delta_2\, m_2+(l+1)\delta_3\, m_8$$

1e) $(1,1,0,0) \longrightarrow (0,1,1,0)$:

$$-2\partial_1\, m_2+2\partial_2\, m_1+\partial_0\,(m_{82}+2m_{94})+2\delta_1\, m_7+2\delta_2\, m_{12}+\delta_3\,(m_{26}-m_{31}+2m_{33})$$

1f) $(0,0,0,-1) \longrightarrow (-1,0,0,0)$:

$$\begin{array}{l} -\partial_1\, m_8 + \partial_2\, m_3 - \partial_3\, m_1 \\ +\partial_0\,(m_{129} + m_{148}) + \delta_1\,(2m_{15} - m_{17}) + \delta_2\,(2m_{31} - m_{33}) + \delta_3\,(m_{46} + m_{54}) \end{array}$$

2a) $\lambda \longrightarrow \lambda + (-2,2,2,2)$, where $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 1$: $\quad \partial_0^2\, m_1$

2b) $(k,k-2,k-2,3) \longrightarrow (k-2,k,k,4)$: $\quad \partial_0^2\, m_7 + 2\partial_0\, \delta_3\, m_1$

2c) $(k,k-2,2,k-1) \longrightarrow (k-2,k,3,k+1)$:

$$\partial_0^2\,(m_{15} + (k-4)m_{17}) - 2(k-3)\partial_0\, \delta_2\, m_1 + 2\partial_0\, \delta_3\, m_3$$

2d) $(k, 1, k-1, k-1) \longrightarrow (k-2, 2, k+1, k+1)$:

$$\partial_0^2\,(m_{26}+m_{31}+(3-k)m_{33})+2(k-2)\partial_0\,\delta_1\,m_1-2\partial_0\,\delta_2\,m_2+2\partial_0\,\delta_3\,m_8$$

2e) $(2, 1, 1, 1) \longrightarrow (0, 2, 2, 2)$:

$$\begin{array}{l}\partial_0^2\,m_{320}+2\partial_1\,\partial_0\,(m_{26}+m_{31})+2\partial_1\,\delta_1\,m_1-2\partial_1\,\delta_2\,m_2\\ +2\partial_1\,\delta_3\,m_8-2\partial_2\,\partial_0\,m_{15}+2\partial_2\,\delta_2\,m_1-2\partial_2\,\delta_3\,m_3\\ +2\partial_3\,\partial_0\,m_7+2\partial_3\,\delta_3\,m_1-2\partial_0\,\delta_1\,m_{57}+\partial_0\,\delta_2\,(-m_{82}+m_{94})\\ +\partial_0\,\delta_3\,(m_{148}+2m_{150})-2\delta_1\,\delta_2\,m_7-2\delta_1\,\delta_3\,m_{17}+2\delta_2\,\delta_3\,m_{33}\end{array}$$

3a) $\lambda \longrightarrow \lambda+(-3,3,3,3)$, where $\lambda_1=\lambda_2+\lambda_3+\lambda_4+2$: $\quad \partial_0^3\,m_1$

3b) $(k, k-3, k-3, 4) \longrightarrow (k-3, k, k, 6)$: $\quad \partial_0^3\,m_7+3\partial_0^2\,\delta_3\,m_1$

3c) $(k, k-3, 3, k-2) \longrightarrow (k-3, k, 5, k+1)$:

$$\partial_0^3\,(m_{15}+(k-6)m_{17})-3(k-5)\partial_0^2\,\delta_2\,m_1+3\partial_0^2\,\delta_3\,m_3$$

3d) $(k, 2, k-2, k-2) \longrightarrow (k-3, 4, k+1, k+1)$:

$$\partial_0^3\,(m_{26}+m_{31}+(5-k)m_{33})+3(k-4)\partial_0^2\,\delta_1\,m_1-3\partial_0^2\,\delta_2\,m_2+3\partial_0^2\,\delta_3\,m_8$$

3e) $(2k, k, k, k) \longrightarrow (2k-3, k+2, k+2, k+2)$:

$$\begin{array}{l}\partial_0^3\,m_{320}+(k+1)\partial_1\,\partial_0^2\,(m_{26}+m_{31})+k(k+1)\partial_1\,\partial_0\,\delta_1\,m_1\\ -k(k+1)\partial_1\,\partial_0\,\delta_2\,m_2+k(k+1)\partial_1\,\partial_0\,\delta_3\,m_8\\ -(k+1)\partial_2\,\partial_0^2\,m_{15}+k(k+1)\partial_2\,\partial_0\,\delta_2\,m_1\\ -k(k+1)\partial_2\,\partial_0\,\delta_3\,m_3+(k+1)\partial_3\,\partial_0^2\,m_7\\ +k(k+1)\partial_3\,\partial_0\,\delta_3\,m_1-(k+1)\partial_0^2\,\delta_1\,m_{57}\\ +\partial_0^2\,\delta_2\,(-m_{82}+km_{94})+\partial_0\,\delta_3\,(m_{148}+(k+1)m_{150})\\ -k(k+1)\partial_0\,\delta_1\,\delta_2\,m_7-k(k+1)\partial_0\,\delta_1\,\delta_3\,m_{17}\\ +k(k+1)\partial_0\,\delta_2\,\delta_3\,m_{33}-(k-1)k(k+1)\delta_1\,\delta_2\,\delta_3\,m_1\end{array}$$

Remarks Cases a) have an obvious generalization to any degree, cf. Ref. 23. Some expressions can be shortened by an appropriate ordering of the elements of the enveloping algebra, in other words, some vectors represent zero, e.g., m_2, m_3, m_8, m_{15} in cases 2e) and 3e). Being way behind the deadline, we did not always perform such renormalization; the cases 2e) and 3e) are left as they are to entertain the reader. The following case — $\mathfrak{g}=\mathfrak{mb}(4|5)$ — is strikingly similar.

5 Singular vectors for $\mathfrak{g} = \mathfrak{mb}(4|5)$ (after Kochetkov)

Here $\mathfrak{g} = \mathfrak{mb}(4|5)$, former $\mathfrak{m}_2$. Recall that in terms of generating functions we identify the $\mathfrak{g}_0$-module $\mathfrak{g}_{-2}$ with $\Pi(\mathbb{C} \cdot 1)$; we denote by $\mathbf{1} \in \mathfrak{m}(4)$ the image of $\Pi(1)$; so $f\mathbf{1}$ denotes $M_f M_1$. We identify $\mathfrak{g}_{-1}$ with $\Pi(\Lambda(\xi))$ by setting

$$\begin{aligned} x_0 &= \Pi(1), \quad x_i = \operatorname{sign}(ijk)\Pi(\xi_j\xi_k) \text{ for } (i,j,k) \in S_3, \\ \eta_0 &= \Pi(\xi_1\xi_2\xi_3), \quad \eta_i = \Pi(\xi_i). \end{aligned}$$

Let V be an irreducible finite dimensional $\mathfrak{g}_0$-module with highest weight Λ, and v_Λ the corresponding vector; let $f \in I(V)$ be a nonzero singular vector.

Theorem ([23]) In $I(V)$, there are only the following singular vectors:

1) $\Lambda = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 3)$, and $m = 1$ or 3

$$\eta_0^m \otimes v + \sum_{1 \le i \le 3} \eta_0^{m-1} x_i \otimes v_i + \sum_{1 \le i \le 3} \eta_0^{m-1} \eta_i \otimes w_i + \eta_0^{m-1} x_0 \otimes v_\Lambda$$

2) $\Lambda = (0,0,0;2)$ and $m = 3$

$$\eta_0 (\sum_{0 \le i \le 3} \eta_i x_i) \otimes v_\Lambda + 2\eta_0 \mathbf{1} \otimes v_\Lambda;$$

3) $\Lambda = (0,0,0;a)$: $\partial_0^2 \otimes v_\Lambda$.

4) $\Lambda = (0,0,-1;0)$ for $m = 1$ or 3

$$\partial_0^m \otimes v_\Lambda + \sum_{1 \le i \le 3} \partial_0^{m-1} \delta_i \otimes w_i + \sum_{1 \le i \le 3} \partial_0^{m-1} \partial_j \otimes v_j, \text{ where } v_3 = v_\Lambda.$$

Remark In Ref. 23 no description of the v_j and w_j is given; now we can compare the above with our latest result:

We give the weights with respect to the following basis of Cartan subalgebra:

$$\begin{aligned} h_1 &= \tau; & h_2 &= -q_0\,\xi_0 + q_1\,\xi_1, \\ h_3 &= -q_0\,\xi_0 + q_2\,\xi_2, & h_4 &= -q_0\,\xi_0 + q_3\,\xi_3. \end{aligned}$$

For the negative elements of $\mathfrak{g}_0$ we take

$$\begin{aligned} y_1 &= q_2\,\xi_1, & y_2 &= q_3\,\xi_2, & y_3 &= -q_0\,q_1 + \xi_2\,\xi_3, \\ y_4 &= -q_3\,\xi_1, & y_5 &= -q_0\,q_2 - \xi_1\,\xi_3, & y_6 &= -q_0\,q_3 + \xi_1\,\xi_2. \end{aligned}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$\begin{array}{lll} m_2 = y_1\,m_1, & m_{17} = y_5\,m_1 & m_{57} = y_3\,y_5\,m_1 \\ m_3 = y_2\,m_1, & m_{24} = y_1\,y_2\,y_3\,m_1 & m_{91} = y_2\,y_3\,y_5\,m_1 \\ m_5 = y_1\,y_2\,m_1 & m_{30} = y_2\,y_5\,m_1 & m_{94} = y_3\,y_6\,m_1 \\ m_7 = y_3\,m_1, & m_{31} = y_3\,y_4\,m_1 & m_{148} = y_3\,y_4\,y_5\,m_1 \\ m_8 = y_4\,m_1, & m_{33} = y_6\,m_1 & m_{150} = y_5\,y_6\,m_1 \\ m_{12} = y_1\,y_3\,m_1 & & \end{array}$$

and $m_{320} = y_3\, y_5\, y_6\, m_1$. Observe that our choice of ordering obscures the fact that some of the vectors either are proportional or represent zero.

Theorem In $I(V)$, there are only the following singular vectors:

1a) $\lambda \longrightarrow \lambda + (-1, 1, 1, 1)$, where $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$: $\xi_0\, m_1$.
1b) $(-k + 2l - 2, k, l, l) \longrightarrow (-k + 2l - 3, k + 1, l, l)$: $-kq_1\, m_1 + \xi_0\, m_7$
1c) $(k, k, k + 1, k + 1) \longrightarrow (k - 1, k, k + 2, k + 1)$:

$$-kq_1\, m_2 - kq_2\, m_1 + \xi_0\, m_{12}$$

1d) $(2k + 1, k, 0, k + 1) \longrightarrow (2k, k, 1, k + 1)$

$$q_1\, m_2 - kq_2\, m_1 + \xi_0\, (m_{12} - (k + 1)m_{17})$$

1e) $(k + 3, k, k, k - 1) \longrightarrow (k + 2, k, k, k)$

$$-(k-3)q_1\, (m_5+m_8)+2(k-3)q_2\, m_3-2(k-3)q_3\, m_1+\xi_0\, (m_{24}-3m_{30}+m_{31}+5m_{33})$$

1f) $(2k + 1, k, k, 1) \longrightarrow (2k, k, k, 2)$

$$q_1\, (m_5+(k-1)m_8)-kq_2\, m_3+k(k-1)q_3\, m_1+\xi_0\, (m_{24}-(k+1)m_{30}+(k-1)m_{31}+(k^2+1)m_{33})$$

1g) $(3, 1, 1, 1) \longrightarrow (2, 0, 0, 0)$

$$\begin{aligned}&q_0\, m_1 - q_1\, (m_{132} + m_{148}) + q_2\, m_{91} - q_3\, m_{57} - \xi_0\, m_{320}\\ &+\xi_1\, m_7 + \xi_2\, m_{12} - \xi_3\, (-m_{30} + m_{31})\end{aligned}$$

2a) $\lambda \longrightarrow \lambda + (-2, 2, 2, 2)$, where $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 2$: $\xi_0^2\, m_1$.
2b) $(2k, -2, k, k) \longrightarrow (2k - 2, 0, k + 1, k + 1)$: $\xi_0^2\, m_7 + 2q_1\, \xi_0\, m_1$.
2c) $(2k, k - 1, -1, k) \longrightarrow (2k - 2, k, 1, k + 1)$

$$\xi_0^2\, (m_{12} - (k + 1)m_{17}) + 2q_1\, \xi_0\, m_2 - 2kq_2\, \xi_0\, m_1$$

2d) $(2k + 2, k, k, 0) \longrightarrow (2k, k + 1, k + 1, 2)$

$$\begin{aligned}&\xi_0^2\, (-m_{30} + m_{31} + (k + 1)m_{33}) + q_1\, \xi_0\, (m_5 + m_8)\\ &-2q_2\, \xi_0\, m_3 + 2kq_3\, \xi_0\, m_1\end{aligned}$$

