SINGLE STATE SUPERMULTIPLET IN 1+1 DIMENSIONS

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We consider multiplet shortening for BPS solitons in $\mathcal{N}=1$ two-dimensional models. Examples of the single-state multiplets were established previously in $\mathcal{N}=1$ Landau-Ginzburg models. The shortening comes at a price of loosing the fermion parity $(-1)^F$ due to boundary effects. This implies the disappearance of the boson-fermion classification resulting in abnormal statistics. We discuss an appropriate index that counts such short multiplets.

A broad class of hybrid models which extend the Landau-Ginzburg models to include a nonflat metric on the target space is considered. Our index turns out to be related to the index of the Dirac operator on the soliton reduced moduli space (the moduli space is reduced by factoring out the translational modulus). The index vanishes in most cases implying the absence of shortening. In particular, it vanishes when there are only two critical points on the compact target space and the reduced moduli space has nonvanishing dimension.

We also generalize the anomaly in the central charge to take into account the target space metric.

Contents

1 Introduction

The minimal supersymmetry, $\mathcal{N}=1$, in two-dimensional field theories has two supercharges Q_α ($\alpha = 1, 2$) which form a Majorana spinor in 1+1 dimensions.[a] The centrally extended superalgebra contains a central charge $\mathcal{Z}$ which has a topological meaning.[1] Solitons can be defined as states with nonvanishing $\mathcal{Z}$. Critical, or BPS (Bogomol'nyi-Prasad-Sommerfield) saturated solitons are such that their mass M and the corresponding central charge $\mathcal{Z}$ are rigidly related,

$$M - |\mathcal{Z}| = 0\,. \tag{1}$$

The BPS saturated solitons preserve 1/2 of the original SUSY: one of two supercharges annihilates the soliton state, while the remaining supergenerator acts nontrivially. The irreducible representation of the superalgebra in this situation is one-dimensional, i.e., the supermultiplet consists of a single state.[2–5] The shortening of the supermultiplet protects the relation (1) against small variations of parameters and quantum corrections.

If such supershort multiplet does exist (and is not accompanied by another short multiplet), the fermion-boson classification, i.e. $(-1)^F$, is lost. In this paper we address this particular issue: whether or not such single-state supermultiplets are dynamically realized. We consider various models in the weak coupling regime using the quasiclassical approach. In this approach the signature of the problem is an odd number of the fermion zero modes. In the simplest case we deal just with one fermion zero mode (one fermion modulus) produced by the action of the remaining supercharge.

Let us elucidate in somewhat more detail the algebraic aspect. The centrally extended $\mathcal{N}=1$ superalgebra has the form

$$\{Q_\alpha\,, \bar{Q}_\beta\} = 2\left(\gamma^\mu P_\mu + \gamma^5 \mathcal{Z}\right)_{\alpha\beta}\,, \qquad [Q_\alpha, P_\mu] = 0\,, \tag{2}$$

where $\bar{Q}_\beta = Q_\alpha (\gamma^0)_{\alpha\beta}$ and

$$\gamma^0 = \sigma_2\,, \qquad \gamma^1 = i\sigma_3\,, \qquad \gamma^5 = i\gamma^0\gamma^1 = -i\sigma_1 \tag{3}$$

are purely imaginary two-by-two matrices.

The states for which $\mathcal{Z} \neq 0$ are called *solitons*. The sign of the central charge $\mathcal{Z}$ differentiates between *solitons*, $\mathcal{Z} > 0$, and *antisolitons*, $\mathcal{Z} < 0$. This definition refers to both, the ground states and excitations. The algebra (2) implies that $P_\mu P^\mu \geq \mathcal{Z}^2$. The solitons which saturate this lower bound, $P_\mu P^\mu = \mathcal{Z}^2$, are called the *BPS states*. They are the ground states in the

[a] By $\mathcal{N} = 1$ we mean what is often denoted in the literature as $(1, 1)$ SUSY, with two supercharges.

soliton sector. In the BPS soliton rest frame, where $P_\mu = \{\mathcal{Z}, 0\}$, it is Q_2 that annihilates the soliton, $Q_2|\text{sol}\rangle = 0$. Correspondingly, in the soliton sector the algebra reduces to

$$(Q_1)^2 = 2\,\mathcal{Z}\,, \qquad (Q_2)^2 = 0\,, \tag{4}$$

so there is only one supercharge Q_1 which is realized nontrivially. In the semiclassical soliton construction it is just this supercharge that generates the fermion zero mode. The irreducible representation of superalgebra consists of a single state (the representation is one-dimensional), and the action of the supercharge reduces to multiplication by a number,

$$Q_1\,|\,\text{sol}\rangle = \pm\sqrt{2\mathcal{Z}}\,|\,\text{sol}\rangle\,. \tag{5}$$

The choice of a particular sign in the irreducible representation breaks Z_2 symmetry which reverses the sign of the fermionic operators, $Q_\alpha \to -Q_\alpha$. Such unusual representation implies that decomposition into bosons and fermions (the standard Z_2 grading) is broken. There are surprising manifestations of this phenomenon. For instance, in analyzing statistics of such solitons one finds[6–9] that an effective multiplicity (defined through the entropy per soliton in the ideal soliton gas) is $\sqrt{2}$ instead of 1.

Note that the problem is not specific for supersymmetric models, it appears always when the total number of the fermion zero modes is odd. In a more general context, the problem exists even in quantum mechanics, when the Clifford algebra has odd number of generators. In fact, an example of such algebra is known since long: it is the Grassmannian description of nonrelativistic spin 1/2 discovered by Berezin and Marinov.[10] These authors introduced three Grassmann variables, ξ_k $(k = 1, 2, 3)$ which were quantized by anticommutators,

$$\{\xi_k\,, \xi_l\} = \delta_{kl}\,. \tag{6}$$

The irreducible representations of the above algebra are two-dimensional. For instance, one can choose $\xi_k = \sigma_k/\sqrt{2}$. There exists a unitary nonequivalent choice, $\xi_k = -\sigma_k/\sqrt{2}$. The Hamiltonian H depends on ξ only through the spin operators, $S_i = -(i/2)\,\epsilon_{ikl}\,\xi_k\,\xi_l$, which are bilinear in ξ_k. Therefore, the change of the sign of ξ_k looks as a classical Z_2 symmetry of the problem.

At the quantum level, however, this symmetry is not realized: there is no operator G such that

$$G^2 = 1\,, \qquad GHG = H\,, \qquad G\xi_k G = -\xi_k\,. \tag{7}$$

Alternatively, the breaking of Z_2 could be seen from the following. The operator $i\xi_1\xi_2\xi_3$ commutes with all ξ_i and its square is equal to 1. Therefore, it can be realized as ± 1. The choice of a particular sign breaks Z_2, the symmetry that interchanges the signs. The absence of Z_2 implies the breakdown of the fermion-boson classification: the operator trilinear in fermions is a number.

Returning to supersymmetric field theories in two dimensions, let us note that the issue of the multiplet shortening and the related loss of the fermion parity, $(-1)^F$, has a long and sometimes confusing history. Probably, the first encounter with the problem occurred in the context of integrable two-dimensional models, in particular, in the supersymmetric sine-Gordon model,[6] and in the tricritical Ising model, field-theoretical limit of which is the Landau-Ginzburg model with a polynomial superpotential.[7] Even earlier the question was discussed[11] in the framework of the Gross-Neveu model which at $N = 3$ is equivalent to the supersymmetric sine-Gordon.

A certain clash was apparent in these considerations: on the one hand, the existence of $(-1)^F$ was taken for granted,[b] implying the absence of the single-state supermultiplets. On the other hand, as was already mentioned, an abnormal multiplicity factor $\sqrt{2}$ was discovered through an entropy calculation. This unusual factor was recently discussed anew by Witten[8] and Fendley and Saleur.[9]

The approach used in the above works was based on the exact S-matrices and the thermodynamic Bethe *ansatz*. Close to this is the approach based on massive deformations of conformal theories.[12] Another line of development is based on the quasiclassical analysis at weak coupling. The main goal was the calculation of quantum corrections to supersymmetric soliton masses, that would generalize the textbook calculations of Ref. 13 done in nonsupersymmetric models. In recent years the interest to this issue was revived by P. van Nieuwenhuizen and collaborators.[14,15] These works were the starting point for investigations of the MIT group[16] and ours.[2,3,5]

Already in Ref. 2 it was mentioned that the irreducible representation of the superalgebra (2) consists of a single state, i.e. the representation is one-dimensional. This issue was not elaborated in detail in Ref. 2 although the assertion was in contradiction with the previous analysis of Ref. 15 where it was assumed that the global Z_2 symmetry (associated with $(-1)^F$) is maintained.

The fact that the irreducible one-dimensional multiplet is realized without doubling was explicitly demonstrated in Ref. 3 where the consideration starts from the $\mathcal{N}=2$ extended version of the model. As was shown in Ref. 17 in such $\mathcal{N}=2$ model the solitonic multiplet is shortened (i.e. it is two- rather than four-dimensional). A soft breaking of $\mathcal{N}=2$ to $\mathcal{N}=1$ was then introduced in Ref. 3. When the breaking parameter reaches some critical value, one of the

[b] In a recent private discussion E. Witten mentioned that he had changed his opinion as to the existence of $(-1)^F$ in the soliton sector.

two soliton states disappears from the physical spectrum[3] via the phenomenon of delocalization. Delocalization means that fields are not localized near the soliton center. The BPS state that remains localized is single. The existence of the supershort multiplets and the loss of $(-1)^F$ was emphasized in the first version of the present paper;[18] in Ref. 4 the authors came to the same conclusion.

In our recent publication[5] the weak coupling consideration was generalized to a wide class of models including those with nonflat target space. The present text combines (and extends) the first version of the paper in Ref. 18 with Ref. 5.

Let us now summarize our main points:

(1) We introduce a new index which counts supershort multiplets in $\mathcal{N}=1$ two-dimensional theories. Let us remind that the first SUSY index, $\mathrm{Tr}\,(-1)^F$, was introduced by Witten twenty years ago[19] to count the number of supersymmetric vacua. About ten years ago, Cecotti, Fendley, Intriligator and Vafa introduced[20] another index, $\mathrm{Tr}\,[F\,(-1)^F]$, counting the number of short multiplets in $\mathcal{N}=2$ theories in two dimensions. No index counting single-state multiplets in $\mathcal{N}=1$ theories in two dimensions was known. This is probably not surprising, since it was always assumed that $(-1)^F$ does exist.

(2) We will show that the appropriate index is $\{\mathrm{Tr}\,Q_1\}^2/2\mathcal{Z}$ — it vanishes for long multiplets and is equal to 1 for one-dimensional multiplets. If the value of this index does not vanish in the given $\mathcal{N}=1$ theory, short multiplets *do exist* with necessity.