2e) $(2k, 0, 0, 0) \longrightarrow (2k - 2, 0, 0, 0)$

$$((3 - k) + q_0\, \xi_0 + q_1\, \xi_1) + q_2\, \xi_2 + q_3\, \xi_3)\, m_1$$

3a) $\lambda \longrightarrow \lambda + (-3, 3, 3, 3)$, where $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4 + 4$: $\xi_0\, m_1$.
3b) $(2k + 1, -3, k, k) \longrightarrow (2k - 2, 0, k + 2, k + 2)$: $\xi_0^3\, m_7 + 3q_1\, \xi_0^2\, m_1$.
3c) $(2k + 1, k + 1, -2, k) \longrightarrow (2k - 2, k + 4, 1, k + 2)$

$$\xi_0^3\, (m_{12} - (k + 2)m_{17}) + 3q_1\, \xi_0^2\, m_2 - 3(k + 1)q_2\, \xi_0^2\, m_1$$

3d) $(2k+3,k,k,-1) \longrightarrow (2k,k+2,k+2,2)$

$$\begin{aligned}&\xi_0^3\,(m_{24}-2m_{30}+(3+k)m_{33})+q_1\,\xi_0^2\,((2-k)m_5+(1+k)m_8)\\&-3q_2\,\xi_0^2\,m_3+3(1+k)q_3\,\xi_0^2\,m_1\end{aligned}$$

3e) $(2,0,0,0) \longrightarrow (-1,1,1,1)$

$$(q_0\,\xi_0^2+3\xi_0^3+q_1\,\xi_0\,\xi_1+q_2\,\xi_0\,\xi_2+q_3\,\xi_0\,\xi_3)\,m_1$$

3f) $(k+2,k,k,k) \longrightarrow (k-1,k+1,k+1,k+1)$

$$\begin{aligned}&\xi_0^3\,m_{320}+(-2+k)q_0\,\xi_0^2\,m_1+q_1\,\xi_0^2\,(m_{132}+m_{148}+(1-k)m_{150})\\&+q_2\,\xi_0^2\,(-m_{91}+(-1+k)m_{94})+(2-k)q_3\,\xi_0^2\,m_{57}\\&-(-3+k)(-2+k)\xi_0\,m_1+(-2+k)\xi_0^2\,\xi_1\,m_7)+(-2+k)\xi_0^2\,\xi_2\,m_{12}\\&+\xi_0^2\,\xi_3\,((-2+k)m_{30}+(2-k)m_{31})-(-2+k)(-1+k)kq_1\,q_2\,q_3\,m_1\\&+q_1\,q_2\,\xi_0\,(m_{24}-km_{30}+(-2+k)m_{31}+(2-2k+k^2)m_{33})\\&+q_1\,q_3\,\xi_0\,((-2+k)m_{12}-(-2+k)km_{17})-(-2+k)(-1+k)q_1\,\xi_0\,\xi_1\,m_1\\&-(-2+k)kq_1\,\xi_0\,\xi_2\,m_2+q_1\,\xi_0\,\xi_3\,((2-k)m_5+(-2+k)km_8)\\&+(-2+k)(-1+k)(q_2\,q_3\,\xi_0\,m_7-(-2+k)(-1+k)q_2\,\xi_0\,\xi_2\,m_1\\&-(-2+k)(-1+k)q_2\,\xi_0\,\xi_3\,m_3-(-2+k)(-1+k)q_3\,\xi_0\,\xi_3\,m_1\end{aligned}$$

6 Singular vectors for $\mathfrak{g}=\mathfrak{mb}(3|8)$

We give the weights with respect to the following basis of Cartan subalgebra:

$$\begin{aligned}&H_1=\tfrac{1}{2}\tau+\tfrac{3}{2}q_1\xi_1-\tfrac{1}{2}q_2\xi_2-\tfrac{1}{2}q_3\xi_3-\tfrac{1}{2}q_4\xi_4;\\&H_2=-q_1\xi_1+q_2\xi_2,\quad H_3=-q_1\xi_1+q_3\xi_3,\quad H_3=-q_1\xi_1+q_4\xi_4.\end{aligned}$$

The basis elements of $\mathfrak{g}_-$ are denoted by

$$\begin{array}{cccc}q_0&q_1&q_2&q_3,\\I&\xi_1&\xi_2&\xi_3,\end{array}$$

and

$$A=-q_0q_1+\xi_2\xi_3,\quad B=-q_0q_2-\xi_1\xi_3,\quad C=-q_0q_3+\xi_1\xi_2.$$

Theorem In $I(V)$, there are only the following singular vectors calculated up to dualization (though some dual vectors are also given):
1a) $(k,0,l,l) \longrightarrow (k+1,0,l-1,l-1)$: $A\otimes m_1$;
1b) $(k,l,-k,l+1) \longrightarrow (k-1,l,-k+1,l+1)$, where $k\neq 0,-l$

$$k\,q_1\otimes(q_2\xi_1\cdot m_1)-(k+l)\,B\otimes(\xi_0\cdot m_1)-k\,(k+l)\,q_2\otimes m_1+A\otimes(\xi_0\cdot q_2\xi_1\cdot m_1)\,.$$

1c) $(k,l,l,2) \longrightarrow (k+1,l-1,l-1,2)$, where $l \neq 2$

$$A\otimes(q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)-B\otimes(q_3\xi_2{\cdot}m_1)-(2-l)\,C\otimes m_1.$$

1d) $(k,l,l,1-k) \longrightarrow (k-1,l,l,2-k)$, where $k+l\neq 1$

$$\begin{aligned}&-k\,q_1\otimes(q_3\xi_1{\cdot}m_1)-B\otimes(\xi_0{\cdot}q_3\xi_2{\cdot}m_1)-k\ (1-k-l)\ q_3\otimes m_1\\&-(1-k-l)\,C\otimes(\xi_0{\cdot}m_1)-k\,q_2\otimes(q_3\xi_2{\cdot}m_1)-A\otimes(\xi_0{\cdot}q_3\xi_1{\cdot}m_1)\,.\end{aligned}$$

1e) $(k,l,1,l+1) \longrightarrow (k+1,l-1,1,l-1)$, where $l\neq 1$

$$(1-l)\,B\otimes m_1+A\otimes(q_2\xi_1{\cdot}m_1)\,.$$

1f) $(k,-k-1,l,l) \longrightarrow (k-1,-k+1,l,l)$, where $k\neq 0$

$$k\,q_1\otimes m_1+A\otimes(\xi_0{\cdot}m_1)\,.$$

2a) $(0,-1,l,l) \longrightarrow (0,0,l-1,l-1)$: $(q_1{\cdot}A)\otimes m_1$;
2b) $(0,-1,1,1) \longrightarrow (0,-1,1,0)$

$$(q_2{\cdot}A+q_1{\cdot}B)\otimes m_1+(q_1{\cdot}A)\otimes(q_2\xi_1{\cdot}m_1)\,.$$

2c) $(2,0,0,-1) \longrightarrow (2,0,-1,-1)$

$$\begin{aligned}&2\,\xi_2\otimes m_1+2\,\xi_3\otimes(q_3\xi_2{\cdot}m_1)-2\ (q_1{\cdot}A)\otimes(q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)+2\ (q_2{\cdot}A)\otimes(q_3\xi_2{\cdot}m_1)\\&-2\ (q_3{\cdot}A)\otimes m_1+(B{\cdot}A)\otimes(\xi_0{\cdot}q_3\xi_2{\cdot}m_1)-(C{\cdot}A)\otimes(\xi_0{\cdot}m_1)\end{aligned}$$

2d) $(-4,3,3,2) \longrightarrow (-4,3,2,2)$

$$\begin{aligned}&-2\ (q_1{\cdot}A)\otimes(q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)-\ \xi_2\otimes m_1-\ \xi_3\otimes(q_3\xi_2{\cdot}m_1)-\ (C{\cdot}A)\otimes(\xi_0{\cdot}m_1)\\&-3\,(\,q_1{\cdot}C)\otimes m_1+\ (q_3{\cdot}A)\otimes m_1+3\ (q_1{\cdot}B)\otimes(q_3\xi_2{\cdot}m_1)-(q_2{\cdot}A)\otimes(q_3\xi_2{\cdot}m_1)\\&+(B{\cdot}A)\otimes(\xi_0{\cdot}q_3\xi_2{\cdot}m_1)\,.\end{aligned}$$

2e) $(0,k,0,k+1) \longrightarrow (0,k-1,1,k)$

$$\begin{aligned}&(q_1{\cdot}A)\otimes\Big((q_2\xi_1)^2{\cdot}m_1\Big)+(1-k)\ (q_1{\cdot}B)\otimes(q_2\xi_1{\cdot}m_1)\\&+(1-k)\ (q_2{\cdot}A)\otimes(q_2\xi_1{\cdot}m_1)+(-1+k)\ k\ (q_2{\cdot}B)\otimes m_1.\end{aligned}$$

2f) $(0,1,0,2) \longrightarrow (0,0,0,2)$

$$\begin{aligned}&(q_1{\cdot}A)\otimes\Big((q_2\xi_1)^2{\cdot}q_3\xi_2{\cdot}m_1\Big)-(q_1{\cdot}B)\otimes(q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)\\&-2\ (q_1{\cdot}C)\otimes(q_2\xi_1{\cdot}m_1)-(q_2{\cdot}A)\otimes(q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)\\&+2\ (q_2{\cdot}B)\otimes(q_3\xi_2{\cdot}m_1)+2\ (q_2{\cdot}C)\otimes m_1-2\ (q_3{\cdot}A)\otimes(q_2\xi_1{\cdot}m_1)+2\ (q_3{\cdot}B)\otimes m_1.\end{aligned}$$

3a) $(-3,2,2,2) \longrightarrow (-2,1,1,1)$

$$\begin{aligned}&-q_0\otimes m_1-(\xi_1{\cdot}A)\otimes m_1-(\xi_2{\cdot}B)\otimes m_1-(\xi_3{\cdot}C)\otimes m_1-(q_1{\cdot}C{\cdot}B)\otimes m_1\\&+(q_2{\cdot}C{\cdot}A)\otimes m_1-(q_3{\cdot}B{\cdot}A)\otimes m_1+(C{\cdot}B{\cdot}A)\otimes(\xi_0{\cdot}m_1)\,.\end{aligned}$$

3a*) $(-2,2,2,2) \longrightarrow (-3,2,2,2)$

$$\begin{aligned}
&-4\, I \otimes m_1 - 2\, q_0 \otimes (\xi_0{\cdot}m_1) + 2\ (\xi_1{\cdot}q_1) \otimes m_1 - 2\ (\xi_1{\cdot}A) \otimes (\xi_0{\cdot}m_1)\\
&+2\ (\xi_2{\cdot}q_2) \otimes m_1 - 2\ (\xi_2{\cdot}B) \otimes (\xi_0{\cdot}m_1) + 2\ (\xi_3{\cdot}q_3) \otimes m_1 - 2\ (\xi_3{\cdot}C) \otimes (\xi_0{\cdot}m_1)\\
&-2\ (q_1{\cdot}C{\cdot}B) \otimes (\xi_0{\cdot}m_1) + 2\ (q_2{\cdot}q_1{\cdot}C) \otimes m_1 + 2\ (q_2{\cdot}C{\cdot}A) \otimes (\xi_0{\cdot}m_1)\\
&-2\ (q_3{\cdot}q_1{\cdot}B) \otimes m_1 + 2\ (q_3{\cdot}q_2{\cdot}A) \otimes m_1 - 2\ (q_3{\cdot}B{\cdot}A) \otimes (\xi_0{\cdot}m_1) + C{\cdot}B{\cdot}A \otimes \left(\xi_0^2{\cdot}m_1\right)
\end{aligned}$$

3b) $(1-k,k,k,k) \longrightarrow (-2-k,k+1,k+1,k+1)$, where $k \neq -1,\ 0,\ 1$

$$\begin{aligned}
&-2\,k\ (1+k)\, I \otimes (\xi_0{\cdot}m_1) - (1+k)\, q_0 \otimes \left({\xi_0}^2{\cdot}m_1\right) + k\ (1+k)\, (\xi_1{\cdot}q_1) \otimes (\xi_0{\cdot}m_1)\\
&-(1+k)\, (\xi_1{\cdot}A) \otimes \left({\xi_0}^2{\cdot}m_1\right) + k\ (1+k)\, (\xi_2{\cdot}q_2) \otimes (\xi_0{\cdot}m_1)\\
&-(1+k)\, (\xi_2{\cdot}B) \otimes \left({\xi_0}^2{\cdot}m_1\right) + k\ (1+k)\, (\xi_3{\cdot}q_3) \otimes (\xi_0{\cdot}m_1)\\
&-(1+k)\, (\xi_3{\cdot}C) \otimes \left({\xi_0}^2{\cdot}m_1\right) - (1+k)\, (q_1{\cdot}C{\cdot}B) \otimes \left({\xi_0}^2{\cdot}m_1\right)\\
&+k\ (1+k)\, (q_2{\cdot}q_1{\cdot}C) \otimes (\xi_0{\cdot}m_1) + (1+k)\, (q_2{\cdot}C{\cdot}A) \otimes \left({\xi_0}^2{\cdot}m_1\right)\\
&-k\ (1+k)\, (q_3{\cdot}q_1{\cdot}B) \otimes (\xi_0{\cdot}m_1)\\
&-(-1+k)\ k\ (1+k)\, (q_3{\cdot}q_2{\cdot}q_1) \otimes m_1\\
&+k\ (1+k)\, (q_3{\cdot}q_2{\cdot}A) \otimes (\xi_0{\cdot}m_1) - (1+k)\, (q_3{\cdot}B{\cdot}A) \otimes \left({\xi_0}^2{\cdot}m_1\right)\\
&+(C{\cdot}B{\cdot}A) \otimes \left({\xi_0}^3{\cdot}m_1\right)
\end{aligned}$$

3b*) $(k,2,2,2) \longrightarrow (k+3,0,0,0)$: $C{\cdot}B{\cdot}A \otimes m_1$.

4a) $(0,2,2,1) \longrightarrow (2,0,0,0)$

$$\begin{aligned}
&(q_0{\cdot}A) \otimes (q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1) - (q_0{\cdot}B) \otimes (q_3\xi_2{\cdot}m_1) + (q_0{\cdot}C) \otimes m_1\\
&+(\xi_1{\cdot}B{\cdot}A) \otimes (q_3\xi_2{\cdot}m_1) - (\xi_1{\cdot}C{\cdot}A) \otimes m_1 + (\xi_2{\cdot}B{\cdot}A) \otimes (q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)\\
&+(\xi_2{\cdot}C{\cdot}A) \otimes (q_2\xi_1{\cdot}m_1) - (\xi_2{\cdot}C{\cdot}B) \otimes m_1 + (\xi_3{\cdot}C{\cdot}A) \otimes (q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)\\
&-(\xi_3{\cdot}C{\cdot}B) \otimes (q_3\xi_2{\cdot}m_1) + (q_1{\cdot}C{\cdot}B{\cdot}A) \otimes (q_2\xi_1{\cdot}q_3\xi_2{\cdot}m_1)\\
&-(q_2{\cdot}C{\cdot}B{\cdot}A) \otimes (q_3\xi_2{\cdot}m_1) + (q_3{\cdot}C{\cdot}B{\cdot}A) \otimes m_1
\end{aligned}$$

7 Singular vectors for $\mathfrak{g} = \mathfrak{ksle}(5|10)$

We set: $\delta_{ij} = \frac{\partial}{\partial\theta_{ij}} + \sum\limits_{\text{even permutations } (ijklm)} \theta_{kl}\partial_m$; e.g.,

$$\begin{aligned}
\delta_{12} &= \tfrac{\partial}{\partial\theta_{12}} + \theta_{34}\partial_5 + \theta_{45}\partial_3 - \theta_{35}\partial_4,\\
\delta_{13} &= \tfrac{\partial}{\partial\theta_{13}} + \theta_{25}\partial_4 - \theta_{24}\partial_5 - \theta_{45}\partial_2,\\
\delta_{14} &= \tfrac{\partial}{\partial\theta_{14}} + \theta_{23}\partial_5 + \theta_{35}\partial_2 - \theta_{25}\partial_3, \text{ etc.}
\end{aligned}$$

The x-part of the elements of $\mathfrak{g}_0 = \mathfrak{sl}(5)$ is obvious. The negative elements are:

$$y_{ij} = x_i\partial_j + \sum_k \theta_{jk}\delta_{ki} \quad \text{for } i<j$$

and the basis of Cartan subalgebra is $h_i = y_{ii} - y_{i+1,i+1}$.

Let us estimate the possible degree of invariant operators. Since $\mathfrak{g}_{\bar 0} \simeq \mathfrak{svect}(5|0)$, and the grading is consistent, wee see that the degree of the singular vector can not exceed $2 \times 2 + 10 = 14$: each element from $\mathfrak{g}_{-1}$ can only contribute once and the degree of singular vector of the $\mathfrak{svect}(5|0)$ modules can not exceed 2; each counted with weight 2. In reality, the degree of singular vectors is much lower, even with infinite dimensional fibers. To compute the singular vectors directly is possible on modern computers, but hardly on a workstation; the inbuilt *Mathematica*'s restrictions aggravate the problem.

Still, even simple-minded direct calculations provide us with several first and second order operators. The only "known" operator, the exterior differential, is inhomogeneous in the consistent grading and consists of parts of degree 1 and parts of degree 2. To match these parts with our operators is a problem.

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

m_1 is the highest weight vector

$$
\begin{aligned}
&m_2 = y_{21} \cdot m_1 && m_{18} = -y_{53} \cdot m_1\\
&m_3 = y_{32} \cdot m_1 && m_{24} = y_{21} \cdot y_{32} \cdot y_{43} \cdot m_1\\
&m_4 = y_{43} \cdot m_1 && m_{25} = y_{21} \cdot y_{32} \cdot y_{54} \cdot m_1\\
&m_5 = y_{54} \cdot m_1 && m_{30} = -y_{21} \cdot y_{42} \cdot m_1\\
&m_7 = y_{21} \cdot y_{32} \cdot m_1 && m_{31} = -y_{21} \cdot y_{53} \cdot m_1\\
&m_8 = y_{21} \cdot y_{43} \cdot m_1 && m_{36} = y_{32} \cdot y_{43} \cdot y_{54} \cdot m_1\\
&m_9 = y_{21} \cdot y_{54} \cdot m_1 && m_{40} = -y_{32} \cdot y_{53} \cdot m_1\\
&m_{11} = y_{32} \cdot y_{43} \cdot m_1 && m_{44} = -y_{43} \cdot y_{31} \cdot m_1\\
&m_{12} = y_{32} \cdot y_{54} \cdot m_1 && m_{48} = -y_{54} \cdot y_{31} \cdot m_1\\
&m_{14} = y_{43} \cdot y_{54} \cdot m_1 && m_{49} = -y_{54} \cdot y_{42} \cdot m_1\\
&m_{16} = -y_{31} \cdot m_1 && m_{51} = -y_{41} \cdot m_1\\
&m_{17} = -y_{42} \cdot m_1 && m_{52} = -y_{52} \cdot m_1
\end{aligned}
$$

$$
\begin{aligned}
&m_{70} = y_{21} \cdot y_{32} \cdot y_{43} \cdot y_{54} \cdot m_1 && m_{175} = -y_{21} \cdot y_{32} \cdot y_{32} \cdot y_{53} \cdot m_1\\
&m_{73} = -y_{21} \cdot y_{32} \cdot y_{42} \cdot m_1 && m_{181} = -y_{21} \cdot y_{32} \cdot y_{43} \cdot y_{53} \cdot m_1\\
&m_{74} = -y_{21} \cdot y_{32} \cdot y_{53} \cdot m_1 && m_{184} = -y_{21} \cdot y_{32} \cdot y_{54} \cdot y_{42} \cdot m_1\\
&m_{83} = -y_{21} \cdot y_{54} \cdot y_{42} \cdot m_1 && m_{187} = -y_{21} \cdot y_{32} \cdot y_{52} \cdot m_1\\
&m_{86} = -y_{21} \cdot y_{52} \cdot m_1 && m_{241} = -y_{32} \cdot y_{43} \cdot y_{54} \cdot y_{31} \cdot m_1\\
&m_{101} = -y_{32} \cdot y_{43} \cdot y_{53} \cdot m_1 && m_{250} = -y_{32} \cdot y_{54} \cdot y_{41} \cdot m_1\\
&m_{115} = -y_{43} \cdot y_{54} \cdot y_{31} \cdot m_1 && m_{254} = y_{32} \cdot y_{31} \cdot y_{53} \cdot m_1\\
&m_{124} = -y_{54} \cdot y_{41} \cdot m_1 && m_{258} = -y_{32} \cdot y_{51} \cdot m_1\\
&m_{127} = y_{31} \cdot y_{42} \cdot m_1 && m_{279} = y_{43} \cdot y_{31} \cdot y_{53} \cdot m_1\\
&m_{128} = y_{31} \cdot y_{53} \cdot m_1 && m_{291} = y_{54} \cdot y_{31} \cdot y_{42} \cdot m_1\\
&m_{130} = y_{42} \cdot y_{53} \cdot m_1 && m_{298} = y_{31} \cdot y_{52} \cdot m_1\\
&m_{132} = -y_{51} \cdot m_1 && m_{301} = y_{53} \cdot y_{41} \cdot m_1\\
&m_{171} = y_{21} \cdot y_{32} \cdot y_{32} \cdot y_{43} \cdot y_{54} \cdot m_1 && m_{397} = -y_{21} \cdot y_{32} \cdot y_{32} \cdot y_{43} \cdot y_{53} \cdot m_1\\
& && m_{539} = y_{32} \cdot y_{43} \cdot y_{31} \cdot y_{53} \cdot m_1
\end{aligned}
$$

The Cartan subalgebra is spanned by

$$
\begin{aligned}
h_1 &= x_1\partial_1 - \theta_{12}\delta_{12} - \theta_{13}\delta_{13} - \theta_{14}\delta_{14} - \theta_{15}\delta_{15} - x_2\partial_2 - \theta_{12}\delta_{12} - \theta_{23}\delta_{23} - \theta_{24}\delta_{24} - \theta_{25}\delta_{25}\\
h_2 &= x_2\partial_2 - \theta_{12}\delta_{12} - \theta_{23}\delta_{23} - \theta_{24}\delta_{24} - \theta_{25}\delta_{25} - x_3\partial_3 - \theta_{13}\delta_{13} - \theta_{23}\delta_{23} - \theta_{34}\delta_{34} - \theta_{35}\delta_{35}\\
h_3 &= x_3\partial_3 - \theta_{13}\delta_{13} - \theta_{23}\delta_{23} - \theta_{34}\delta_{34} - \theta_{35}\delta_{35} - x_4\partial_4 - \theta_{14}\delta_{14} - \theta_{24}\delta_{24} - \theta_{34}\delta_{34} - \theta_{45}\delta_{45}\\
h_4 &= x_4\partial_4 - \theta_{14}\delta_{14} - \theta_{24}\delta_{24} - \theta_{34}\delta_{34} - \theta_{45}\delta_{45} - \xi_1\partial_5 - \theta_{15}\delta_{15} - \theta_{25}\delta_{25} - \theta_{35}\delta_{35} - \theta_{45}\delta_{45}
\end{aligned}
$$

Theorem In $I(V)$ in degree d), there are only the following singular vectors (computed for degree 2 up to dualization):

1a) $(k,l,0,0) \longrightarrow (k,l+1,0,0)$: $\delta_{12}\otimes m_1$;

1a*) $(0,0,k,l) \longrightarrow (0,0,k-1,l)$, where $k\neq 0$ and $k+l+1\neq 0$ [b]

$$
\begin{aligned}
&2k\,(1+k+l)\,\delta_{45}\otimes\, m_1 - 2\,(1+k+l)\,\delta_{35}\otimes\, m_4\\
&+2\delta_{25}\otimes((1+k-l)\ m_{11}+2\,l\,m_{17})\\
&+2\delta_{15}\otimes(3\ (-1+k-l)\ m_{24}-2\ (-1+2\,k-2\,l)\ m_{44}+2\ (-1+2\,k-l)\ m_{51})\\
&+2\delta_{34}\otimes(m_{14}-(1+k)\ m_{18})+2\delta_{24}\otimes(m_{36}+(1+k)\ m_{40}-2\,m_{49}+2\,m_{52})\\
&-2\delta_{23}\otimes(m_{101}+2\,m_{130})\\
&+2\delta_{14}\otimes(3\,m_{70}+3\,(-1+k)\,m_{74}-4\,m_{115}+2\,m_{124}-2\,(-1+2\,k)\,m_{128}-2\,m_{132})\\
&-2\delta_{13}\otimes(3\,m_{181}-4\,m_{279}+2\,m_{301})+\delta_{12}\otimes(m_{397}-4\,m_{539})
\end{aligned}
$$

1b) $(k,l,-k-1,0) \longrightarrow (k+1,l-1,-k,0)$, where $l\neq 0$

$$-l\,\delta_{13}\otimes m_1+\delta_{12}\otimes m_3$$

1b*) $(0,k,l,-k-1) \longrightarrow (0,k-1,l+1,-k)$, where $k\neq 0,-1$ and $l\neq -1$

$$
\begin{aligned}
&k(1+k)(1+l)\delta_{35}\otimes\, m_1-(1+k)(1+l)\delta_{25}\otimes\, m_3\\
&+\delta_{15}\otimes((1+k)\ (1+k-l)\ m_7-(1+k)\ (k-2\,l)\ m_{16})\\
&+k(1+l)\delta_{34}\otimes m_5-(1+l)\delta_{24}\otimes\, m_{12}\\
&+\delta_{23}\otimes(m_{36}-(1+k)\ m_{40}-(1+k)\ m_{49}-k\ (1+k)\ m_{52})\\
&+\delta_{14}\otimes((1+k-l)\ m_{25}-(k-2\,l)\ m_{48})\\
&+\delta_{13}\otimes(m_{70}-m_{74}+(1+k)\ m_{83}+k\,m_{86}-2\,m_{115}+2\,m_{124}+(2+k)\ m_{128}\\
&-\left(2+k^2\right)\ m_{132})-\delta_{12}\otimes(m_{184}-k\,m_{258}-2\,m_{291}-k\,m_{298})
\end{aligned}
$$

1c) $(k,0,l,-l-1) \longrightarrow (k+1,0,l-1,-l)$, where $l\neq 0$

$$l\,\delta_{14}\otimes m_1-\delta_{13}\otimes m_4+\delta_{12}\otimes m_{17}$$

1c*) $(k,-k-1,0,l) \longrightarrow (k-1,-k,0,l-1)$, where $k\neq 0$ and $l\neq 0$

$$
\begin{aligned}
&-k\,l\delta_{25}\otimes m_1+l\,\delta_{15}\otimes m_2+k\,\delta_{24}\otimes m_5-k\,\delta_{23}\otimes m_{18}-\delta_{14}\otimes m_9\\
&+\delta_{13}\otimes m_{31}+\delta_{12}\otimes(m_{86}+(1+k)\ m_{132})
\end{aligned}
$$

1d) $(k,l,-k-l-2,0) \longrightarrow (k-1,l,-k-l-1,0)$, where $k\neq 0$ and $k+l+1\neq 0$

$$
\begin{aligned}
&k(1+k+l)\delta_{23}\otimes\, m_1-(1+k+l)\delta_{13}\otimes\, m_2\\
&+\delta_{12}\otimes(m_7+(-1-k)\ m_{16})
\end{aligned}
$$

[b] Hereafter in similar statements the reader can check our restrictions: the coefficient of $\otimes\, m_1$ must not vanish.