(3) We consider a wide class of $\mathcal{N}=1$ hybrid models which include, along with a superpotential $\mathcal{W}(\phi)$, a nonflat metric $g_{ab}(\phi)$ of the target space for the fields ϕ^a. These are hybrids between the sigma models and the Landau-Ginzburg models. The result depends on the soliton moduli space. More exactly, what counts is a *reduced* moduli space, namely the moduli space with the translational modulus (corresponding to the motion of the center of inertia) factored out, together with its fermionic superpartner. The index we introduce, $\{\mathrm{Tr}\,Q_1\}^2/2\mathcal{Z}$, turns out to be a square of the index of the Dirac operator defined on the reduced moduli space.

(4) We find, with surprise, that the existence of the BPS solitons belonging to one-dimensional supermultiplets is quite a rare occasion. It happens only in the problems with a single modulus, translational. The reduced moduli space is then trivial. Although the index of Dirac operator on a generic compact space may be nonvanishing, the reduced moduli spaces arising in the hybrid models have a special geometry (similar to spherical) for which the index vanishes. If the reduced moduli space has dimension 1 or larger and is compact there are no BPS solitons.

(5) Another issue we address is the generalization of the anomaly in the central charge found previously in the Landau-Ginzburg models,[2] where the target space metric is flat, to the hybrid models with a nonflat metric. Our result for the anomaly in this case is a straightforward extension of Ref. 2 and can be formulated as a substitution

$$\mathcal{W}(\phi) \longrightarrow \widetilde{\mathcal{W}}(\phi) = \mathcal{W}(\phi) + \frac{1}{4\pi}\,\nabla^a\nabla_a\mathcal{W}(\phi) \tag{8}$$

for the superpotential. The quantum anomaly is represented by the second term, with the covariant Laplacian on the target space, $\nabla^a\nabla_a \equiv g^{ab}\nabla_a\nabla_b$. The anomaly-corrected superpotential enters into the energy-momentum tensor, the supercharges and the central charge. In particular, the operator of the central charge becomes

$$\mathcal{Z} = \widetilde{\mathcal{W}}(\phi(z\to\infty)) - \widetilde{\mathcal{W}}(\phi(z\to-\infty))\,. \tag{9}$$

The paper is organized as follows. In Sec. 2 we thoroughly discuss algebraic aspects. The relation between the multiplet shortening and the loss of $(-1)^F$ is explained. We also make a remark on short multiplets in 2+1 dimensions (Sec. 2.5). Section 3 treats $\operatorname{Tr} Q_1$ as an index. In Sec. 4 we begin a systematic consideration of various models. We start from a generic hybrid model combining the Ginzburg-Landau and sigma models, with $\mathcal{N}=1$ supersymmetry. The target space $\mathcal{T}$ is an arbitrary Riemann manifold. Section 4.1 presents generalities. In Sec. 4.2 we derive the quantum anomaly in the central charge for the hybrid model of the general form. Classification of the models is presented in Sec. 4.3. In Sec. 5 we treat particular examples with flat target space. Section 6 is devoted to a model in which both, the target space and the spatial coordinate are circles. Section 7 is devoted to nonflat target spaces. It is demonstrated that the index $\operatorname{Tr} Q_1$ is related to the index of the Dirac operator on the soliton moduli space. The main example here is S^3 as the target space where we find the index $\operatorname{Tr} Q_1$ to be vanishing. Correspondingly, all supermultiplets are long, there are no BPS solitons in this model. Sections 8 summarizes our findings and results. Appendix presents a proof that in any hybrid model the situation is similar: the reduced moduli space is such that the index of the Dirac operator on it (and, hence, $\operatorname{Tr} Q_1$) vanishes provided that the reduced moduli space is a compact manifold of positive dimension.

2 Representations of superalgebra

2.1 Automorphisms of superalgebra

Let us address the question: what extra symmetries are compatible with the centrally extended algebra (2)? It is clear that the Lorentz boost can be included,

$$
\begin{aligned}
&Q_\alpha \to \left[\exp\left(-\frac{i}{2}\,\beta\,\gamma^5\right)\right]_{\alpha\beta} Q_\beta\,, \qquad \bar{Q}_\alpha \to \left[\exp\left(\frac{i}{2}\,\beta\,\gamma^5\right)\right]_{\alpha\beta} \bar{Q}_\beta\,,\\
&P_\mu(\gamma^\mu)_{\alpha\beta} \to \left[\exp\left(-\frac{i}{2}\,\beta\,\gamma^5\right)\right]_{\alpha\gamma} \left[\exp\left(\frac{i}{2}\,\beta\,\gamma^5\right)\right]_{\beta\delta} P_\mu(\gamma^\mu)_{\gamma\delta}\,,\\
&\mathcal{Z} \to \mathcal{Z}\,,
\end{aligned}
\tag{10}
$$

where β is a real parameter (in our convention $i\gamma^5$ is Hermitean).

If the supercharges Q_α were complex, as in $\mathcal{N}=2$, an extra symmetry would emerge — in addition to the real β one could consider transformations with purely imaginary β. This symmetry would express the conservation of the fermion charge F. In $\mathcal{N}=1$ where the supercharges are real there is no fermion charge. What survives, however, is the fermion parity $G=(-1)^F$. The action of G (given by putting $\beta = 2i\,\pi$) reduces to changing the sign for the fermion operators leaving the boson operators intact,

$$
G\,Q_\alpha\,G^{-1} = -Q_\alpha\,, \qquad G\,P_\mu\,G^{-1} = P_\mu\,. \tag{11}
$$

The fermion parity G realizes Z_2 symmetry associated with changing the sign of the fermion fields. From this standpoint it seems that this symmetry is guaranteed. In fact we will show in some examples that in the soliton sector the very classification of states as either bosonic or fermionic is broken. In constructing representations of superalgebra we would not necessarily assume that the fermion parity $(-1)^F$ is the valid symmetry but the Lorentz symmetry is certainly assumed.

2.2 Beginning the construction

Now let us start the construction. The Lorentz symmetry implies that $P^\mu P_\mu$ is invariant, $P^\mu P_\mu = M^2$ where M is the mass of the state. In what follows we will treat M and $\mathcal{Z}$ as c-numbers characterizing each given irreducible representation. It is convenient to choose the rest frame where $P_\mu = (M, 0)$ and the algebra (2) takes the form

$$
(Q_1)^2 = M + \mathcal{Z}\,, \qquad (Q_2)^2 = M - \mathcal{Z}\,, \qquad \{Q_1\,, Q_2\} = 0\,. \tag{12}
$$

Positive definiteness leads to

$$M^2 \geq \mathcal{Z}^2 .$$

The analysis bifurcates at this point: one should consider separately the non-BPS cases, $M^2 > \mathcal{Z}^2$, and the BPS case, $M^2 = \mathcal{Z}^2$.

Let us note that for solitons for which $M - \mathcal{Z} \ll \mathcal{Z}$ we are in domain of nonrelativistic description (it is implied that $\mathcal{Z} > 0$). The quantity which plays the role of the Hamiltonian in the supersymmetric quantum mechanics of solitons is then $M - \mathcal{Z}$, where M should be understood as the operator $\sqrt{P^\mu P_\mu}$,

$$H_{\rm SQM} = M - \mathcal{Z} = (Q_2)^2 . \tag{13}$$

The relation $H_{\rm SQM} = (Q_2)^2$ is a subalgebra of the SUSY algebra which is stationary for the BPS states.[c]

In constructing $H_{\rm SQM}$ we separate dynamics of the center of mass, passing to the rest frame. The total spatial momentum P_z is conjugated to the center of mass coordinate, Q_1 is its superpartner. Both, P_z and Q_1 commute with $H_{\rm SQM}$. Their partnership becomes evident in the moduli dynamics example which will be considered in Sec. 7.1, zero modes generated by P_z and Q_1 have the same dependence on the coordinate z.

2.3 Non-BPS multiplets

If $M^2 \neq \mathcal{Z}^2$ Eq. (12) represents the Clifford algebra with two generators, its irreducible representation is two-dimensional. For instance, one can choose

$$Q_1 = \sigma_1 \sqrt{M + \mathcal{Z}} , \qquad Q_2 = \sigma_2 \sqrt{M - \mathcal{Z}} , \tag{14}$$

where $\sigma_{1,2}$ are the Pauli matrices.

There is an obvious automorphism: one can substitute σ_1, σ_2 by rotated matrices $\tilde\sigma_1 = \sigma_1 \cos\alpha + \sigma_2 \sin\alpha$ and $\tilde\sigma_2 = -\sigma_1 \sin\alpha + \sigma_2 \cos\alpha$. It means that the two-dimensional representation at hand admits introduction of the fermion number $F = (1 - \sigma_3)/2$. Generically, the fermion number operator F can be expressed in terms of the supersymmetry generators Q_α. First, let us consider a bosonic operator

$$S = -\frac{i}{2} \, [Q_1 \, , Q_2] = \frac{1}{2} \, \bar{Q} \, Q \, , \tag{15}$$

which is an element of the enveloping algebra. The expression after the second equality sign is not bound to the rest frame. This operator has the following

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[c] The construction (13) is called $\mathcal{N}=1/2$ quantum mechanics with $\mathcal{N}$ counting the number of *pairs* of supercharges that square to $H_{\rm SQM}$.

features:

$$
\begin{aligned}
&S^2 = P_\mu P^\mu - \mathcal{Z}^2\,, \quad \{S, Q_\alpha\} = 0\,,\\
&[S\,, Q_\alpha] = -2\,\left(\gamma^\mu P_\mu + \gamma^5 \mathcal{Z}\right)_{\alpha\beta} Q_\beta\\
&[S\,, P_\mu] = 0\,, \quad [S\,, \mathcal{Z}] = 0\,.
\end{aligned}
\tag{16}
$$

In the representation (14) the operator S has the form,

$$S = \sqrt{M^2 - \mathcal{Z}^2}\,\sigma_3\,. \tag{17}$$

This expression shows that the operator S can be introduced only for non-BPS representations, $M^2 \neq \mathcal{Z}^2$. Then $S/\sqrt{M^2 - \mathcal{Z}^2}$ is a generator of SO(2) rotations, associated with the fermion charge F,

$$\frac{S}{\sqrt{M^2 - \mathcal{Z}^2}} = 1 - 2\,F\,, \qquad F = \frac{1}{2} - \frac{\bar{Q}Q}{4\,\sqrt{M^2 - \mathcal{Z}^2}}\,. \tag{18}$$

The fact that S is bilinear in Q_α results in $F^2 = F$, i.e. the operator F acts as a projection operator. Its eigenvalues are 0 and 1 and it measures the number of fermions modulo two. It means that the very same operator S defines also the fermion parity G of state

$$G = (-1)^F = \frac{\bar{Q}Q}{2\,\sqrt{M^2 - \mathcal{Z}^2}}\,. \tag{19}$$

Let us emphasize that in the $\mathcal{N}=1$ models there is no local current associated with the fermion charge. Therefore, the fermion charge we have introduced has no local representation. A local current does exists in the case of extended $\mathcal{N}=2$ supersymmetry. The corresponding fermion charge is different from F defined in Eq. (18). It is known that the fermion charge defined by the local current is noninteger for solitons[17] (the fractional fermion charge of the soliton was discovered by Jackiw and Rebbi[21]). At the same time the fermion charge (18) is always integer. In the topologically trivial one-particle sector of $\mathcal{N}=2$ theories both fermion charges coincide.

Note an analogy between the introduction of the operator S above, in constructing representations of the superalgebra, and the introduction of the Pauli-Lubanski spin operator $\Gamma^\mu = \epsilon^{\mu\nu\gamma\delta} M_{\nu\gamma} P_\delta$ for the Poincaré group. In this case there is no local current too, and the Pauli-Lubanski operator is not defined for massless particles, $P_\sigma P^\sigma = 0$. Let us emphasize once more that S vanishes for the BPS states.

2.4 BPS representations

Now let us consider the special case $M^2 = \mathcal{Z}^2$. Assuming $\mathcal{Z}$ to be positive we consider the BPS soliton, $M = \mathcal{Z}$, for which the supercharge Q_2 is trivial, $Q_2 = 0$. Thus, we are left with a single supercharge Q_1 realized nontrivially. The algebra reduces to a single relation

$$(Q_1)^2 = 2\,\mathcal{Z}\,. \tag{20}$$

The irreducible representations of this algebra are one-dimensional, there are two such representations,

$$Q_1 = \pm\sqrt{2\mathcal{Z}}\,, \tag{21}$$

i.e., two types of solitons,

$$Q_1|\,\mathrm{sol}_+\,\rangle = \sqrt{2\mathcal{Z}}\,|\,\mathrm{sol}_+\,\rangle\,, \qquad Q_1|\,\mathrm{sol}_-\,\rangle = -\sqrt{2\mathcal{Z}}\,|\,\mathrm{sol}_-\,\rangle\,. \tag{22}$$

It is clear that these two representations are unitary nonequivalent.

The one-dimensional irreducible representation implies multiplet shortening: the short BPS supermultiplet contains only one state while non-BPS supermultiplets contain two. However, the possibility of such supershort one-dimensional multiplets is usually discarded. It is for a reason: while the fermion parity $(-1)^F$ is granted in any local field theory based on fermionic and bosonic fields, it is not defined in the one-dimensional irreducible representation. Indeed, if it were defined, it would be -1 for Q_1, which is incompatible with any of the equations (22). The only way to recover $(-1)^F$ is to have a reducible representation containing both $|\,\mathrm{sol}_+\,\rangle$ and $|\,\mathrm{sol}_-\,\rangle$. Then,

$$Q_1 = \sigma_3\sqrt{2\mathcal{Z}}\,, \qquad (-1)^F = \sigma_1\,. \tag{23}$$

Does it mean that the one-state multiplet is not a possibility in the local field theory? It was argued in Refs. 2,3,4 (and we are going to review this again in Sec. 5) that solitons in certain models do realize such supershort multiplets indeed defying $(-1)^F$.

Thus, for the BPS representations we come to two scenarios:

- **(i)** Fermion parity $(-1)^F$ is broken and the irreducible representation is realized. The supermultiplet is short, contains only one state.
- **(ii)** Fermion parity $(-1)^F$ is not broken, the representation is reducible. The multiplet of degenerate states is not short, containing bosonic and fermionic components.

The important point is that only short multiplets of the BPS states are protected against becoming non-BPS under small perturbations. It is clearly not the case in the scenario **(ii)**. This leads us to introduction of a new index, $(\mathrm{Tr}\, Q_1)^2$, which counts short multiplets with broken $(-1)^F$. This index will be carefully defined in Sec. 3.

The discrete Z_2 symmetry $(-1)^F$ discussed above is nothing but the change of sign of all fermion fields, $\psi \to -\psi$. This symmetry is seemingly present in any theory with fermions. How this symmetry can be lost in the soliton sector will be explained later. Here we would like to mention the following. Although the overall sign of Q_1 on the irreducible representation is not observable, the relative sign is. For instance, there are two types of reducible representations of dimension two: one is $\{+,-\}$ (see Eq. (23)), and another $\{+,+\}$ (equivalent to $\{-,-\}$). Our index $(\mathrm{Tr}\, Q_1)^2/2\mathcal{Z}$ discriminates between these two cases — it vanishes in the first case and equals to 4 in the second. Another example of dimension three is the reducible representation $\{+,+,-\}$. The index is 1 in this case, implying that the pair $\{+,-\}$ can leave the BPS bound leaving a single BPS state.

2.5 Massless supermultiplets in 2+1 dimensions

The SUSY algebra (2) with the central charge $\mathcal{Z}$ we have considered in 1+1 dimensions becomes identical to the superalgebra in 2+1 dimensions (without central extension) provided one identifies the central charge $\mathcal{Z}$ with the momentum P_2 in the extra spatial dimension. Indeed, after this identification the algebra (2) can be rewritten as

$$\{Q_\alpha\,, \bar{Q}_\beta\} = 2\left(\gamma^M\right)_{\alpha\beta} P_M\,, \qquad (M=0,1,2)\,, \tag{24}$$

where γ^2 coincides with γ^5 from Eq. (3).

The one-dimensional representation we have constructed for the BPS states, $P_\mu P^\mu = \mathcal{Z}^2$, in 1+1 dimensions, in 2+1 becomes a representation $|P_1, P_2\rangle$ for the massless particle, $P_M P^M = 0$. Assume that we choose a Lorentz frame where $P_M = (E, 0, E)$. In this frame the supercharges are represented as

$$Q_1 = \pm\sqrt{2E}\,, \qquad Q_2 = 0\,, \tag{25}$$

cf. Eq. (21). Again, although irreducible representations are one-dimensional, maintaining $(-1)^F$ makes the representation two-dimensional and reducible, see e.g. Ref. 22.

Is it possible to break $(-1)^F$ in 2+1D similarly to solitons in 1+1? We are aware of no dynamical example of this type.[d] For example, in the free massless supersymmetric theory there are two states of the $|\pm\rangle$ type,

$$|P_1 = 0, P_2 = E\rangle_\pm = \int \mathrm{d}^2x\, \mathrm{e}^{i\vec{P}\vec{x}} \left(\sqrt{2E}\,\phi \pm \psi_1\right)|0\rangle\,, \tag{26}$$

which are irreducible representations. Here ϕ and ψ_α are free field operators and we chose $P_1 = 0$. Together they form the reducible representation for which $(-1)^F$ is well defined.

Note that models where the breaking of $(-1)^F$ may occur, emerge for domain wall junctions in 3+1 dimensions.[23] In these models the junctions effectively reduce to 1+1 dimensional objects.

3 Tr Q_1 as index

In the context of supersymmetric theories Tr $(-1)^F$ as an index was introduced by Witten. The Witten index counts the difference between the numbers of the bosonic and fermionic states of zero energy, i.e. vacua which are annihilated by supercharges. For all supermultiplets with nonzero energy this difference vanishes.

In particular, it vanishes in the soliton sector in $\mathcal{N}=2$ two-dimensional theories. However, the BPS solitons are annihilated by a part of supercharges and form short multiplets. This is counted[24] by the Cecotti-Vafa-Fendley-Intriligator index Tr $F\,(-1)^F$. The fermion number F is well defined in $\mathcal{N}=$ 2 theories.

In $\mathcal{N} = 1$ theories the fermion number F, and even the fermion parity $(-1)^F$, are not defined for short multiplets. Is there an index in the $\mathcal{N}=$ 1 soliton problems which would count the supershort multiplets? We assert that $\{\,\mathrm{Tr}\,Q_1\}^2$ does the job. More exactly, the definition of the index is as follows

$$\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1) = \frac{1}{2\,\mathcal{Z}}\left\{\lim_{\beta\to\infty} \mathrm{Tr}\left[Q_1\,\exp(-\beta\,(Q_2)^2)\right]\right\}^2 . \tag{27}$$

The exponential factor in Eq. (27) is introduced for the UV regularization. The necessity of taking the $\beta \to \infty$ limit is due to continuous spectrum, as explained in Ref. 20.

This index vanishes for non-BPS multiplets for which the fermion parity $(-1)^F$ can be consistently defined. Equation (14) demonstrates this explicitly.

[d] Of course, here we speak of spatially localized states, rather then of extended objects which can appear as an elevation from 1+1D.

For each irreducible BPS representation the index is unity,

$$\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1)\,[\,\text{irreducible BPS}\,] = 1\,.$$

If a reducible representation contains a few irreducible BPS multiplets the index may or may not vanish depending on the numbers of the $\{+\}$ and $\{-\}$ irreps. In the case of the vanishing index one can introduce $(-1)^F$, and small deformations or quantum corrections can destroy the BPS saturation.

Note, that our index is not additive: it is not equal to the sum of the indices of the irreducible representations. An interesting example of an $\mathcal{N}=1$ reducible BPS representation is provided by solitons in $\mathcal{N}=2$ models. The $\mathcal{N}=2$ BPS multiplet consists of two $\mathcal{N}=1$ multiplets of the opposite types, leading to the vanishing of the index (27).

The definition (27) has a technical drawback — it refers to the soliton rest frame. It is simple to make it Lorentz invariant,

$$\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1) = \frac{1}{2\,\mathcal{Z}^2}\left(\mathrm{Tr}\,\bar{Q}\right) \not{P} \left(\mathrm{Tr}\,Q\right)\,, \tag{28}$$

where the trace refers to the Hilbert space but not to the Lorentz indices of the supercharges Q_α and $\bar{Q}_\alpha = Q_\beta(\gamma^0)_{\beta\alpha}$. Here we have omitted the regularizing exponent.

4 Theories with $\mathcal{N}=1$ supersymmetry in 1+1 dimensions

In this section we consider a generic $\mathcal{N}=1$ field theory in 1+1 dimensions, presenting a realization of SUSY with two real supercharges Q_α and the central charge, see Eq. (2). The two-dimensional space $x^\mu=(t,z)$ is flat. The time $t\in R$ while the spatial coordinate z lives either on the line R (noncompact), or on the circle S^1 (compact). We will deal with n superfields

$$\Phi^a = \phi^a + \bar\theta\psi^a + \frac{1}{2}\bar\theta\theta F^a\,, \qquad (a=1,\dots,n)\,. \tag{29}$$

Each superfield contains a real boson field ϕ^a, a two-component Majorana spinor ψ^a_α ($\alpha=1,2$) and an auxiliary field F^a. The target space formed by ϕ^a is an arbitrary Riemann manifold $\mathcal{T}$ endowed with the metric $g_{ab}(\phi)$. Moreover, we introduce a superpotential $\mathcal{W}(\phi)$. Thus, the generic model is a hybrid between the σ-model and the Landau-Ginzburg theory.