1d*) $(0,k,l,-k-l-2) \longrightarrow (0,k-1,l,-k-l-1)$, where $k\neq 0$ and $k+l+1\neq 0$

$$k(1+k+l)\delta_{34}\otimes m_1 - (1+k+l)\delta_{24}\otimes m_3 + \delta_{23}\otimes(m_{11}-(1+k)\,m_{17})$$
$$+\delta_{14}\otimes((1+k-l)\,m_7 + 2\,l\,m_{16}) + \delta_{13}\otimes(m_{24}+(1+k)\,m_{30} - 2\,m_{44} + 2\,m_{51})$$
$$-\delta_{12}\otimes(m_{73} - 2\,m_{127})$$

1e) $(k,0,0,l) \longrightarrow (k+1,0,0,l-1)$, where $l\neq 0$

$$l\,\delta_{15}\otimes m_1 - \delta_{14}\otimes m_5 + \delta_{13}\otimes m_{18} + \delta_{12}\otimes m_{52}$$

1f) $(k,-k-1,l,-l-1) \longrightarrow (k-1,-k,l-1,-l)$, where $k\neq 0$ and $l\neq 0,-1$

$$(k\,l-1-l)\delta_{24}\otimes m_1 + k(1+l)\delta_{23}\otimes m_4 + l(1+l)\delta_{14}\otimes m_2$$
$$-(1+l)\,\delta_{13}\otimes m_8 + \delta_{12}\otimes(m_{24}+l\,m_{30} - (1+k)\,m_{44} + (1+k)\,(1+l)\,m_{51})$$

2a) $(k,0,0,1) \longrightarrow (k+1,1,0,0)$

$$\delta_{15}\delta_{12}m_1 - \delta_{14}\delta_{12}m_5 + \delta_{13}\delta_{12}m_{18}$$

2b) $(k,-k-1,0,1) \longrightarrow (k-1,-k+1,0,0)$, where $k\neq 0$

$$-k\,\delta_{25}\delta_{12}m_1 + \delta_{15}\delta_{12}m_2 + k\,\delta_{24}\delta_{12}m_5 - \delta_{14}\delta_{12}m_9 - k\,\delta_{23}\delta_{12}m_{18} + \delta_{13}\delta_{12}m_{31}$$

8 Singular vectors for $\mathfrak{g}=\mathfrak{ksle}(9|11)$

Consider the following negative operators from $\mathfrak{g}_0$:

$$y_1 = x_2\partial_1 - \theta_{13}\delta_{23} - \theta_{14}\delta_{24} - \theta_{15}\delta_{25}$$
$$y_2 = x_3\partial_2 - \theta_{12}\delta_{13} - \theta_{24}\delta_{34} - \theta_{25}\delta_{35}$$
$$y_3 = \delta_{12} + \theta_{34}\partial_5 + \theta_{45}\partial_3 - \theta_{35}\partial_4$$
$$y_4 = x_5\partial_4 - \theta_{14}\delta_{15} - \theta_{24}\delta_{25} - \theta_{34}\delta_{35}$$
$$y_5 = -x_3\partial_1 - \theta_{12}\delta_{23} + \theta_{14}\delta_{34} + \theta_{15}\delta_{35}$$
$$y_6 = -\delta_{13} - \theta_{25}\partial_4 + \theta_{24}\partial_5 + \theta_{45}\partial_2$$
$$y_7 = -\delta_{23} - \theta_{14}\partial_5 - \theta_{45}\partial_1 + \theta_{15}\partial_4$$

and the operators from $\mathfrak{g}_-$:

$$n_1 = \partial_4$$
$$n_2 = \partial_5$$
$$n_3 = \delta_{14} + \theta_{23}\partial_5 + \theta_{35}\partial_2 - \theta_{25}\partial_3$$
$$n_4 = \delta_{15} + \theta_{24}\partial_3 - \theta_{23}\partial_4 - \theta_{34}\partial_2$$
$$n_5 = \delta_{24} + \theta_{15}\partial_3 - \theta_{13}\partial_5 - \theta_{35}\partial_1$$
$$n_6 = \delta_{25} + \theta_{13}\partial_4 + \theta_{34}\partial_1 - \theta_{14}\partial_3$$
$$n_7 = \delta_{34} + \theta_{12}\partial_5 + \theta_{25}\partial_1 - \theta_{15}\partial_2$$
$$n_8 = \delta_{35} + \theta_{14}\partial_2 - \theta_{12}\partial_4 - \theta_{24}\partial_1$$
$$n_9 = x_4\partial_1 + \theta_{12}\delta_{24} + \theta_{13}\delta_{34} - \theta_{15}\delta_{45}$$
$$n_{10} = \xi_1\partial_1 + \theta_{12}\delta_{25} + \theta_{13}\delta_{35} + \theta_{14}\delta_{45}$$
$$n_{11} = x_4\partial_2 - \theta_{12}\delta_{14} + \theta_{23}\delta_{34} - \theta_{25}\delta_{45}$$
$$n_{12} = \xi_1\partial_2 - \theta_{12}\delta_{15} + \theta_{23}\delta_{35} + \theta_{24}\delta_{45}$$
$$n_{13} = x_4\partial_3 - \theta_{13}\delta_{14} - \theta_{23}\delta_{24} - \theta_{35}\delta_{45}$$
$$n_{14} = \xi_1\partial_3 - \theta_{13}\delta_{15} - \theta_{23}\delta_{25} + \theta_{34}\delta_{45}$$
$$n_{17} = \partial_1$$
$$n_{18} = \partial_2$$
$$n_{19} = \partial_3$$
$$n_{20} = \delta_{45} + \theta_{12}\partial_3 + \theta_{23}\partial_1 - \theta_{13}\partial_2$$

$$n_{15} = -\xi_1\theta_{12}\partial_3 + \xi_1\theta_{13}\partial_2 - \xi_1\theta_{23}\partial_1 + 2\,\theta_{12}\theta_{13}\theta_{23}\partial_4 - \theta_{12}\theta_{13}\theta_{24}\partial_3 +$$
$$\theta_{12}\theta_{13}\theta_{34}\partial_2 - \theta_{12}\theta_{14}\theta_{23}\partial_3 - \theta_{12}\theta_{23}\theta_{34}\partial_1 + \theta_{13}\theta_{14}\theta_{23}\partial_2 + \theta_{13}\theta_{23}\theta_{24}\partial_1 - \xi_1\delta_{45} +$$
$$2\,\theta_{12}\theta_{13}\delta_{15} + 2\,\theta_{12}\theta_{23}\delta_{25} - \theta_{12}\theta_{34}\delta_{45} + 2\,\theta_{13}\theta_{23}\delta_{35} + \theta_{13}\theta_{24}\delta_{45} + \theta_{14}\theta_{23}\delta_{45}$$

$$n_{16} = -x_4\theta_{12}\partial_3 + x_4\theta_{13}\partial_2 - x_4\theta_{23}\partial_1 + 2\,\theta_{12}\theta_{13}\delta_{14} + 2\,\theta_{12}\theta_{23}\delta_{24} + \theta_{12}\theta_{35}\delta_{45} +$$
$$2\,\theta_{13}\theta_{23}\delta_{34} - \theta_{13}\theta_{25}\delta_{45} - \theta_{15}\theta_{23}\delta_{45} - 2\,\theta_{12}\theta_{13}\theta_{23}\partial_5 + \theta_{12}\theta_{13}\theta_{25}\partial_3 - \theta_{12}\theta_{13}\theta_{35}\partial_2 + \theta_{12}\theta_{15}\theta_{23}\partial_3 +$$
$$\theta_{12}\theta_{23}\theta_{35}\partial_1 - \theta_{13}\theta_{15}\theta_{23}\partial_2 - \theta_{13}\theta_{23}\theta_{25}\partial_1 - x_4\delta_{45}$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

m_1 is the highest weight vector	$m_{15} = y_5 \cdot m_1$	$m_{88} = y_4 \cdot y_7 \cdot m_1$
$m_2 = y_1 \cdot m_1$	$m_{21} = y_1 \cdot y_2 \cdot y_2 \cdot m_1$	$m_{91} = y_1 \cdot y_1 \cdot y_1 \cdot y_1 \cdot y_1 \cdot m_1$
$m_3 = y_2 \cdot m_1$	$m_{22} = y_1 \cdot y_2 \cdot y_3 \cdot m_1$	$m_{92} = y_1 \cdot y_1 \cdot y_1 \cdot y_1 \cdot y_2 \cdot m_1$
$m_4 = y_3 \cdot m_1$	$m_{27} = y_1 \cdot y_6 \cdot m_1$	$m_{93} = y_1 \cdot y_1 \cdot y_1 \cdot y_1 \cdot y_3 \cdot m_1$
$m_5 = y_4 \cdot m_1$	$m_{31} = y_2 \cdot y_3 \cdot y_4 \cdot m_1$	$m_{94} = y_1 \cdot y_1 \cdot y_1 \cdot y_1 \cdot y_4 \cdot m_1$
$m_7 = y_1 \cdot y_2 \cdot m_1$	$m_{36} = y_3 \cdot y_5 \cdot m_1$	$m_{95} = y_1 \cdot y_1 \cdot y_1 \cdot y_2 \cdot y_2 \cdot m_1$
$m_9 = y_1 \cdot y_4 \cdot m_1$	$m_{39} = y_4 \cdot y_5 \cdot m_1$	$m_{96} = y_1 \cdot y_1 \cdot y_1 \cdot y_2 \cdot y_3 \cdot m_1$
$m_{10} = y_2 \cdot y_2 \cdot m_1$	$m_{41} = y_7 \cdot m_1$	$m_{97} = y_1 \cdot y_1 \cdot y_1 \cdot y_2 \cdot y_4 \cdot m_1$
$m_{11} = y_2 \cdot y_3 \cdot m_1$	$m_{56} = y_1 \cdot y_2 \cdot y_3 \cdot y_4 \cdot m_1$	$m_{98} = y_1 \cdot y_1 \cdot y_1 \cdot y_3 \cdot y_4 \cdot m_1$
$m_{12} = y_2 \cdot y_4 \cdot m_1$	$m_{65} = y_1 \cdot y_4 \cdot y_6 \cdot m_1$	$m_{99} = y_1 \cdot y_1 \cdot y_1 \cdot y_4 \cdot y_4 \cdot m_1$
$m_{13} = y_3 \cdot y_4 \cdot m_1$	$m_{82} = y_3 \cdot y_4 \cdot y_5 \cdot m_1$	$m_{100} = y_1 \cdot y_1 \cdot y_1 \cdot y_5 \cdot m_1$

Theorem In $I(V)$ in degree 1) (higher degrees were not considered), there are only the following singular vectors:

1a) $\lambda \longrightarrow \lambda + (0, 0, -2, 1)$: $n_{16} \otimes m_1$ for ANY λ;

1b) $\lambda \longrightarrow \lambda + (0, 0, -1, -1)$: $-\lambda_4 n_{15} \otimes m_1 + n_{16} \otimes m_5$ for ANY λ;

1c) $(k, l, 1, 1) \longrightarrow (k, l + 1, 0, 0)$

$$- n_{15} \otimes m_4 + 2\, n_{14} \otimes m_1 + n_{16} \otimes m_{13} - 2\, n_{13} \otimes m_5$$

1d) $(k, l, 2, 0) \longrightarrow (k, l + 1, 0, 1)$: $n_{16} \otimes m_4 - 2\, n_{13} \otimes m_1$;

1e) $(k, -1, 1, 1) \longrightarrow (k, 0, 1, 0)$

$$- n_{15} \otimes m_{11} - 2\, n_{12} \otimes m_1 + 2\, n_{14} \otimes m_3 + n_{16} \otimes m_{31} + 2\, n_{11} \otimes m_5 - 2\, n_{13} \otimes m_{12}$$