The generic form of the Lagrangian is (for a review see Ref. 25)

$$\begin{aligned}\mathcal{L} \;&=\; \frac{1}{2}\,g_{ab}\left[\partial_\mu\phi^a\,\partial^\mu\phi^b + \bar\psi^a\,i\gamma^\mu\mathcal{D}_\mu\psi^b + F^aF^b\right] + \frac{1}{12}\,R_{abcd}\,(\bar\psi^a\psi^c)(\bar\psi^b\psi^d)\\ &+\; F^a\partial_a\mathcal{W} - \frac{1}{2}\,(\nabla_a\partial_b\mathcal{W})\,\bar\psi^a\psi^b\,,\end{aligned} \tag{30}$$

where $\bar{\psi} = \psi^T \gamma^0$, and $\Gamma^b_{c\,d}(\phi)$ and $R_{abcd}(\phi)$ are the Christoffel symbols and the Riemann tensor, respectively. Furthermore, $\mathcal{W}(\phi)$ is the superpotential, and ∂_a and ∇_a denote usual and covariant derivatives in the target space, e.g.,

$$\mathcal{W}_{,\,a} = \nabla_a \mathcal{W} = \partial_a \mathcal{W}\,; \qquad \nabla_a \mathcal{W}_{,b} = \partial_a \mathcal{W}_{,b} - \Gamma^c_{ab}\, \mathcal{W}_{,c}\,, \tag{31}$$

while the covariantized space-time derivative $\mathcal{D}_\mu$ is

$$\mathcal{D}_\mu\, \psi^b = \frac{\partial\, \psi^b}{\partial x^\mu} + \Gamma^b_{cd}\, \frac{\partial\, \phi^c}{\partial x^\mu}\, \psi^d\,. \tag{32}$$

The Lagrangian (30) implies that the auxiliary field $F^a = -g^{ab}\partial_b \mathcal{W}$.

4.1 SUSY, central charge and BPS saturation

The $\mathcal{N}{=}1$ supersymmetry of the model is expressed by two supercharges,

$$Q_\alpha = \int \mathrm{d}z\, S^0_\alpha\,, \qquad S^\mu = g_{ab}\, (\partial\!\!\!/\, \phi^a - i\, F^a)\, \gamma^\mu\, \psi^b\,, \tag{33}$$

where S^μ is the conserved supercurrent. These supercharges form the $\mathcal{N}{=}1$ algebra (2) with the metric independent central charge

$$\mathcal{Z} = \int \mathrm{d}z\, \partial_z \phi^a\, \partial_a \mathcal{W}\,. \tag{34}$$

The central charge does not vanish for classical solitons interpolating between different vacua of the theory. These vacua correspond to critical points of the superpotential $\mathcal{W}(\phi)$ at which $\partial_a \mathcal{W} = 0$. For the soliton interpolating between critical points $\phi = A$ and $\phi = B$ the central charge is equal to

$$\mathcal{Z}_0 = \Delta\mathcal{W} = \mathcal{W}(A) - \mathcal{W}(B)\,, \tag{35}$$

where we assume, by convention, that $\mathcal{Z} > 0$. Certainly, the inverse interpolation (antisoliton) with negative $\mathcal{Z}$ also exists.

For the BPS saturated soliton and antisoliton their masses are equal to $|\mathcal{Z}|$. The BPS soliton configuration ϕ_0 satisfies the following equation:

$$\frac{\mathrm{d}\phi^a_0}{\mathrm{d}z} = g^{ab}\, \partial_b\, \mathcal{W}(\phi_0)\,. \tag{36}$$

4.2 Ultraviolet aspects and quantum anomaly

The expressions (33), (34) and (35) above are obtained at the classical level. If the target space manifold $\mathcal{T}$ is flat, i.e., we deal with the Landau-Ginzburg model, the theory is superrenormalizable: logarithmic divergences appear only at one loop. For generic nonflat $\mathcal{T}$ the theory is nonrenormalizable: as well known, loops will generate an infinite series of new structures in the target space metric. However, if $\mathcal{T}$ is symmetric, the number of structures is finite, and the theory is renormalizable: divergences can be absorbed into a finite number of parameters.

Already in the superrenormalizable Landau-Ginzburg models (where $g_{ab} = \delta_{ab}$) loop corrections lead to a quantum anomaly.[2] The anomaly occurs in the energy-momentum tensor, supercurrent and the central charge density. In fact, these anomalies form a supermultiplet. Moreover, the impact of the anomaly is local and universally expressed through the substitution

$$\mathcal{W} \longrightarrow \widetilde{\mathcal{W}} = \mathcal{W} + \frac{1}{4\pi}\,\partial_a\partial_a\mathcal{W}\,. \tag{37}$$

It is clear, then, that a similar anomaly must occur in the hybrid models as well. Below we present its form for the generic model. Note, first, that the distinction between the superrenormalizable Landau-Ginzburg models and renormalizable hybrid models is irrelevant for the analysis of the one loop anomaly. The form of the anomaly is severely limited by the following considerations: (i) dimension and locality; (ii) general covariance in the target space; (iii) the flat metric limit. The one-loop calculation presented in Ref. 2 can be readily extended to include the target space metric and leads to

$$\mathcal{W} \longrightarrow \widetilde{\mathcal{W}} = \mathcal{W} + \frac{1}{4\pi}\,g^{ab}\,\nabla_a\nabla_b\mathcal{W}\,. \tag{38}$$

This differs from the Landau-Ginzburg case only by covariantization of the Laplacian,

$$\delta^{ab}\partial_a\partial_b\mathcal{W} \to g^{ab}\,\nabla_a\nabla_b\mathcal{W} = g^{-1/2}\,\partial_a\,g^{1/2}g^{ab}\partial_b\,\mathcal{W}\,. \tag{39}$$

The corrected superpotential $\widetilde{\mathcal{W}}$ should be substituted into the expressions for the energy-momentum tensor, supercurrent and the central charge.

In particular, the central charge $\mathcal{Z}$ becomes

$$\mathcal{Z} = \widetilde{\mathcal{W}}(A) - \widetilde{\mathcal{W}}(B)\,. \tag{40}$$

A novel feature compared with the flat target space is the occurrence of a metric dependence. Let us remind that at the loop level the metric "runs". It is this running metric that enters the anomaly (38). As a result, the anomaly which was one-loop in the flat target space[2] becomes multi-loop.

It is worth singling out interesting cases of superpotentials which are eigenfunctions of the covariant Laplacian,

$$g^{ab}\nabla_a\nabla_b\mathcal{W} = c\mathcal{W}\,. \tag{41}$$

The simplest example of this type is provided by the sine-Gordon model, $\mathcal{W} = mv^2\sin(\phi/v)$ (see Sec. 5). The impact of the anomaly is the following replacement of the classical central charge $\mathcal{Z}_0$:

$$\mathcal{Z} = \mathcal{Z}_0\left[1 - \frac{1}{4\pi v^2}\right]\,.$$

The equivalent replacement $v^2 \to v^2 - (1/4\pi)$ is well-known in the framework of the CFT treatment of the integrable models.[12]

Another similar example is given by the S^3 model of Sec. 7.1. In this model the superpotential satisfies Eq. (41) with $c = -3f$ where f is the running coupling constant.[e] The anomaly shifts $\mathcal{Z}_0$,

$$\mathcal{Z} = \mathcal{Z}_0\left(1 - \frac{3f}{4\pi}\right)\,. \tag{42}$$

Equation (42) implies that the combination on the right-hand side is renormalization-group invariant.

4.3 Classification of models

The models to be considered in this paper fall into several distinct categories characterized by the following features:

(i) The geometry of the spatial coordinate z: compact versus noncompact (in both cases the metric is flat for one spatial coordinate);

(ii) The geometry of the target manifold $\mathcal{T}$: compact versus noncompact.

(iii) The metric of $\mathcal{T}$: flat versus nonflat.

Our examples represent almost all combinations of the classes above. We start with the flat target space. In this case the generic classical Lagrangian (30) takes the form

$$\mathcal{L} = \frac{1}{2}\left\{\partial_\mu\phi^a\,\partial^\mu\phi^a + i\bar\psi^a\gamma^\mu\partial_\mu\psi^a - \frac{\partial\mathcal{W}}{\partial\phi^a}\,\frac{\partial\mathcal{W}}{\partial\phi^a} - \frac{\partial^2\mathcal{W}}{\partial\phi^a\partial\phi^b}\,\bar\psi^a\psi^b\right\}\,. \tag{43}$$

[e] At one-loop level, $1/f(\mu) = 1/f_0 - (1/\pi)\,\ln(\mu_0/\mu)$.

Some of such models are known to be exactly integrable.[6,12,17] However, following Refs. 2, 3, we will limit our consideration to the quasiclassical regime assuming that the expansion parameter is small.

5 Flat target manifold, noncompact space

In this section we consider the $\mathcal{N}=1$ Landau-Ginzburg models (43) with one or two superfields with the spatial coordinate $z \in R$.

5.1 One-superfield models

Our presentation in this section follows Ref. 2. Although the superpotential $\mathcal{W}(\phi)$ can be arbitrary, for classification purposes we will discuss two representative examples: the polynomial (PM) model,

$$\mathcal{W}_{\rm PM}(\phi) = \frac{m^2}{4\lambda}\,\phi - \frac{\lambda}{3}\,\phi^3\,, \tag{44}$$

and the sine-Gordon (SG) model,

$$\mathcal{W}_{\rm SG}(\phi) = mv^2 \sin\frac{\phi}{v}\ . \tag{45}$$

The target space is noncompact in the PM case and compact S^1 in the SG model. The classical BPS equation

$$\frac{{\rm d}\phi_0}{{\rm d}z} = \mathcal{W}'(\phi_0)\,, \tag{46}$$

has the following solutions:

$$\begin{aligned} \phi_0 &= \tfrac{m}{2\lambda}\tanh\tfrac{m\,z}{2}\,, && \text{(PM)} \\ \phi_0 &= v\,\arcsin[\tanh(mz)]\,, && \text{(SG)} \end{aligned} \tag{47}$$

interpolating between two neighboring vacua.

For infrared regularization the system is placed in a large spatial box, i.e., the boundary conditions at $z = \pm L/2$ are imposed. The conditions we choose are

$$\begin{aligned} &[\partial_z\phi - \mathcal{W}'(\phi)]_{z=\pm L/2} = 0\,, \qquad \psi_1|_{z=\pm L/2} = 0\,, \\ &[\partial_z - \mathcal{W}''(\phi)]\,\psi_2|_{z=\pm L/2} = 0\,, \end{aligned} \tag{48}$$

where $\psi_{1,2}$ denote the components of the spinor ψ_α. The first line is nothing but a supergeneralization of the BPS equation, $D_1\Phi(t, z = \pm L/2, \theta) = 0$ at the boundary. The second line is the consequence of the Dirac equation of motion, if ψ satisfies the Dirac equation there are essentially no boundary conditions for ψ_2 . Therefore, it is not an independent boundary condition in the solution of the classical equations of motion. We will use these boundary conditions later for the construction of modes in the differential operators of the second order.

The above choice is particularly convenient because it is compatible with the residual supersymmetry in the presence of the BPS soliton. The boundary conditions (48) are consistent with the classical solutions, both for the spatially constant vacuum configurations and for the kink. In particular, the soliton solution ϕ_0 of Eq. (47) satisfies $\partial_z\phi - \mathcal{W}' = 0$ everywhere. Note that the conditions (48) are not periodic.

The next step is to introduce the expansion in modes for deviations from the soliton solution (47). For the mode expansion we use the second order Hermitean differential operators L_2 and $\tilde{L}_2$,

$$L_2 = P^\dagger P\,, \qquad \tilde{L}_2 = PP^\dagger\,, \tag{49}$$

where

$$P = \partial_z - \mathcal{W}''|_{\phi=\phi_0(z)}\,, \qquad P^\dagger = -\partial_z - \mathcal{W}''|_{\phi=\phi_0(z)}\,. \tag{50}$$

The operator L_2 defines the modes of $\chi \equiv \phi - \phi_0$, and those of the fermion field ψ_2, while $\tilde{L}_2$ does this job for ψ_1. The boundary conditions for $\psi_{1,2}$ are given in Eq. (48), for $\phi - \phi_0$ they follow from the expansion of the first condition in Eq. (48),

$$\left[\partial_z - \mathcal{W}''(\phi_0(z))\right] \chi|_{z=\pm L/2} = 0\,. \tag{51}$$

It is easy to verify that there is only one zero mode $\chi_0(z)$ for the operator L_2 which has the form,

$$\chi_0 \propto \frac{\mathrm{d}\phi_0}{\mathrm{d}z} \propto \mathcal{W}'|_{\phi=\phi_0(z)} \propto \begin{cases} \dfrac{1}{\cosh^2(mz/2)} & (\mathrm{PM})\,. \\[2ex] \dfrac{1}{\cosh{(mz)}} & (\mathrm{SG})\,. \end{cases} \tag{52}$$

This is the zero mode for the boson field χ (translational mode) and for fermion ψ_2 (supersymmetric mode).

The operator $\tilde{L}_2$ has no zero modes at all. Let us emphasize that the absence of the zero modes for $\tilde{L}_2$ is *not* because the solution of $\tilde{L}_2\,\tilde{\chi} = 0$

is non-normalizable (we keep the size of the box finite) but because of the boundary conditions $\tilde{\chi}(z = \pm L/2) = 0$.

The translational and supersymmetric zero modes discussed above imply that the soliton is described by two collective coordinates: its center z_0 and a "fermionic" center η,

$$\phi = \phi_0(z - z_0) + \text{nonzero modes}\,, \quad \psi_2 = \eta\,\chi_0 + \text{nonzero modes}\,, \tag{53}$$

where χ_0 is the normalized mode given by Eq. (52). The nonzero modes are those of the operator L_2. As for ψ_1 it is given by the sum over the nonzero modes of the operator $\tilde{L}_2$.

Substituting the mode expansion in the supercharges (33) we arrive at

$$Q_1 = 2\sqrt{\mathcal{Z}}\,\eta + \text{nonzero modes}\,, \qquad Q_2 = \sqrt{\mathcal{Z}}\,\dot{z}_0\,\eta + \text{nonzero modes}\,. \tag{54}$$

Now we can proceed to the quasiclassical quantization. Projecting the canonical equal-time commutation relations for the fields ϕ and ψ on the zero modes we get

$$[\,p, z_0] = -i\,, \qquad \eta^2 = \frac{1}{2}\,, \tag{55}$$

where $p = \mathcal{Z}\,\dot{z}_0$ is the canonical momentum conjugated to z_0. It means that in quantum dynamics of the soliton moduli z_0 and η the operators p and η can be realized as

$$p = -i\,\frac{\mathrm{d}}{\mathrm{d}z_0}\,, \qquad \eta = \frac{1}{\sqrt{2}}\,. \tag{56}$$

It is clear that we could have chosen $\eta = -1/\sqrt{2}$. This is the same unobservable ambiguity that was discussed in Sec. 2, the supercharge Q_1 is linear in η.

Thus, the supercharges depend only on the canonical momentum p,

$$Q_1 = \sqrt{2\mathcal{Z}}\,, \qquad Q_2 = \frac{p}{\sqrt{2\mathcal{Z}}}\,. \tag{57}$$

In the rest frame in which we perform our consideration $\{Q_1, Q_2\} = 0$, and the only value of p consistent with it is $p = 0$. Thus, for the soliton $Q_1 = \sqrt{2\mathcal{Z}}$, $Q_2 = 0$ in full agreement with the general construction discussed in Sec. 2.

Note that the representation (57) can be used at nonzero p as well. It reproduces the superalgebra (2) in the nonrelativistic limit, with p having the meaning of the total spatial momentum P_1.

In passing from Eq. (54) to (57) we have omitted the nonzero modes. For each given nonzero eigenvalue there is one bosonic eigenfunction (in the operator L_2), the same eigenfunction in ψ_2 and one eigenfunction in ψ_1 (of the operator $\tilde{L}_2$). The quantization of the nonzero modes is quite standard. The corresponding additional terms in $Q_{1,2}$ can be easily written in term of the creation and annihilation operators. They describe excitations of the BPS solitons. These excitations form long (two-dimensional) multiplets. Both supercharges do not vanish and one can introduce the fermion number (15), (18).

The multiplet shortening guarantees that the equality $M = \mathcal{Z}$ is not corrected. For the exactly solvable $\mathcal{N}=1$ models,[6,12,17] such as that with the superpotential $\mathcal{W} = mv^2 \sin(\phi/v)$, the soliton mass is known exactly. In Ref. 2 it was explicitly checked that M is equal to the matrix element of $\mathcal{Z}$ (see Eq. (35) with the account for the anomaly (37)) up to two loops. Moreover, it was seen that the coupling constant expansion has a finite radius of convergence (no essential singularity at small coupling).

What lessons can one draw from the considerations of this section? In the case of the polynomial model the target space is noncompact, while the sine-Gordon case can be viewed as a compact target manifold S^1. In these both cases we found one and the same result: short (one-dimensional) soliton multiplet defying the fermion parity. It is clear that this conclusion remains valid for a general choice of the superpotential $\mathcal{W}(\phi)$ admitting classical BPS solitons.

We would like to emphasize the following point. Although we started from a noncompact spatial coordinate technically our analysis was performed in the finite box (with specific boundary conditions). Thus, the infrared regularization was guaranteed. However, the theory is not ultraviolet finite, only super-renormalizable. This circumstance turn out to be crucial, as we will see in the next section where a finite model will be considered. We will see there that a heavy field which acts as an UV regulator develops a fermion zero mode localized on the boundary of the box.

5.2 Two-superfield model

We start from the Landau-Ginzburg model with the extended $\mathcal{N}=2$ supersymmetry which is ultraviolet finite theory. Then a soft breaking down to $\mathcal{N}=1$ by a mass term preserves finiteness. Our presentation in this section follows Ref. 3.

The Lagrangian (43) with two real superfields $\Phi^a = \{\Phi, \widetilde{\Phi}\}$ has $\mathcal{N}=2$ supersymmetry if the superpotential $\mathcal{W}(\phi, \widetilde{\phi})$ is a harmonic function,

$$\Delta_\phi \mathcal{W} \equiv \frac{\partial^2 \mathcal{W}}{\partial\phi^a \partial\phi^a} = 0 \qquad \text{for } \mathcal{N}=2\,. \tag{58}$$

It means, in particular, the absence of the anomaly in the central charge – the superpotential is not changed by radiative corrections. The $\mathcal{N} =$

2 supersymmetry makes the model finite, while in $\mathcal{N}=1$ it was superrenormalizable. A polynomial example of the harmonic superpotential is

$$\mathcal{W}(\phi,\widetilde{\phi}) = \frac{m^2}{4\lambda}\,\phi - \frac{\lambda}{3}\,\phi^3 + \lambda\,\phi\,\widetilde{\phi}^{\,2}\,. \tag{59}$$

How can one introduce breaking of $\mathcal{N}=2$? To this end, consider a more general case of nonharmonic $\mathcal{W}\,(\phi_1,\phi_2)$,

$$\mathcal{W}\,(\phi,\widetilde{\phi}) = \frac{m^2}{4\lambda}\,\phi - \frac{\lambda}{3}\,\phi^3 + \lambda\,\phi\,\widetilde{\phi}^{\,2} + \frac{p\,m}{2}\,\widetilde{\phi}^{\,2} + \frac{q\lambda}{3}\,\widetilde{\phi}^{\,3}\,, \tag{60}$$

where p and q are dimensionless parameters. For $p,q\neq 0$, the extended $\mathcal{N}=2$ supersymmetry is explicitly broken down to $\mathcal{N}=1$. The parameter p introduces soft breaking of $\mathcal{N}=2$ which preserves finiteness of the theory and the absence of the anomaly.[f] The nonvanishing q breaks the finiteness (the theory stays superrenormalizable, however) and introduces the anomaly, $\Delta_\phi\mathcal{W} = 2q\lambda\widetilde{\phi}$.

The classical solution for the kink is the same as in the one-field PM model, see the first line in Eq. (47), with second field $\widetilde{\phi}$ vanishing,

$$\phi_{\rm sol} = \phi_0(z) = \frac{m}{2\lambda}\tanh\frac{mz}{2}\,,\qquad \widetilde{\phi}_{\rm sol} = 0\,. \tag{61}$$

It satisfies supersymmetric boundary conditions in the finite box, which has the following form in terms of superfields Φ^a:

$$D_1\Phi^a(t, z=\pm L/2,\theta) = 0\,,\qquad a=1,2\,. \tag{62}$$

It is a straightforward generalization of the one-field case (48).