1f) $(-1, k, k + 2, 1) \longrightarrow (-2, k, k + 2, 0)$, where $k \neq 0$

$$\begin{array}{l} - n_{15} \otimes (m_{22} - (1 + k)\, m_{27}) - 2\, k\, n_{10} \otimes m_1 + 2\, k\, n_{12} \otimes m_2 + 2\, n_{14} \otimes m_7 \\ + n_{16} \otimes (m_{56} - (1 + k)\, m_{65}) + 2\, k\, n_9 \otimes m_5 - 2\, k\, n_{11} \otimes m_9 - 2\, n_{13} \otimes m_{23} \end{array}$$

1g) $(k, 0, -k - 1, 1) \longrightarrow (k - 1, 0, -k - 1, 0)$, where $k \neq 0, -1$

$$\begin{array}{l} - n_{15} \otimes (m_{22} - (k + 2)\, m_{27} - (k + 1)\, m_{36} - (k + 1)^2\, m_{41}) \\ + 2\, k\, (k + 1)\, n_{10} \otimes m_1 + 2\, (k + 1)\, n_{12} \otimes m_2 \\ + n_{14} \otimes (-2(3 + 2\, k)\, m_7 - 2\, (1 + k)\, m_{15}) \\ + n_{16} \otimes (m_{56} - (k + 2)\, m_{65} - (k + 1)\, m_{82} - (k + 1)^2\, m_{88}) \\ - 2\, k\, (1 + k)\, n_9 \otimes m_5 - 2\, (1 + k)\, n_{11} \otimes m_9 + 2\, (1 + k)\, n_{13} \otimes (m_{23} + m_{39}) \end{array}$$

1h) $(k, -k - 2, 1, 1) \longrightarrow (k - 1, -k - 2, 1, 0)$, where $k \neq 0$

$$\begin{array}{l} - n_{15} \otimes (m_{22} - (1 + k)\, m_{36} + (1 + k)\, m_{41}) - 2\, k\, n_{10} \otimes m_1 - 2\, n_{12} \otimes m_2 \\ + 2\, n_{14} \otimes (m_7 - (1 + k)\, m_{15}) + n_{16} \otimes (m_{56} - (1 + k)\, m_{82} + (1 + k)\, m_{88}) \\ + 2\, k\, n_9 \otimes m_5 + 2\, n_{11} \otimes m_9 - 2\, n_{13} \otimes (m_{23} - (1 + k)\, m_{39}) \end{array}$$

9 Singular vectors for $\mathfrak{g} = \mathfrak{ksle}(11|9)$

Here we realize the elements of $\mathfrak{g}$, as in Ref. 19, as divergence-free vector fields and closed 2-forms with shifted parity. We consider the following negative operators from $\mathfrak{g}_0$:

$$y_1 = x_2\partial_1 \qquad y_2 = x_4\partial_3 \qquad y_3 = x_5\partial_4$$
$$y_4 = \pi dx_1 dx_2 \qquad y_5 = -x_5\partial_3$$

and the elements of Cartan subalgebra

$$h_1 = x_1\partial_1 - x_2\partial_2 \qquad h_2 = -\tfrac{1}{2}(x_1\partial_1 + x_2\partial_2) + x_3\,\partial_3,$$
$$h_3 = -\tfrac{1}{2}(x_1\partial_1 + x_2\partial_2) + x_4\,\partial_4, \qquad h_4 = -\tfrac{1}{2}(x_1\partial_1 + x_2\partial_2) + x_5\,\partial_5$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$m_2 = y_1\,m_1$	$m_{16} = y_1\,y_2\,y_3\,m_1$	$m_{42} = (y_2)^2\,(y_3)^2\,m_1$
$m_3 = y_2\,m_1$	$m_{18} = y_1\,y_5\,m_1$	$m_{45} = y_2\,y_3\,y_5\,m_1$
$m_4 = y_3\,m_1$	$m_{20} = (y_2)^2\,y_3\,m_1$	$m_{49} = y_3\,y_4\,m_1$
$m_6 = y_1\,y_2\,m_1$	$m_{21} = y_2\,(y_3)^2\,m_1$	$m_{50} = (y_5)^2\,m_1$
$m_7 = y_1\,y_3\,m_1$	$m_{22} = y_2\,y_5\,m_1$	$m_{74} = y_1\,y_3\,y_4\,m_1$
$m_8 = (y_2)^2\,m_1$	$m_{24} = y_3\,y_5\,m_1$	$m_{85} = y_2\,y_3\,y_4\,m_1$
$m_9 = y_2\,y_3\,m_1$	$m_{25} = y_4\,m_1$	$m_{89} = (y_3)^2\,y_4\,m_1$
$m_{10} = (y_3)^2\,m_1$	$m_{34} = y_1\,(y_2)^2\,y_3\,m_1$	$m_{126} = y_1\,y_2\,y_3\,y_4\,m_1$
$m_{11} = y_5\,m_1$	$m_{36} = y_1\,y_2\,y_5\,m_1$	$m_{146} = y_2\,(y_3)^2\,y_4\,m_1$
$m_{15} = y_1\,(y_2)^2\,m_1$	$m_{39} = y_1\,y_4\,m_1$	$m_{231} = (y_2)^2\,(y_3)^2\,y_4\,m_1$

Theorem In $I(V)$ in degrees d), there are only the following singular vectors:

1a) $(2k, -k, l, m) \longrightarrow (2k+1, -k+\frac{3}{2}, l+1/2, m+\frac{1}{2})$: $x_3\,\partial_2\,m_1$;

1b) $(2k, l, 1-k, m) \longrightarrow (2k+1, l+\frac{1}{2}, -k+\frac{5}{2}, m+\frac{1}{2})$:

$$(x_3\,\partial_2)\,m_3 + (1-k-l)(x_4\,\partial_2)\,m_1;$$

1c) $(2k, l, m, 2-k) \longrightarrow (2k+1, l+\frac{1}{2}, m+\frac{1}{2}, -k+\frac{7}{2})$

$(x_3\,\partial_2)\,(m_{11}+(-2+k+m)m_{15})+(1-k-l)(x_4\,\partial_2)\,m_4+(-1+k+l)(-2+k+m)(x_5\,\partial_2)\,m_1$

1d) $(2k, 3-k, 3-k, 2-k) \longrightarrow (2k+1, \frac{5}{2}-k, \frac{5}{2}-k, \frac{5}{2}-k)$

$(\pi dx_1 dx_3)\,m_{11} - (\pi dx_1 dx_4)\,m_4 + (\pi dx_1 dx_5)\,m_1) - (x_3\partial_2)\,m_{85} + (x_4\partial_2)\,m_{49}) - (x_5\partial_2)\,m_{25}$

1e) $(2k, k+1, l, m) \longrightarrow (2k-1, k+\frac{5}{2}, l+\frac{1}{2}, m+\frac{1}{2})$: $2k(x_3\,\partial_1)\,m_1 + (x_3\,\partial_2)\,m_2$

1f) $(2k, l, 2+k, m) \longrightarrow (2k-1, l+\frac{1}{2}, k+\frac{7}{2}, m+\frac{1}{2})$

$$2k(x_3\,\partial_1)\,m_3) + (x_3\,\partial_2)\,m_6 + 2k(2+k-l)(x_4\,\partial_1)\,m_1 + (2+k-l)(x_4\,\partial_2)\,m_2$$

1g) $(2k, l, m, 3+k) \longrightarrow (2k-1, l+\frac{1}{2}, m+\frac{1}{2}, k+\frac{9}{2})$

$$(x_3\,\partial_1)\,(2km_9 - 2k(3+k-m)m_{11}) + (x_3\,\partial_2)\,(m_{16} + (-3-k+m)m_{18})$$
$$+2k(2+k-l)(x_4\,\partial_1)\,m_4 + (2+k-l)(x_4\,\partial_2)\,m_7$$
$$+2k(2+k-l)(3+k-m)(x_5\,\partial_1)\,m_1 + (2+k-l)(3+k-m)(x_5\,\partial_2)\,m_2$$

1h) $(2k, 4+k, 4+k, 3+k) \longrightarrow (2k-1, \frac{7}{2}-k, \frac{7}{2}-k, \frac{7}{2}-k)$

$$(\pi\, dx_1 dx_3)\, m_{18} - (\pi\, dx_1 dx_4)\, m_7 (\pi\, dx_1 dx_5)\, m_2 - 2k(\pi\, dx_2 dx_3)\, m_{11} 2k(\pi\, dx_2 dx_4)\, m_4$$
$$-2k(\pi\, dx_2 dx_5)\, m_1 - 2k(x_3\, \partial_1)\, m_{85} - (x_3\, \partial_2)\, m_{126} + 2k(x_4\, \partial_1)\, m_{49} + (x_4\, \partial_2)\, m_{74})$$
$$-2k(x_5\, \partial_1)\, m_{25} - (x_5\, \partial_2)\, m_{39}$$

2a) $(2k, -k-1, l, m) \longrightarrow (2k+2, -k+2, l+1, m+1)$: $(x_3\, \partial_2)^2\, m_1$

2b) $(2k, -k-1, -k+1, l) \longrightarrow (2k+2, -k+1, -k+3, l+1)$:

$$(x_3\, \partial_2)^2\, m_3 + 2x_3\partial_2\, x_4\partial_2\, m_1$$

2c) $(2k, -k-1, l, -k+2) \longrightarrow (2k+2, -k+1, l+1, -k+4)$

$$(x_3\partial_2)^2\, (m_9 + (-2+k+l)m_{11}) + 2x_3\partial_2\, x_4\partial_2\, m_4 - 2(-2+k+l)x_3\partial_2\, x_5\partial_2\, m_1$$

2d) $(2k, l, -k, m) \longrightarrow (2k+2, l+1, -k+3, m+1)$

$$(x_3\partial_2)^2\, m_8 + (-1+k+l)(k+l)(x_4\partial_2)^2\, m_1 - 2(-1+k+l)x_3\partial_2\, x_4\partial_2\, m_3$$

2e) $(2k, l, -k, -k+2) \longrightarrow (2k+2, l+1, -k+2, -k+4)$

$$(x_3\partial_2)^2\, (m_{20} - (2m_{22})) + (-1+k+l)(k+l)(x_4\partial_2)^2\, m_4$$
$$-2(-1+k+l)x_3\partial_2\, x_4\, \partial_2\, (m_9 - m_{11}) - 2(-1+k+l)x_3\, \partial_2\, x_5\partial_2\, m_3$$
$$+2(-1+k+l)(k+l)x_4\partial_2\, x_5\partial_2\, m_1$$

In particular, 2ea) $l = 1-k$:

$$(x_3\partial_2)^2\, m_{22} + (x_4\partial_2)^2\, m_4 - x_3\partial_2\, x_4\partial_2\, m_9 - 2x_3\partial_2\, x_5\partial_2\, m_3 + 2x_4\partial_2\, x_5\partial_2\, m_1$$

2f) $(2k, l, m, -k+1) \longrightarrow (2k+2, l+1, m+1, -k+4)$

$$(x_3\partial_2)^2\, (m_{42} + 2(-2+k+m)m_{45} + (-2+k+m)(-1+k+m)m_{50})$$
$$+(-1+k+l)(k+l)(x_4\partial_2)^2\, m_{10} + (-1+k+l)(k+l)(-2+k+m)(-1+k+m)(x_5\partial_2)^2\, m_1$$
$$-2(-1+k+l)x_3\partial_2\, x_4\partial_2\, (m_{21} + (-2+k+m)m_{24})$$
$$+2(-1+k+l)(-2+k+m)x_3\partial_2\, x_5\partial_2\, (m_9 + (-1+k+m)m_{11}) -$$
$$2(-1+k+l)(k+l)(-2+k+m)x_4\partial_2\, x_5\partial_2\, m_4$$

In particular, 2fa) $(2k, -k+1, -k, -k+1) \longrightarrow (2k+2, -k+2, -k+1, -k+4)$:

$$(x_3\partial_2)^2\, (m_{42} - 4m_{50}) + 2(x_4\partial_2)^2\, m_{10} + 4(x_5\partial_2)^2\, m_1$$
$$+4x_3\partial_2\, x_4\partial_2\, (-m_{21} + 2m_{24}) + 8x_3\partial_2\, x_5\partial_2)\, (-m_9 + m_{11}) + 8x_4\partial_2\, x_5\partial_2\, m_4$$

2fb) $(2k, -k, -k, -k+1) \longrightarrow (2k+2, -k+1, -k+1, -k+4)$:

$$(x_3\partial_2)^2\, m_{45} + (x_4\partial_2)^2\, m_{10} + 2(x_5\partial_2)^2\, m_1 - x_3\partial_2\, x_4\partial_2\, m_{21} - 4x_3\partial_2\, x_5\partial_2\, m_9 + 4x_4\partial_2\, x_5\partial_2\, m_4$$