The mode expansion is again based on operators $L_2 = P^\dagger P$, $\tilde{L}_2 = PP^\dagger$ where the operator

$$P_{ab} = \delta_{ab}\,\partial_z - \partial_a\partial_b\mathcal{W}(\phi=\phi_{\rm sol}) \tag{63}$$

now has a matrix form. The matrix is diagonal in our case,

$$P_{ab} = \begin{pmatrix} \partial_z - 2\lambda\phi_0(z) & 0 \\ 0 & \partial_z + 2\lambda\phi_0(z) + pm \end{pmatrix}. \tag{64}$$

The zero modes for the fields ϕ, ψ are the same as in Sec. 5. A new zero mode appears in the field $\widetilde{\psi}$,

$$\left(\widetilde{\psi}\right)_{\rm zero\ mode} = \xi\,N\,\frac{\exp\left(-p\,mz\right)}{\cosh^2\left(mz/2\right)}\begin{pmatrix}0\\1\end{pmatrix}, \tag{65}$$

[f] The term $p\,m\,\phi_2^2/2$ leads to a constant in $\Delta_\phi\mathcal{W}$ which shifts the superpotential by an unobservable constant.

where ξ is the operator coefficient, N is the normalization factor. At $p = 0$ it has the same functional form as the old fermionic mode in ψ. This is not surprising because of $\mathcal{N}=2$ supersymmetry at $p = 0$. What is crucial is that the zero mode (65) is not lifted even at nonvanishing p. This feature is due to the Jackiw-Rebbi theorem.[21]

Let us consider first $q = 0$ and $|p| < 1$; the second condition ensures localization of the zero mode (65). One boson and two fermion zero modes mean that we have two soliton states which are the BPS states (within our approximation). They form a reducible multiplet which preserves the fermion parity. The fact that the multiplet is not short implies that its BPS nature can be lost. It could be demonstrated, for instance, by introducing a nonvanishing q.

Indeed, let us show that at $q \neq 0$ the one-loop anomaly makes $Q_2 \neq 0$. The anomalous part in Q_2 is

$$
\begin{aligned}
Q_2 &= -\frac{1}{4\pi}\int \mathrm{d}z \left[\frac{\partial}{\partial\phi_2}\,\Delta_\phi \mathcal{W}\right]\left(\tilde{\psi}_2\right)_{\text{zero mode}} \\
&= -\xi\,\frac{q\lambda\,N}{2\pi}\int \mathrm{d}z\,\frac{\exp\left(-p\,mz\right)}{\cosh^2\left(mz/2\right)} = -\xi\,\frac{q\lambda}{\pi m}\sqrt{\frac{3\,m\,p\,\pi}{2\left(1-4p^2\right)\tan\left(p\pi\right)}}\,, \qquad (66)
\end{aligned}
$$

where ξ is the second fermion modulus, $\xi^2 = 1/2$. Correspondingly, the shift of the soliton mass from $\mathcal{Z}$ is

$$
M - \mathcal{Z} = Q_2^2 = \frac{3\,q^2\lambda^2}{4\pi^2 m}\,\frac{p\,\pi}{\left(1-4p^2\right)\tan\left(p\pi\right)}\,. \qquad (67)
$$

Note the absence of singularity at $p = 1/2$.

Moreover, even at $q = 0$ when there is no anomaly (and Q_2 remains zero at one loop) we conjecture that a nonvanishing Q_2 is generated by nonperturbative effects. If it is the case, $M - \mathcal{Z} \propto \exp(-c/\lambda)$.

Now let us discuss what happens when $|p| \geq 1$. It is clear that in this interval the zero mode (65) delocalizes: depending on the sign of p it runs to the left or right wall of the box. It becomes non-normalizable in the limit $L \to \infty$; there is no normalization problem at finite L, however. If one considers the entire system which includes the box, the supermultiplet continues to be reducible even at $|p| \geq 1$, i.e., unprotected against leaving the BPS bound.

However, physically we would like to limit ourselves to experiments which are insensitive to the boundaries in the limit of large L. Then we loose one state (associated with the boundaries) as well as the fermion parity; the multiplet becomes short and BPS saturated at $L \to \infty$.

In fact, at $|p| \gg 1$, when the mass of the second superfield is large, this field can be viewed as an ultraviolet regulator for the one-field model of Sec. 5.1. The two-field model demonstrates that the short multiplets appear at a price of running away from the soliton to the boundaries. The states which run away are associated with the heavy (regulator) fields.

6 Circle in the target and coordinate spaces

In the model with Lagrangian (43) with **one** superfield let us assume that the field ϕ lives on the circle S^1 of circumference $2\pi v$. This implies that $\mathcal{W}'(\phi)$ is periodic, with the period $2\pi v$. Moreover, we assume that the spatial coordinate z is also compact and defined on a circle S^1 of circumference L, i.e. the points z and $z + L$ are identified.

As was shown in Ref. 26, the BPS saturated solitons are possible provided the superpotential $\mathcal{W}$ is a multivalued function such that $\mathcal{W}'$ is single-valued. Let us take, for instance

$$\mathcal{W}(\phi) = c\phi + w(\phi)\,, \qquad \mathcal{W}'(\phi) = c + w'(\phi)\,, \tag{68}$$

where $w(\phi)$ is a $2\pi v$ periodic function and c is an appropriately chosen numerical coefficient. The central charge will be equal to $2\pi vc$. As an example one can have in mind $w = mv^2 \sin(\phi/v)$.

The BPS equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \mathcal{W}'(\phi) \tag{69}$$

has an implicit solution

$$\int_{\phi(0)}^{\phi(z)} \frac{\mathrm{d}\phi}{\mathcal{W}'(\phi)} = z\,. \tag{70}$$

The function $\mathcal{W}'(\phi)$ must be positive everywhere on the target space circle. We choose the value of $\phi(0)$ such that $\mathcal{W}'(\phi(0)) = \mathrm{Max}\{\mathcal{W}'\}$, it puts the center of the soliton at $z = 0$.

The condition of periodicity

$$\int_0^{2\pi v} \frac{\mathrm{d}\phi}{\mathcal{W}'(\phi)} = L\,, \tag{71}$$

fixes the value of c, assuming that Eq. (71) has a solution, which is a generic situation. We denote the solution $\phi_0(z)$.

The mode expansion of $\phi - \phi_0$ and $\psi_{1,2}$ is performed in the eigenmodes of differential operators L_2 and $\tilde{L}_2$, in the same way it was done in the previous section. The only difference is in the boundary conditions. Now, instead of Eq. (48), we require periodicity. In noncompact space the operator L_2 had a zero mode while $\tilde{L}_2$ had no zero mode. Now, in the compact space, both have zero modes, we denote them as χ_0 for L_2 and $\tilde{\chi}_0$ for $\tilde{L}_2$,

$$\begin{aligned}\chi_0 &\propto \exp\left\{\int_0^z \mathcal{W}''(\phi_0(z))\mathrm{d}z\right\} \propto \frac{\mathrm{d}\phi_0}{\mathrm{d}z} \propto \mathcal{W}'(\phi_0)\,,\\ \tilde{\chi}_0 &\propto \exp\left\{-\int_0^z \mathcal{W}''(\phi_0(z))\mathrm{d}z\right\} \propto \frac{1}{\mathcal{W}'(\phi_0)}\,.\end{aligned} \tag{72}$$

Note that while the zero mode χ_0 (in ϕ and ψ_2 fields) is localized on the kink, the mode $\tilde{\chi}_0$, i.e. that of ψ_1 is localized off the kink. The zero mode balance is the same as for nonzero modes: we have one bosonic mode and two fermionic. Retaining only the zero modes we have the following expansion for the bosonic and fermionic fields:

$$\phi(z) = \phi_0(z - z_0)\,, \quad \psi_1 = \xi\,\tilde{\chi}_0\,, \quad \psi_2 = \eta\,\chi_0\,, \tag{73}$$

where ξ and η are the fermion collective coordinates. This leads to exactly the same supercharges as in Eq. (54). The difference lies in the quantization relations,

$$[p, z_0] = -i\,, \quad \eta^2 = \xi^2 = \frac{1}{2}\,, \quad \{\eta, \xi\} = 0\,. \tag{74}$$

Due to $\{\eta, \xi\} = 0$ the representation now is two-dimensional.

In the leading approximation above both soliton states are BPS since $Q_2 = 0$. However, shortly we will show that already at the one-loop level the supercharge Q_2 does not vanish. Thus, the long (two-dimensional) multiplet is formed. The states are non-BPS, their mass exceeds the central charge by a two-loop correction.

The easiest way to demonstrate the phenomenon is the explicit calculation of Q_2 with account of the anomaly (37),

$$Q_2 = \xi \int \mathrm{d}z \left[\partial_z\phi_0 - \mathcal{W}'(\phi_0) - \frac{\mathcal{W}'''(\phi_0)}{4\pi}\right]\tilde{\chi}_0(z)\,, \tag{75}$$

where we substituted the classical soliton solution for ϕ and the zero modes for ψ in the definition (33). The zero mode of ψ_2 drops out from Q_2 at $p = \mathcal{Z}\dot{z}_0 = 0$. The term $\mathcal{W}'''(\phi_0)/4\pi$ is due to the anomaly. On the classical solution the first two terms in the square brackets cancel each other, only the anomalous term survives. Thus, we see that $Q_2 \neq 0$,

$$Q_2 = -\frac{1}{4\pi}\,\xi\left[\int \frac{\mathrm{d}\phi}{(\mathcal{W}')^3}\right]^{-1/2}\int \mathrm{d}\phi\,\frac{\mathcal{W}'''}{(\mathcal{W}')^2}\,, \tag{76}$$

where we used expression (72) for the zero mode $\tilde{\chi}_0$. It means that the excess of the soliton mass over the central charge is

$$M - \mathcal{Z} = Q_2^2 = \frac{1}{32\pi^2}\left[\int \frac{\mathrm{d}\phi}{(\mathcal{W}')^3}\right]^{-1}\left[\int \mathrm{d}\phi\,\frac{\mathcal{W}'''}{(\mathcal{W}')^2}\right]^2 . \tag{77}$$

Note that taking account of the anomaly in the model of Sec. 5 (in the box) does not lead to nonvanishing Q_2 because of the absence of the zero mode in ψ_2. Its effect on Q_1,

$$\Delta Q_1 = \frac{1}{\sqrt{2}}\int \mathrm{d}z\left[\frac{\mathcal{W}'''(\phi_0)}{4\pi}\right]\chi_0(z) = \frac{1}{\sqrt{2\mathcal{Z}}}\,\frac{1}{4\pi}\big[\mathcal{W}''(z\to\infty) - \mathcal{W}''(z\to-\infty)\big]\,, \tag{78}$$

amounts to the shift in the classical value of $\mathcal{Z}$ (see Eq. (35)) caused by the anomaly by virtue of the substitution (37).

7 Nonflat target space: $\mathrm{Tr}\, Q_1$ as index of the Dirac operator on the reduced moduli space

In this section we treat target spaces with nonflat metric. Our central point is to show that the index $\mathrm{Tr}\, Q_1$ introduced above is, in fact, the index of a Dirac operator defined on the soliton moduli space. More exactly, the index defined in Eq. (27) coincides with the square of the index of a Dirac operator on the reduced moduli space of solitons (see Sec. 7.2 for the definition). The latter was studied by mathematicians. Thus, it is possible to determine in which $\mathcal{N} = 1$ models $\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1) = 0$, i.e. the multiplet shortening does *not* take place (in the general situation). In particular, the index vanishes provided that the reduced moduli space is not a point, i.e., its dimension nonvanishing, and is compact. This happens, for instance, in the following situation. In terms of the superpotential $\mathcal{W}$ the soliton sweeps the interval $[\mathcal{W}(A), \mathcal{W}(B)]$. If there are no other critical points in this interval the reduced moduli space is compact.

A representative example of nonflat target space $\mathcal{T}$ is sphere S^{n+1} with a superpotential producing only two critical points coinciding with the poles of the sphere. For instance, $\mathcal{W} = \cos\xi$, where ξ is a polar angle, does the job. The case $n = 0$ (the circle S^1) was considered in Sec. 5.1; this is the sine-Gordon model in which we observed short multiplets, i.e., $\mathrm{Tr}\, Q_1 \neq 0$. However, in the case at hand, when we deal with a single field, the target space metric is necessarily flat. Note, that for higher spheres, $n \geq 1$, the metric is necessarily nonflat.

The first nonflat example is S^2. In this case we deal with two fermion moduli from the very beginning: hence, $(-1)^F$ is defined and all representations are even-dimensional. The multiplet shortening cannot occur. This is obviously true for all odd n. The first example with an odd number of fermion moduli is S^3 (i.e. $n = 2$), from which we start.

7.1 Superpotential on S^3 target space

In this section we consider solitons in the model with the sphere S^3 as a target space and some specific superpotential. The target space S^3 is symmetric so the theory contains only one running coupling. This example is of special interest for us because, as we will see, it leads to an odd number of fermionic zero modes, similar to the one-field model of Sec. 5. We will show, however, that unlike the one-field model of Sec. 5.1, in the case of S^3 there will be no BPS solitons.

The generic form of the Lagrangian of the sigma model is given by Eq. (30). The metric in this case is given by the following expression for the interval,

$$\mathrm{d}s_3^2 = g_{ab}\,\mathrm{d}\phi^a\mathrm{d}\phi^b = \frac{1}{\lambda}\left[\mathrm{d}\xi^2 + \sin^2\xi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\right]\,, \tag{79}$$

where λ is the coupling constant and we choose the angle coordinates ξ, θ and φ to parameterize S^3.

The superpotential is

$$\mathcal{W}(\phi) = \frac{1}{2}\,M_0\cos\xi\,. \tag{80}$$

The superpotential has maximum at $\xi = 0$ and minimum at $\xi = \pi$ and no other critical points. The critical points are two vacua of the theory. Excitations at these vacua form three boson-fermion supermultiplets with the mass $\lambda M_0/2$.

In the classical approximation the model has a family of BPS solitons interpolating between the maximum and the minimum. The mass of the soliton is given by the central charge (34)

$$M_{\rm sol} = \mathcal{Z} = \Delta\mathcal{W} = M_0\,. \tag{81}$$

Its profile as a function of the coordinate z is determined by the BPS equations,

$$\frac{\mathrm{d}\phi^a_{\rm sol}}{\mathrm{d}z} = g^{ab}\,\partial_b\,\mathcal{W}(\phi_{\rm sol})\,. \tag{82}$$

On the soliton trajectories ξ changes between 0 and π at fixed θ and ϕ,

$$\begin{aligned}&\xi_{\rm sol}(z) = 2\arctan\left(\exp\left[-\frac{1}{2}\,\lambda M_0\,(z - z_0)\right]\right)\,,\\ &\theta_{\rm sol}(z) = \theta_0\,,\quad \varphi_{\rm sol}(z) = \varphi_0\,,\end{aligned} \tag{83}$$

so in addition to the soliton center z_0 there are two extra moduli, θ_0 and φ_0. We denote the set of all three moduli by $m^i = \{\theta_0, \varphi_0, z_0\}$ where $i = 1, 2, 3$.

There is a fermionic partner η^i to each bosonic modulus and the fields in the soliton sector are represented as

$$\phi^a(z,t) = \phi^a_{\rm sol}(z,\, m^i) + \text{nonzero modes}\,, \qquad a = 1,2,3 \tag{84}$$

$$\psi^a_1(z,t) = \text{nonzero modes}\,, \quad \psi^a_2(z,t) = \eta^i\, \frac{\partial \phi^a_{\rm sol}}{\partial m^i} + \text{nonzero modes}\,,$$

where the time dependence enters through collective coordinates. Equation (84) implies that we are in the rest frame of the soliton.

Substituting these expressions in the Lagrangian (30) and neglecting all nonzero modes we obtain the Lagrangian for dynamics of moduli,

$$L = \frac{1}{2}\, h_{ij}(m) \left(\dot m^i\, \dot m^j + i\, \eta^i\, \mathcal{D}_t\, \eta^j\right) - M_0\,, \tag{85}$$

where the induced metric $h_{ij}(m)$ refers to the $S^2 \times R$ geometry of the moduli space,

$$\mathrm{d}s^2_m = h_{ij}(m)\, \mathrm{d}m^i\, \mathrm{d}m^j = \frac{4}{\lambda^2\, M_0}\left[\mathrm{d}\theta_0^2 + \sin^2\!\theta_0\, \mathrm{d}\varphi_0^2\right] + M_0\, \mathrm{d}z_0^2\,. \tag{86}$$

The coordinate on R is $m^3 = z_0$ and $m^{1,2} = \{\theta_0, \varphi_0\}$ are the angles on S^2. Moreover, the covariant derivative $\mathcal{D}_t$ is defined as

$$\mathcal{D}_t\, \eta^j = \dot\eta^j + \dot m^k\, \tilde\Gamma^j_{kl}\, \eta^l\,. \tag{87}$$

This is in correspondence with the field-theoretic definition (32), but the Christoffel symbols $\tilde\Gamma^j_{kl}$ refer, of course, to the moduli metric h_{ij}. We put the tilde to differentiate from the field theory ones.

By the same token we get also expressions for the supercharges,

$$Q_1 = 2\mathcal{Z}\, \eta^1\,, \tag{88}$$

$$Q_2 = h_{ij}\, \dot m^i\, \eta^j = \dot m_j\, \eta^j\,, \tag{89}$$

in terms of bosonic and fermionic moduli where $\mathcal{Z} = M_0$.

The next step is to quantize the moduli dynamics.[g] To this end one introduces the canonical momenta conjugated to the coordinates and imposes

[g] The procedure of quantization has a rich literature. In our presentation we follow the one given in Witten's lecture.[27] As far as technical details are concerned, we closely follow Ref. 28 where they are thoroughly discussed.

commutation relations. The crucial point is to establish the ordering of non-commuting operators. The ordering is completely fixed by general covariance in the target space and supersymmetry. Namely, unlike Hamiltonian, quantization of the supercharge is uniquely defined.

For the bosonic coordinates m^i one has

$$p_i = \frac{\partial L}{\partial \dot{m}^i} = h_{ij}\,\dot{m}^j + \frac{i}{2}\,\eta_j\,\tilde{\Gamma}^j_{i\,l}\,\eta^l\,, \qquad [p_i\,, m^j] = -i\delta^j_i\,. \tag{90}$$

For the fermion coordinates η^i,

$$\zeta_i = \frac{\partial L}{\partial \dot{\eta}^i} = i\,h_{ij}\,\eta^j\,, \qquad \{\zeta_i\,, \eta^j\} = i\,\delta^j_i\,, \qquad \{\eta^i\,, \eta^j\} = h^{ij}\,. \tag{91}$$

A subtlety in this case is that the canonical momenta are function of coordinates and are not independent. The validity of the anticommutation relations (91) can be verified by substituting expressions (84) into field-theoretic commutators. Alternatively, one can check them considering Green functions in quantum mechanics (see, e.g., Ref. 28).

We can realize the algebra of the commutation relations (90), (91) in the Hilbert space of two-component spinor wave functions $\Psi_a(m)$ with the scalar product

$$\langle \Phi \mid \Psi \rangle = \int \Pi \mathrm{d}m^i\,\sqrt{h(m)}\;\Phi^\dagger(m)\,\Psi(m) \tag{92}$$

in the following way

$$\begin{aligned} p_i &= -i\,\delta_{ab}\,h^{-1/4}\,\frac{\partial}{\partial m^i}\,h^{1/4}\,, \qquad (i=1,2,3)\,, \quad (a,b=1,2)\,, \\ \eta^i &\equiv \frac{1}{\sqrt{2}}\,\sigma^i \equiv \frac{1}{\sqrt{2}}\,e^i_A\left(\sigma^A\right)_{ab}\,, \qquad\qquad (A=1,2,3)\,, \end{aligned} \tag{93}$$

where σ^A are the Pauli matrices and we introduce frames e^i_A, satisfying the conditions $e^i_A e^i_B\,\delta^{AB} = h^{ij}$. A possible choice for e^i_A is

$$e^i_A = \mathrm{diag}\left\{\frac{\lambda\sqrt{\mathcal{Z}}}{2}\,, \frac{\lambda\sqrt{\mathcal{Z}}}{2\,\sin\theta_0}\,, \frac{1}{\sqrt{\mathcal{Z}}}\right\}\,. \tag{94}$$

In fact, what we need is the quantum version of the classical supercharges presented in Eqs. (88), (89),

$$Q_1 = 2\mathcal{Z}\,\eta^1\,, \tag{95}$$

$$Q_2 = \frac{1}{2}\left(\eta^i\,\pi_i + \pi_i\,\eta^i\right)\,, \tag{96}$$

where the operator of covariant momentum π_i (a quantum version of the velocity operator $\dot{m}_i = h_{ij}\dot{m}^j$, see Eq. (90)) is defined as

$$\pi_i = p_i - \frac{i}{4}\,\tilde{\Gamma}_{j,i\,l}\left[\eta^j , \eta^l\right] . \tag{97}$$

In fact, π_i reduces to the covariant derivative on the spin manifold, $\pi_i = -i\nabla_i$, the fermion term in Eq. (97) represent the spin connection. In terms of ∇_i the supercharge (96) can be rewritten as

$$Q_2 = \eta^j\,(-i\nabla_j) \equiv \frac{1}{\sqrt{2}}\,\sigma^j\,(-i\nabla_j)\,, \qquad j = 1,2\,. \tag{98}$$

This is nothing but the Dirac operator $\not{D}$ on the manifold.

Let us stress that the dynamics of the moduli m^3, η^3 along the R direction is factored out, this is just a free motion of the center of mass (together with its fermionic partner). In particular $\pi^3 \equiv p^3$ is conserved and we set it to zero by choosing the rest frame. It means that the sum in Eqs. (96), (98) for Q_2 runs only over the S^2 coordinates, $i = 1, 2$. We will show below that the situation is general: the moduli space always factorizes as $\mathcal{R} \otimes \mathcal{M}$.