2g) $(2k, 3-k, 3-k, 1-k) \longrightarrow (2k+2, 3-k, 3-k, 3-k)$

$$(x_3\partial_2)^2\, m_{231} + 2(x_4\partial_2)^2\, m_{89} + 4(x_5\,\partial_2)^2\, m_{25}$$
$$-4\pi dx_1dx_3\, x_3\partial_2\, m_{45} + 4\pi dx_1dx_3\, x_4\partial_2\, m_{24} - 4\pi dx_1dx_3\, x_5\partial_2\, m_{11}$$
$$+4\pi dx_1dx_4\, x_3\partial_2\, (m_{21} - m_{24}) - 4\pi dx_1dx_4\, x_4\partial_2\, m_{10} + 4\pi dx_1dx_4\, x_5\partial_2\, m_4$$
$$+4\pi dx_1dx_5\, x_3\partial_2\, (-3m_9 + 2m_{11}) + 4\pi dx_1dx_5\, x_4\partial_2\, m_4 - 8\pi dx_1dx_5\, x_5\partial_2\, m_1$$
$$-2x_3\partial_2\, x_4\partial_2\, m_{146} + 4x_3\partial_2\, x_5\partial_2\, m_{85}$$
$$-4x_4\partial_2\, x_5\partial_2\, m_{49}$$

2h) $(-2, 0, k, l) \longrightarrow (-2, 3, k+1, l+1)$: $(x_3\partial_2)^2\, m_2 - 2x_3\partial_1\, x_3\partial_2\, m_1$

2i) $(2k, -k, l, m) \longrightarrow (2k, -k+2, l+2, m+1)$

$$(x_3\partial_2)^2\, m_6 + 2kx_3\partial_1\, x_3\partial_2\, m_3 + 4k(1+k)x_3\partial_2\, x_4\partial_1\, m_1 + 2(1+k)x_3\partial_2\, x_4\partial_2\, m_2$$

2j) $(2k, k+1, l, m) \longrightarrow (2k, k+3, l+2, m+1)$

$$(x_3\partial_2)^2\, m_6 + 2kx_3\partial_1\, x_3\partial_2\, m_3 - 2k(1+2k)x_3\partial_1\, x_4\partial_2\, m_1$$
$$+2kx_3\partial_2\, x_4\partial_1\, m_1 - 2kx_3\partial_2\, x_4\partial_2\, m_2 - 2(-1+k+l)(k+l)(-2+k+m)x_4\partial_2\, x_5\partial_2\, m_4$$

In particular, 2ja) $(0, 1, 1, m) \longrightarrow (0, 3, 3, m+1)$:

$$-x_3\partial_1\, x_4\partial_2\, m_1 + x_3\partial_2\, x_4\partial_1\, m_1$$

2k) $(2k, -k, l, k+3) \longrightarrow (2k, -k+2, l+1, k+4)$

$$(x_3\partial_2)^2\, (m_{16} + (-3-k+l)m_{18}) + 2kx_3\partial_1\, x_3\partial_2\, (m_9 - (3+k-l)m_{11})$$
$$+4k(1+k)x_3\partial_2\, x_4\partial_1\, m_4 + 2(1+k)x_3\partial_2\, x_4\partial_2\, m_7$$
$$+4k(1+k)(3+k-l)x_3\partial_2\, x_5\partial_1\, m_1 + 2(1+k)(3+k-l)x_3\partial_2\, x_5\partial_2\, m_2$$

2l) $(2k, k+1, l, 2-k) \longrightarrow (2k, k+3, l+1, 4-k)$

$$(x_3\partial_2)^2\, (m_{16} + (-2+k+l)m_{18}) + 2kx_3\partial_1\, x_3\partial_2\, (m_9 + (-2+k+l)m_{11})$$
$$-2k(1+2k)x_3\partial_1\, x_4\partial_2\, m_4 + 2k(1+2k)(-2+k+l)x_3\partial_1\, x_5\partial_2\, m_1$$
$$+2kx_3\partial_2\, x_4\partial_1\, m_4 - 2kx_3\partial_2\, x_4\partial_2\, m_7$$
$$-2k(-2+k+l)x_3\partial_2\, x_5\partial_1\, m_1 + 2k(-2+k+l)x_3\partial_2\, x_5\partial_2\, m_2$$

In particular, 2la) $(0, 1, l, 2) \longrightarrow (0, 3, l+1, 4)$:

$$x_3\partial_1\, x_4\partial_2\, m_4 - (l-2)x_3\partial_1\, x_5\partial_2\, m_1 - x_3\partial_2\, x_4\partial_1\, m_4 + (l-2)x_3\partial_2\, x_5\partial_1\, m_1$$

2m) $(-2, k, 1, l) \longrightarrow (-2, k+1, 4, l+1)$

$$(x_3\partial_2)^2\, m_{15} + (-2+k)(-1+k)(x_4\partial_2)^2\, m_2 - 2x_3\partial_1\, x_3\partial_2\, m_8 + 2(-2+k)x_3\partial_1\, x_4\partial_2\, m_3$$
$$+2(-2+k)x_3\partial_2\, x_4\partial_1\, m_3 - 2(-2+k)x_3\partial_2\, x_4\partial_2\, m_6 - 2(-2+k)(-1+k)x_4\partial_1\, x_4\partial_2\, m_1$$

2n) $(2k, l, -k+1, k+3) \longrightarrow (2k, l+1, -k+3, k+5)$

$$(x_3\partial_2)^2\,(m_{34} - 2(1+k)m_{36}) - (2+k-l)(-1+k+l)(x_4\partial_2)^2\,m_7 +$$
$$2kx_3\partial_1\,x_3\partial_2\,(m_{20} - 2(1+k)m_{22}) + 2k(-1+k+l)x_3\partial_1\,x_4\partial_2\,(-m_9$$
$$+2(1+k)m_{11}) + 2kx_3\partial_2\,x_4\partial_1\,((3+k-l)m_9 - 2(1+k)m_{11}) +$$
$$2x_3\partial_2\,x_4\partial_2\,(-(-2+l)m_{16} + (1+k)(-2+k+l)m_{18}) + 4k(1+k)(2+k-l)x_3\partial_2\,x_5\partial_1\,m_3$$
$$+2(1+k)(2+k-l)x_3\partial_2\,x_5\partial_2\,m_6 - 2k(2+k-l)(-1+k+l)x_4\partial_1\,x_4\partial_2\,m_4 -$$
$$4k(1+k)(2+k-l)(-1+k+l)x_4\partial_2\,x_5\partial_1\,m_1 - 2(1+k)(2+k-l)(-1+k+l)x_4\partial_2\,x_5\partial_2\,m_2$$

2o) $(2k, l, k+2, -k+2) \longrightarrow (2k, l+1, k+4, -k+4)$

$$(x_3\partial_2)^2\,(m_{34} + (2k)m_{36}) - (2+k-l)(-1+k+l)(x_4\partial_2)^2\,m_7 + 2kx_3\partial_1\,x_3\partial_2\,(m_{20} + 2km_{22})$$
$$+2k(-1+k+l)x_3\partial_1\,x_4\partial_2\,(-m_9 + m_{11}) + 2k(1+2k)(-1+k+l)x_3\partial_1\,x_5\partial_2\,m_3$$
$$+2kx_3\partial_2\,x_4\partial_1\,((3+k-l)m_9 + (1+5k+2k^2-l-2kl)m_{11})$$
$$+2x_3\partial_2\,x_4\partial_2\,(-(-2+l)m_{16} + k(3+k-l)m_{18}) - 2k(-1+k+l)x_3\partial_2\,x_5\partial_1\,m_3$$
$$+2k(-1+k+l)x_3\partial_2\,x_5\partial_2\,m_6 - 2k(2+k-l)(-1+k+l)x_4\partial_1\,x_4\partial_2\,m_4$$
$$+2k(1+2k)(2+k-l)(-1+k+l)x_4\partial_1\,x_5\partial_2\,m_1 - 2k(2+k-l)(-1+k+l)x_4\partial_2\,x_5\partial_1\,m_1$$
$$+2k(2+k-l)(-1+k+l)x_4\partial_2\,x_5\partial_2\,m_2$$

10 Singular vectors for $\mathfrak{g} = \mathfrak{kas}$ and $\mathfrak{g} = \mathfrak{k}(1|n)$

The coordinates of the weights are given with respect to the following basis of $\mathfrak{g}_0$:

$$(K_t, K_{\xi_1\eta_1}, \dots, K_{\xi_s\eta_s}), \quad s = [\frac{n}{2}].$$

Theorem In $I(V)$, there are only the following singular for $\mathfrak{k}(1|3)$ vectors:
1a) $(k, -k) \longrightarrow (k-1, -k+1)$: $\xi_1 \otimes m_1$;
1a*) $(k+1, k) \longrightarrow (k, k-1)$

$$\xi_1 \otimes \Big((\eta_1\theta_1)^2 \cdot m_1\Big) - k\ (-1+2\,k)\ \eta_1 \otimes m_1 + (1-2\,k)\,\theta_1 \otimes (\eta_1\theta_1 \cdot m_1)$$

1b) $(1, -1) \longrightarrow (0, -1)$: $\xi_1 \otimes (\eta_1\theta_1 \cdot m_1) + \theta_1 \otimes m_1$;
2a) $\frac{1}{2}(3, 1) \longrightarrow \frac{1}{2}(-1, 1)$

$$I \otimes m_1 - 2\ (\xi_1 \cdot \theta_1) \otimes (\eta_1\theta_1 \cdot m_1) + (\eta_1 \cdot \xi_1) \otimes m_1$$

Theorem In $I(V)$, there are only the following singular for $\mathfrak{k}(1|4)$ vectors:
1a) $(k, -k, 0) \longrightarrow (k-1, -k+1, 0)$: $\xi_1 \otimes m_1$;
1a*) $(k+1, -1, k) \longrightarrow (k, -1, k-1)$, where $k \neq 1$

$$\xi_1 \otimes (\eta_1\eta_2 \cdot m_1) + (1-k)\,\eta_2 \otimes m_1$$

1b) $(k, -1, 1-k) \longrightarrow (k-1, -1, 2-k)$, where $k \neq 2$

$$(-2+k)\,\xi_2 \otimes m_1 + \xi_1 \otimes (\xi_2\eta_1 \cdot m_1)$$

1c) $(k+2, k, 0) \longrightarrow (k+1, k-1, 0)$, where $k \neq 0$

$$\xi_1 \otimes (\eta_1\eta_2 \cdot m_1) - k\,\eta_2 \otimes m_1$$

2a) $(1, -1, -1) \longrightarrow (-1, 0, 0)$: $(\xi_2 \cdot \xi_1) \otimes m_1$;
2b) $(2, -1, 1) \longrightarrow (0, 0, 0)$: $(\eta_2 \cdot \xi_1) \otimes m_1$;
2c) $(2, -1, -1) \longrightarrow (0, -1, -1)$

$$I \otimes m_1 - (\xi_2 \cdot \xi_1) \otimes (\eta_1\eta_2 \cdot m_1) + (\eta_1 \cdot \xi_1) \otimes m_1 + (\eta_2 \cdot \xi_2) \otimes m_1$$

Theorem In $I(V)$, there are only the following singular for $\mathfrak{k}(1|6)$ and $\mathfrak{kas}$ vectors:
1a) $\lambda = (k, -k, l, l) \longrightarrow \lambda + (-1, 1, 0, 0)$; $\mathfrak{kas}$ and $\mathfrak{k}(1|6)$: $\xi_1 \otimes m_1$ (for $\mathfrak{k}(1|6)$ only if $l = 0$; for $\mathfrak{kas}$ without restrictions);
1a*) $\lambda = (k, l, 1-k, l+1) \longrightarrow \lambda + (-1, 0, 1, 0)$

$$\begin{array}{ll} \mathfrak{kas}: & (-1+k+l)\,\xi_2 \otimes m_1 + \xi_1 \otimes (\xi_2\eta_1 \cdot m_1) \\ \mathfrak{k}(1|6): & \text{the above for } l = -1 \end{array}$$

1b) $\lambda = (k, l, l, 2-k) \longrightarrow \lambda + (-1, 0, 0, 1)$, where $l + k \neq 2$

$$\begin{array}{ll} \mathfrak{kas}: & \xi_1 \otimes (\xi_2\eta_1 \cdot \xi_3\eta_2 \cdot m_1) + \xi_2 \otimes (\xi_3\eta_2 \cdot m_1) + (-2+k+l)\,\xi_3 \otimes m_1 \\ \mathfrak{k}(1|6): & \text{the above for } l = -1 \end{array}$$

1c) $\lambda = (k, l, -l-2, k-2) \longrightarrow \lambda + (-1, 0, 0, -1)$, where $l + k \neq 1$ and $k - l \neq 4$

$$\begin{array}{ll} \mathfrak{kas}: & \xi_1 \otimes (\xi_2\eta_1 \cdot \eta_2\eta_3 \cdot m_1) + (-4+k-l)\,\xi_1 \otimes (\eta_1\eta_3 \cdot m_1) \\ & +(-1+k+l)\,\xi_2 \otimes (\eta_2\eta_3 \cdot m_1) - (-4+k-l)\,(-1+k+l)\,\eta_3 \otimes m_1 \\ \mathfrak{k}(1|6): & \text{the above for } l = -1 \end{array}$$

The singular vectors of degree 2 for $\mathfrak{kas}$ and $\mathfrak{k}(1|6)$ are the same:
2a) $(3, -1, -1, -1) \longrightarrow (1, -1, -1, -1)$

$$\begin{array}{l} -2\,I \otimes m_1 + (\xi_2 \cdot \xi_1) \otimes (\xi_3\eta_2 \cdot \eta_1\eta_3 \cdot m_1) + (\xi_3 \cdot \xi_1) \otimes (\eta_1\eta_3 \cdot m_1) \\ + (\xi_3 \cdot \xi_2) \otimes (\eta_2\eta_3 \cdot m_1) - (\eta_1 \cdot \xi_1) \otimes m_1 - (\eta_2 \cdot \xi_1) \otimes (\xi_2\eta_1 \cdot m_1) \\ - (\eta_2 \cdot \xi_2) \otimes m_1 + (\eta_3 \cdot \xi_1) \otimes (\xi_2\eta_1 \cdot \xi_3\eta_2 \cdot m_1) \\ - (\eta_3 \cdot \xi_1) \otimes (\xi_3\eta_1 \cdot m_1) - (\eta_3 \cdot \xi_3) \otimes m_1 \end{array}$$

2a*) $(3,-1,0,0) \longrightarrow (1,-1,0,0)$

$$-I\otimes m_1 + (\xi_2\cdot\xi_1)\otimes(\xi_3\eta_2\cdot\eta_1\eta_3\cdot m_1) + (\xi_3\cdot\xi_1)\otimes(\eta_1\eta_3\cdot m_1)$$
$$+(\xi_3\cdot\xi_2)\otimes(\eta_2\eta_3\cdot m_1) - (\eta_1\cdot\xi_1)\otimes m_1 - (\eta_2\cdot\xi_1)\otimes(\xi_2\eta_1\cdot m_1)$$
$$+(\eta_3\cdot\xi_1)\otimes(\xi_2\eta_1\cdot\xi_3\eta_2\cdot m_1) - (\eta_3\cdot\xi_1)\otimes(\xi_3\eta_1\cdot m_1) - (\eta_3\cdot\xi_2)\otimes(\xi_3\eta_2\cdot m_1)$$

2b) $(3,-1,-1,1) \longrightarrow (1,-1,-1,1)$

$$-I\otimes m_1 + (\xi_2\cdot\xi_1)\otimes(\eta_2\eta_3\cdot\xi_3\eta_1\cdot m_1) - (\xi_3\cdot\xi_1)\otimes(\xi_2\eta_1\cdot\eta_2\eta_3\cdot m_1)$$
$$+(\xi_3\cdot\xi_1)\otimes(\eta_1\eta_3\cdot m_1) - (\eta_1\cdot\xi_1)\otimes m_1 - (\eta_2\cdot\xi_1)\otimes(\xi_2\eta_1\cdot m_1)$$
$$-(\eta_2\cdot\xi_2)\otimes m_1 - (\eta_3\cdot\xi_1)\otimes(\xi_3\eta_1\cdot m_1)$$
$$-(\eta_3\cdot\xi_2)\otimes(\xi_3\eta_2\cdot m_1) + (\eta_3\cdot\xi_3)\otimes m_1$$

11 Singular vectors for $\mathfrak{g} = \mathfrak{kas}(1|6;3\xi)$

Set

m_1 is the highest weight vector	$m_{15} = \xi_1\eta_2\,(\xi_3\eta_1)^2\,m_1$
$m_2 = \xi_1\eta_2\,m_1$	$m_{16} = \xi_1\eta_2\,\xi_3\eta_1\,\eta_3\,m_1$
$m_3 = \xi_3\eta_1\,m_1$	$m_{18} = \xi_1\eta_2\,\eta_1\,m_1$
$m_4 = \eta_3\,m_1$	$m_{23} = \eta_3\,\xi_3\eta_2\,m_1$
$m_6 = \xi_1\eta_2\,\xi_3\eta_1\,m_1$	$m_{25} = -(\eta_2\,m_1)$
$m_7 = \xi_1\eta_2\,\eta_3\,m_1$	$m_{34} = \xi_1\eta_2\,(\xi_3\eta_1)^2\,\eta_3\,m_1$
$m_9 = \xi_3\eta_1\,\eta_3\,m_1$	$m_{36} = \xi_1\eta_2\,\xi_3\eta_1\,\eta_1\,m_1$
$m_{10} = \xi_3\eta_2\,m_1$	$m_{46} = -(\xi_3\eta_1\,\eta_2\,m_1)$
$m_{11} = \eta_1\,m_1$	$m_{49} = \xi_3\eta_2\,\eta_1\,m_1$

Theorem In $I(V)$ in degrees d), there are only the following singular vectors:

1a) $\lambda \longrightarrow \lambda + (1,1,1,1)$ for ANY λ: $\xi_1\xi_2\xi_3\,m_1$;

1b) $(k,l-1,-l-1,k) \longrightarrow (k,l,-l,k)$: $\xi_1\xi_2\,m_1 + \xi_1\xi_2\xi_3\,m_4$;

1c) $(k,l,k-1,-l-1) \longrightarrow (k,l,k,-l)$, where $k+l \neq 0$

$$\xi_1\xi_2\,m_3 - (k+l)\xi_2\xi_3\,m_1 + \xi_1\xi_2\xi_3\,(m_9 - (k+l+1)m_{11})$$

1d) $(k+1,k-1,l,-l) \longrightarrow (k+1,k-1,l+1,-l+1)$, where $k+l \neq 0$, $k \neq l+1$

$$\xi_1\xi_2\,(m_6 + (-k+l)m_{10}) - (-1+k-l)(k+l)\xi_1\xi_3\,m_1$$
$$-(k+l)\xi_2\xi_3\,m_2 + \xi_1\xi_2\xi_3\,(m_{16} - (k+l+1)m_{18} + (-k+l)m_{23} - (k-l)(k+l)m_{25})$$

1e) $(k+1,k-1,0,0) \longrightarrow (k+1,k-1,1,1)$, where $k \neq 1$ (new for $k=0$ only)

$$\xi_1\xi_2\,(m_6 - 2m_{10}) + 2(1-k)\xi_1\xi_3\,m_1$$
$$-2\xi_2\xi_3\,m_2 - 2\xi_1\xi_2\xi_3\,(m_{18} + m_{23} + km_{25})$$

2a) $\lambda \longrightarrow \lambda + (1,2,2,1)$, where $\lambda_4 = -2-\lambda_3$: $\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,m_1$;

2b) $\lambda \longrightarrow \lambda + (1,2,1,2)$, where $\lambda_4 = -1-\lambda_2$, $\lambda_2+\lambda_3 \neq -1$

$$\xi_1\xi_2\cdot\xi_1\xi_2\xi_3\,m_3 + (-1-\lambda_2-\lambda_3)(\xi_2\xi_3\cdot\xi_1\xi_2\xi_3\,m_1)$$

2c) $\lambda \longrightarrow \lambda + (1,1,2,2)$, where $\lambda_4 = -\lambda_3$, $\lambda_2 \neq \lambda_3, \lambda_2+\lambda_3 \neq -1$

$$(\xi_1\xi_2\cdot\xi_1\xi_2\xi_3)\,(m_6 + (-1-\lambda_2+\lambda_3)m_{10}) - (\lambda_2-\lambda_3)(1+\lambda_2+\lambda_3)$$
$$\times(\xi_1\xi_3\cdot\xi_1\xi_2\xi_3)\,m_1 + (-1-\lambda_2-\lambda_3)(\xi_2\xi_3\cdot\xi_1\xi_2\xi_3)\,m_2$$

2d) $(k, l, 0, 0) \longrightarrow (k+1, l+1, 2, 2)$, where $l \neq 0$ (new for $l = -1$ only)

$$(\xi_1\xi_2 \cdot \xi_1\xi_2\xi_3)\, m_{10} + l(\xi_1\xi_3 \cdot \xi_1\xi_2\xi_3)\, m_1 + (\xi_2\xi_3 \cdot \xi_1\xi_2\xi_3)\, m_2$$

2e) $(k, -k, k-3, k-1) \longrightarrow (k, -k+2, k-2, k)$

$$(\xi_1\xi_2)^2\, m_3 - 2\xi_2 \cdot \xi_1\xi_2\xi_3\, m_1 + 2\xi_1\xi_2 \cdot \xi_2\xi_3\, m_1 + \xi_1\xi_2 \cdot \xi_1\xi_2\xi_3\, (m_9 - m_{11}) + 2\xi_2\xi_3 \cdot \xi_1\xi_2\xi_3\, m_4$$

2f) $(k, -k, l, l) \longrightarrow (k, -k+2, l+1, l+1)$, where $k \neq l$

$$(\xi_1\xi_2)^2\, m_3 - 4(k-l)\xi_2 \cdot \xi_1\xi_2\xi_3\, m_1 - 4\xi_1\xi_2 \cdot \xi_1\xi_2\xi_3\, m_{11} + 4\xi_2\xi_3 \cdot \xi_1\xi_2\xi_3\, m_4$$

2g) $(1+k, l, -k, 1+l) \longrightarrow (1+k, l+1, 2-k, 2+l)$

$$\begin{aligned}
&(k-l)(1+k-l)(k+l)\xi_1 \cdot \xi_1\xi_2\xi_3\, m_1 + (k-l)(1+k-l)\xi_2 \cdot \xi_1\xi_2\xi_3\, m_2 \\
&+\xi_1\xi_2 \cdot \xi_1\xi_2\xi_3\, (m_{16} + (k-l)m_{18} + (-1-k-l)m_{23} + (1+k-l)(1+k+l)m_{25}) \\
&-(1+k-l)(k+l)\xi_1\xi_3 \cdot \xi_1\xi_2\xi_3\, m_4 + (-1-k+l)\xi_2\xi_3 \cdot \xi_1\xi_2\xi_3\, m_7
\end{aligned}$$

2h) $(k+2, k-2, k, -k) \longrightarrow (k+2, k-1, k+1, -k+2)$

$$\begin{aligned}
&(\xi_1\xi_2)^2\, m_{15} + 2k(-1+2k)(\xi_2\xi_3)^2\, m_2 - 2(-1+2k)\xi_1 \cdot \xi_1\xi_2\xi_3\, m_3 \\
&-2\xi_2 \cdot \xi_1\xi_2\xi_3\, (m_6 + 2km_{10}) - 4k(-1+2k)\xi_3 \cdot \xi_1\xi_2\xi_3\, m_1 + 2(-1+2k)\xi_1\xi_2 \cdot \xi_1\xi_3\, m_3 \\
&-2(-1+2k)\xi_1\xi_2 \cdot \xi_2\xi_3\, m_6 + \xi_1\xi_2 \cdot \xi_1\xi_2\xi_3\, (m_{34} - 2(1+k)m_{36} - 2km_{46} - 2km_{49}) \\
&-4k(-1+2k)\xi_1\xi_3 \cdot \xi_2\xi_3\, m_1 + 2(-1+2k)\xi_1\xi_3 \cdot \xi_1\xi_2\xi_3\, (m_9 - (1+2k)m_{11}) \\
&+2\xi_2\xi_3 \cdot \xi_1\xi_2\xi_3\, (-(-1+k)m_{16} + (-1+2k^2)m_{18} + km_{23} + 2(-1+2k)m_{25})
\end{aligned}$$

2i) $(-l+1, k, k, l+1) \longrightarrow (-l+2, k+2, k+2, l+2)$

$$\begin{aligned}
&(k-l)(\xi_2\xi_3)^2\, m_2 - 2(k+l)\xi_1 \cdot \xi_1\xi_2\xi_3\, m_3 + 2\xi_2 \cdot \xi_1\xi_2\xi_3\, (-m_6 + (1-k-l)m_{10}) \\
&-2(-1+k-l)(k+l)\xi_3 \cdot \xi_1\xi_2\xi_3\, m_1 + 2\xi_1\xi_2 \cdot \xi_2\xi_3\, (-m_6 + m_{10}) \\
&+2\xi_1\xi_2 \cdot \xi_1\xi_2\xi_3\, (m_{46} + m_{49}) + 2\xi_1\xi_3 \cdot \xi_1\xi_2\xi_3\, (-m_9 + (k-l)m_{11}) \\
&+2\xi_2\xi_3 \cdot \xi_1\xi_2\xi_3\, (m_{18} - m_{23} + (2-k+l)m_{25})
\end{aligned}$$

12 Singular vectors for $\mathfrak{g} = \mathfrak{kas}(1|6; 3\eta)$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$m_2 = \xi_2\, m_1$	$m_{11} = \xi_3\eta_2\, m_1$	$m_{30} = \xi_2\, \xi_1\eta_2\, \xi_3\eta_2\, m_1$
$m_3 = \xi_1\eta_2\, m_1$	$m_{13} = \xi_2\, \xi_1\eta_2\, \xi_3\eta_1\, m_1$	$m_{34} = -(\xi_2\, \xi_3\, m_1)$
$m_4 = \xi_3\eta_1\, m_1$	$m_{16} = \xi_2\, \xi_3\eta_2\, m_1$	$m_{37} = (\xi_1\eta_2)^2\, (\xi_3\eta_1)^2\, m_1$
$m_5 = \xi_2\, \xi_1\eta_2\, m_1$	$m_{18} = (\xi_1\eta_2)^2\, \xi_3\eta_1\, m_1$	$m_{41} = \xi_1\eta_2\, \xi_3\eta_1\, \xi_1\, m_1$
$m_6 = \xi_2\, \xi_3\eta_1\, m_1$	$m_{19} = \xi_1\eta_2\, (\xi_3\eta_1)^2\, m_1$	$m_{42} = \xi_1\eta_2\, \xi_3\eta_1\, \xi_3\eta_2\, m_1$
$m_7 = (\xi_1\eta_2)^2\, m_1$	$m_{21} = \xi_1\eta_2\, \xi_3\eta_2\, m_1$	$m_{43} = -(\xi_1\eta_2\, \xi_3\, m_1)$
$m_8 = \xi_1\eta_2\, \xi_3\eta_1\, m_1$	$m_{23} = \xi_3\eta_1\, \xi_1\, m_1$	$m_{48} = \xi_1\, \xi_3\eta_2\, m_1$
$m_9 = (\xi_3\eta_1)^2\, m_1$	$m_{24} = \xi_3\eta_1\, \xi_3\eta_2\, m_1$	$m_{49} = (\xi_3\eta_2)^2\, m_1$
$m_{10} = \xi_1\, m_1$	$m_{25} = -(\xi_3\, m_1)$	$m_{86} = -(\xi_1\, \xi_3\, m_1)$
	$m_{27} = \xi_2\, (\xi_1\eta_2)^2\, \xi_3\eta_1\, m_1$	

Theorem In $I(V)$ in degrees d), there are only the following singular vectors:

1a) $(k,l,m,-m) \longrightarrow (l,l,m-1,-m-1)$: $\eta_1\eta_3\, m_1$;

1b) $(k,l,m,-l-1) \longrightarrow (k,l-1,m,-l-2)$: $\eta_1\eta_3\, m_3 + (-l+m)\eta_2\eta_3\, m_1$;

1c) $(k,l,-l-2,m) \longrightarrow (k,l-1,-l-3,m)$

$(1{+}l{-}m)(2{+}l{+}m)\eta_1\eta_2\, m_1{+}\eta_1\eta_3\,(m_8{+}({-}2{-}l{-}m)m_{11}){+}({-}1{-}l{+}m)\eta_2\eta_3\, m_4$

1d) $(k+3,-k-2,k,k-1) \longrightarrow (k+2,-k-2,k-1,k-1)$

$$2k\eta_1\, m_1 - 2k\eta_3\, m_4 + 2k\eta_1\eta_2\, m_2 + \eta_1\eta_3\,(m_{13}+m_{16}) + 2k\eta_2\eta_3\, m_6$$

1e) $(k+3,k-1,-k-1,k-1) \longrightarrow (k+2,k-2,-k-1,k-1)$

$$\begin{aligned}&2k\eta_1\, m_3 - 4k^2\eta_2\, m_1 + 2k\eta_3\,(-m_8 + (1+2k)m_{11})\\ &+2k\eta_1\eta_2\,(m_5 + 2km_{10}) + \eta_1\eta_3\,(m_{27} - 2km_{30} + 2km_{41} - 2km_{43} - 4k^2m_{48})\\ &+2k\eta_2\eta_3\,(-m_{16} - m_{23} + m_{25})\end{aligned}$$

1f) $(4,0,-1,-1) \longrightarrow (3,-1,-1,-1)$

$$\begin{aligned}&-2\eta_1\, m_3) + 2\eta_2\, m_1 + \eta_3\,(m_8 - 2m_{11}) - \eta_1\eta_2\,(m_5 + m_{10})\\ &+\eta_1\eta_3\,(m_{27} + m_{43} + m_{48}) - \eta_2\eta_3\,(2m_{13} - m_{16} + m_{23})\end{aligned}$$

1g) $(4,0,0,0) \longrightarrow (2,0,0,0)$

$$6\, m_1{+}\eta_1\,(m_5{+}3m_{10}){+}3\eta_2\, m_2{-}\eta_3\,(m_{16}{+}m_{23}){+}\eta_1\eta_2\, m_{15}{+}\eta_1\eta_3\, m_{86}{+}\eta_2\eta_3\, m_{34}$$

2a) $(k,l,m,2-m) \longrightarrow (k,l,m-2,-m)$: $(\eta_1\eta_3)^2\, m_1$;

2b) $(k,l,l+2,-l-1) \longrightarrow (0,l-1,l+1,-l-3)$

$$(\eta_1\eta_3)^2\, m_3 + 2\eta_1\eta_3\cdot\eta_2\eta_3\, m_1$$

2c) $(k,l-2,-l,l+1) \longrightarrow (k,l-3,-l-2,l)$, where $l \neq -\frac{1}{2}$

$$(\eta_1\eta_3)^2\,(m_8 - (1+2l)m_{11}) - 2(1+2l)\eta_1\eta_2\cdot\eta_1\eta_3\, m_1 + 2\eta_1\eta_3\cdot\eta_2\eta_3\, m_4$$

2d) $(k,l,m,-l) \longrightarrow (k,l-2,m,-l-2)$

$$(\eta_1\eta_3)^2\, m_7 + (-1+l-m)(l-m)(\eta_2\eta_3)^2\, m_1 - 2(-1+l-m)\eta_1\eta_3\cdot\eta_2\eta_3\, m_3$$

2e) $(k,l-1,-l-1,-l+1) \longrightarrow (k,l-3,-1-2,-l)$

$$\begin{aligned}&(\eta_1\eta_3)^2\,(m_{18} - 2m_{21}) + 2l(-1+2l)(\eta_2\eta_3)^2\, m_4 + 2(-1+2l)\eta_1\eta_2\cdot\eta_1\eta_3\, m_3\\ &-4l(-1+2l)\eta_1\eta_2\cdot\eta_2\eta_3\, m_1 + 2(-1+2l)\eta_1\eta_3\cdot\eta_2\eta_3\,(-2m_8 + m_{11})\end{aligned}$$

2ea) Particular solution for $l = \frac{1}{2}$:

$$(\eta_1\eta_3)^2\, m_{21} + (\eta_2\eta_3)^2\, m_4 + 2\eta_1\eta_2\cdot\eta_1\eta_3\, m_3 - 2\eta_1\eta_2\cdot\eta_2\eta_3\, m_1 - \eta_1\eta_3\cdot\eta_2\eta_3\, m_8$$

2f) $(k, l, -l-1, m) \longrightarrow (k, l-2, -l-3, m)$, where $m \neq l, l+1, -l-1, -l-2$

$$
\begin{aligned}
&(l-m)(1+l-m)(1+l+m)(2+l+m)(\eta_1\eta_2)^2\, m_1\\
&+(\eta_1\eta_3)^2\,(m_{37} - 2(2+l+m)m_{42} + (1+l+m)(2+l+m)m_{49})\\
&+(l-m)(1+l-m)(\eta_2\eta_3)^2\, m_9\\
&+2(l-m)(2+l+m)\eta_1\eta_2\cdot\eta_1\eta_3\,(m_8 - (1+l+m)m_{11})\\
&-2(l-m)(1+l-m)(2+l+m)\eta_1\eta_2\cdot\eta_2\eta_3\, m_4\\
&+2(l-m)\eta_1\eta_3\cdot\eta_2\eta_3\,(-m_{19} + (2+l+m)m_{24})
\end{aligned}
$$

2fa) Particular solution for $l = m = 0$:

$$
\begin{aligned}
&4(\eta_1\eta_2)^2\, m_1 + (\eta_1\eta_3)^2\,(m_{37} - 4m_{49}) + 2(\eta_2\eta_3)^2\, m_9\\
&+8\eta_1\eta_2\cdot\eta_1\eta_3\,(m_8 - m_{11}) - 8\eta_1\eta_2\cdot\eta_2\eta_3\, m_4 + 4\eta_1\eta_3\cdot\eta_2\eta_3\,(-m_{19} + 2m_{24})
\end{aligned}
$$

2fb) Particular solution for $l = -\frac{1}{2}$, $m = \frac{1}{2}$:

$$
\begin{aligned}
&2(\eta_1\eta_2)^2\, m_1 + (\eta_1\eta_3)^2\, m_{42} + (\eta_2\eta_3)^2\, m_9 + 4\eta_1\eta_2\cdot\eta_1\eta_3\, m_8\\
&-4\eta_1\eta_2\cdot\eta_2\eta_3\, m_4 - \eta_1\eta_3\cdot\eta_2\eta_3\, m_{19}
\end{aligned}
$$

2g) $(4, -3, 1, 0) \longrightarrow (3, -3, -1, -1)$

$$(\eta_1\eta_3)^2\,(m_{13} + m_{16}) + 2\eta_1\cdot\eta_1\eta_3\, m_1 - 2\eta_3\cdot\eta_1\eta_3\, m_4 + 2\eta_1\eta_2\cdot\eta_1\eta_3\, m_2 + 2\eta_1\eta_3\cdot\eta_2\eta_3\, m_6$$

13 Singular vectors for $\mathfrak{g} = \mathfrak{vas}(4|4)$

We consider the following negative operators from $\mathfrak{g}_0$:

$$
\begin{array}{ll}
a_5 = x_2\delta_3 + x_3\delta_2 & a_{14} = -x_4\partial_1 + \xi_1\delta_4\\
a_6 = x_3\delta_3 & a_{15} = -2x_3\delta_4 - \xi_1\partial_2 + \xi_2\partial_1\\
a_8 = x_2\delta_4 + x_4\delta_2 & a_{18} = -x_3\partial_2 + \xi_2\delta_3\\
a_9 = x_3\delta_4 + x_4\delta_3 & a_{19} = -x_4\partial_2 + \xi_2\delta_4\\
a_{10} = x_4\delta_4 & a_{20} = 2x_2\delta_4 - \xi_1\partial_3 + \xi_3\partial_1\\
a_{12} = -x_2\partial_1 + \xi_1\delta_2 & a_{25} = -x_4\partial_3 + \xi_3\delta_4\\
a_{13} = -x_3\partial_1 + \xi_1\delta_3 & a_{26} = -2x_2\delta_3 - \xi_1\partial_4 + \xi_4\partial_1
\end{array}
$$

For the basis of Cartan subalgebra we take

$$
\begin{array}{l}
a_{11} = -\frac{1}{2}x_1\partial_1 + \frac{1}{2}x_2\partial_2 + \frac{1}{2}x_3\partial_3 + \frac{1}{2}x_4\partial_4 + \xi_1\delta_1)\\
a_{17} = \frac{1}{2}x_1\partial_1 - \frac{1}{2}x_2\partial_2 + \frac{1}{2}x_3\partial_3 + \frac{1}{2}x_4\partial_4 + \xi_2\delta_2\\
a_{24} = \frac{1}{2}x_1\partial_1 + \frac{1}{2}x_2\partial_2 - \frac{1}{2}x_3\partial_3\frac{1}{2}x_4\partial_4 + \xi_3\delta_3\\
a_{32} = \frac{1}{2}x_1\partial_1 + \frac{1}{2}x_2\partial_2 + \frac{1}{2}x_3\partial_3 - \frac{1}{2}x_4\partial_4 + \xi_4\delta_4)
\end{array}
$$

The m_i are the following elements of the irreducible $\mathfrak{g}_0$-module V:

$$
\begin{array}{ll}
m_1 \text{ is the highest weight vector} & m_{10} = a_8\cdot m_1\\
m_2 = a_5\cdot m_1 & m_{11} = a_{20}\cdot m_1\\
m_3 = a_{25}\cdot m_1 & m_{24} = a_{26}\cdot a_{20}\cdot m_1\\
m_4 = a_{26}\cdot m_1 & m_{27} = -\, a_{12}\cdot m_1\\
m_8 = a_{25}\cdot a_{26}\cdot m_1 &
\end{array}
$$

Theorem In $I(V)$, there are only the following singular vectors:

1a) $(k, l, l, l) \longrightarrow (k+1, l, l, l)$: $\delta_1 \otimes m_1$;

1b) $(-1, 0, 0, 0) \longrightarrow \frac{1}{2}(-1, 1, 1, -1)$: $\partial_4 m_1 + \delta_1 m_4$

1c) $\frac{1}{2}(-1,1,1,-1) \longrightarrow (0,1,0,0)$: $\qquad \partial_3 m_1 - \partial_4 m_3 + \delta_1 m_{11}$

1d) $(l,k+l,l,l) \longrightarrow (l,k+l+1,l,l)$; two particular cases:

1da) $l \neq 0 \Longrightarrow k \neq -1$:

$-\partial_3\,(4\,l\,m_2 + m_4) - \partial_4\,(-\,m_8 + 4\,l\,m_{10}) + \delta_1\,(m_{24} - 4\,l\,m_{27}) - 4\,(1+k)\,l\,\delta_2 m_1$

1db) $l = 0 \Longrightarrow k \neq 0$: $\qquad \partial_3 m_2 + \partial_4 m_{10} + \delta_1 m_{27} + k\,\delta_2 m_1$

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PS. After the deadline for submission to the Marinov Memorial Volume there appeared a preprint[21] with some singular vectors for $\mathfrak{mb}(3|8)$ and $\mathfrak{mb}(5|10)$ (finite dimensional fibers). We use the opportunity to add the reference; comparison will be done elsewhere.

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