The commutators

$$\left[\pi_i\,, m^j\right] = -i\delta_i^j\,, \qquad \left[\pi_i\,, \pi_j\right] = -\frac{1}{2}\,\tilde{R}_{ijkl}\,\eta^k\,\eta^l\,, \qquad \left[\pi_i\,, \eta^j\right] = i\,\tilde{\Gamma}^j_{i\,l}\,\eta^l \tag{99}$$

allow one to calculate the commutators of supercharges (95), (96) with the coordinates m^i, η^i,

$$\begin{aligned} &\left[Q_1, m^i\right] = 0\,, \qquad \left\{Q_1, \eta^1\right\} = 2\,, \qquad \left\{Q_1, \eta^{2,3}\right\} = 0\,, \\ &\left[Q_2, m^i\right] = -i\,\eta^i\,, \qquad \left\{Q_2, \eta^i\right\} = \frac{1}{2}\left(h^{ij}\,\pi_j + \pi_j\,h^{ij}\right) . \end{aligned} \tag{100}$$

These commutators match the classical supersymmetry transformations.

Finally, the algebra of the supercharges $Q_{1,2}$ is

$$(Q_1)^2 = 2\,\mathcal{Z}\,, \qquad \{Q_1, Q_2\} = 0\,, \tag{101}$$

$$(Q_2)^2 = H - \mathcal{Z} = \frac{1}{2}\,h^{-1/4}\,\pi_i\,h^{1/2}\,h^{ij}\,\pi_j\,h^{-1/4} + \frac{1}{8}\,\tilde{R}\,. \tag{102}$$

The expression (102) for $(Q_2)^2 = (1/2)(\sigma^j\ i\nabla_j)^2$ is a particular case of the famous Lichnerowicz formula: the first term can be written as $-h^{ij}\nabla_i\nabla_j/2$, i.e., it represents the invariant Laplacian in application to spinors, and $\tilde{R}$ denotes the scalar curvature for the moduli metric h_{ij}. In our example it is the curvature of the S^2 sphere,

$$\tilde{R} = \frac{\lambda^2}{2}\,\mathcal{Z}\,. \tag{103}$$

Although our derivation was framed in terms of a concrete metric the results (101), (102) are perfectly general and can be applied to σ model on arbitrary manifold. The geometry of the moduli space, i.e., h_{ij}, depends on both: the field-theoretic metric g_{ab} and on the form of the superpotential $\mathcal{W}$.

Let us return to our example. In this case the scalar curvature $\tilde{R}$ is clearly a positive constant. This provides a positive term in the Hamiltonian, the last term in Eq. (102). The first term in $H - \mathcal{Z}$, which coincides (up to fermion terms) with the invariant Laplacian, is positive definite by itself. Thus, there can be no zero eigenvalues of $H - \mathcal{Z}$. In other words, there are no states which are annihilated by the supercharge Q_2. The states which were BPS saturated at the classical level cease to be BPS at the quantum level.

It is not difficult to determine a complete spectrum of $H - \mathcal{Z}$ in the S^2 case. We limit ourselves to the lowest eigenvalue. After some simple algebra we get

$$\frac{4}{\tilde{R}}\, h^{-1/4}\, \pi_i\, h^{1/2}\, h^{ij}\, \pi_j\, h^{-1/4} = -\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(i\,\frac{\partial}{\partial\varphi}+\frac{\sigma^3}{2}\cos\theta\right)^2 .$$

The φ dependence is just $\exp(i\,m\,\varphi)$ and for the ground state $m = 0$. The θ dependence is given by the associated Legendre functions $P_{1/2}^{\pm 1/2}$, so we get the lowest eigenvalue equal to $3\tilde{R}/16$ and doubly degenerate ground state,

$$\Psi_{1/2} = \sqrt{\frac{\sin\theta}{\pi}}\begin{pmatrix}1\\0\end{pmatrix}, \quad \Psi_{-1/2} = \sqrt{\frac{\sin\theta}{\pi}}\begin{pmatrix}0\\1\end{pmatrix},$$
$$(H-\mathcal{Z})\,\Psi_{\pm 1/2} = \frac{3}{4}\cdot\frac{\tilde{R}}{4}\,\Psi_{\pm 1/2}\,. \tag{105}$$

The moduli dynamics we have considered is a nice example of the $\mathcal{N}=1/2$ supersymmetric quantum mechanics. The possibility of a nontrivial $\mathcal{N}=1/2$ construction is due to the fact that the interaction enters through kinetic terms rather than through potential. The same moduli dynamics can be viewed as a theory based on the Dirac operator defined on the curved moduli space. Indeed, the realization of $H - \mathcal{Z} = (Q_2)^2$ by matrix-valued differential operators in the moduli dynamics allows to interpreted the supercharge Q_2 as the Dirac operator on the moduli manifold.

The Dirac operators on manifolds were extensively studied in the mathematical literature. In particular, the absence of zero modes of the Dirac operator on S^n spheres (in our language the absence of the BPS states) follows from the Lichnerowicz formula (102) as was mentioned above. More generally, one can introduce an index of the Dirac operator $\not{D}$ which counts the difference of left and right chiral zero modes,

$$\mathrm{ind}\,(\not{D}) = \mathrm{Tr}\left[\sigma^3 \exp\left(-\beta\,\not{D}^2\right)\right]\,, \tag{106}$$

where the matrix σ^3 anticommutes with the Dirac operator $\not{D}$ defined by Eq. (98). For S^2 the Dirac operator has no zero modes at all, so the index vanishes.

The matrix σ^3 in the definition (106) is a realization of γ^5 in our S^2 case. Moreover, in the moduli dynamics σ^3 is a realization of the supercharge $Q_1 = \sqrt{2\mathcal{Z}}\,\sigma^3$, therefore the index can be rewritten as

$$\mathrm{ind}\,(\not{D}) = \mathrm{Tr}\left[\frac{Q_1}{\sqrt{2\mathcal{Z}}}\,\exp\left(-\beta\,Q_2^2\right)\right]\,. \tag{107}$$

7.2 Generic target space

Let us pass now to the general case: $\mathcal{T}$ is an arbitrary Riemann manifold endowed with a metric $g_{ab}(\phi)$ and a superpotential $\mathcal{W}(\phi)$. The classical vacua are the critical points of the superpotential, $\partial_a \mathcal{W} = 0$. Classical BPS solitons interpolate between vacua A and B and satisfy the first order differential equations (36).

We start from briefly reviewing elements of the Morse theory (see e.g. Refs. 29, 30). For every critical point A the Morse index of this point $\nu(A)$ is defined as the number of the negative eigenvalues in the matrix of the second derivatives

$$H_{ab}(\phi) = \nabla_a \partial_b \mathcal{W}(\phi) \tag{108}$$

at $\phi = A$. At the critical points the covariant derivative ∇_a coincides with the regular ∂_a. For solitons interpolating between two critical points, $\phi = A$ at $z \to -\infty$ and $\phi = B$ at $z \to \infty$ one can determine the relative Morse index ν_{BA},

$$\nu_{BA} = \nu(B) - \nu(A)\,. \tag{109}$$

This relative Morse index counts the difference between the numbers of the zero modes of the operators P and $P^\dagger$,

$$\nu_{BA} = \ker\{P\} - \ker\left\{P^\dagger\right\}\,, \tag{110}$$

where P and $P^\dagger$ are

$$P_{ab} = g_{ab} D_z - H_{ab}\,, \qquad P^\dagger_{ab} = -g_{ab} D_z - H_{ab}\,. \tag{111}$$

Here D_z is defined in Eq. (32), and the field ϕ is taken to be $\phi_{\rm sol}(z)$.

Note that $\nu_{BA} = 0$ in the $\mathcal{N}{=}2$ case and its small deformations. Indeed, due to the harmonicity of $\mathcal{W}$ in this case, $\Delta_\phi \mathcal{W} = 0$, which leads to one negative eigenvalue in the matrix of the second derivatives in each vacua (per pair of fields related by $\mathcal{N}{=}2$).

For the BPS soliton, satisfying Eq. (36), one zero mode certainly present in P is the translational mode. It corresponds to the soliton center z_0, one of the coordinates in the soliton moduli space. The same zero mode of P is the fermion zero mode — the corresponding modulus η is the superpartner of z_0.

We will limit ourselves to the case when $\ker P^\dagger = 0$. (Note that even if that is not the case, one can get rid of the zero modes in $P^\dagger$ by small deformations of the superpotential). Then, the Morse index

$$\nu_{BA} \equiv n + 1 \geq 1 \tag{112}$$

counts the dimension of the soliton moduli space M^{n+1}. Thus, we arrive at quantum mechanics of $n+1$ bosonic and $n+1$ fermionic moduli on M^{n+1}.

As was mentioned above, one of $n+1$ bosonic moduli is z_0, the coordinate of the soliton center. This is a cyclic coordinate conjugated to the generator P_z of the spatial translations, $z_0 \in \mathcal{R}$. Note an ambiguity in z_0 — one can add to z_0 an arbitrary function of other moduli. This ambiguity is fixed by the definition given below, see Eq. (117). Thus, the moduli space M^{n+1} is a direct product

$$M^{n+1} = \mathcal{R} \otimes \mathcal{M}^n \tag{113}$$

of $\mathcal{R}$ and the manifold $\mathcal{M}^n$ with coordinates $m^1, ..., m^n$ describing internal degrees of freedom of the soliton. This manifold $\mathcal{M}^n$ is what we call the *reduced moduli space*.

It is instructive to elucidate the factorization (113) in more detail.[5] We must show that the moduli space metric h_{ij},

$$h_{ij}(m) = \int \mathrm{d}z\, g_{ab}(\phi_{\rm sol})\, \frac{\partial \phi^a_{\rm sol}}{\partial m^n}\, \frac{\partial \phi^b_{\rm sol}}{\partial m^k}\,, \qquad i,j = 0,1,2,...,n\,, \tag{114}$$

where $m^0 \equiv z_0$, has a block form, i.e. $h_{0j} = 0$ for $j = 1,2,...,n$. Indeed,

$$h_{0j}(m) \;= -\int \mathrm{d}z\, \partial_b \mathcal{W}(\phi_{\rm sol})\, \frac{\partial \phi^b_{\rm sol}}{\partial m^j} \;= -\frac{\partial}{\partial m^j} \int \mathrm{d}z\, \left[\mathcal{W}(\phi_{\rm sol}) - \mathcal{W}(\phi_{\rm sol})_{m=m_*}\right]\,, \tag{115}$$

where we use the fact that the soliton solution depends on the spatial coordinate only through the combination $z - z_0$, to replace $\partial\phi^a_{\rm sol}/\partial m^0$ by $\partial\phi^a_{\rm sol}/\partial z$, which, in turn, can be replaced by $g^{ac}\,\partial_c\,\mathcal{W}(\phi_{\rm sol})$ by virtue of Eq. (36). We also regularized the integral on the right-hand side of Eq. (115) by subtracting from the integrand the superpotential at some fixed values of the moduli $m = m_*$.

Considering Eq. (115) for h_{00} we get

$$h_{00} = -\frac{\partial}{\partial m^0}\int \mathrm{d}z\,[\mathcal{W}(\phi_{\rm sol}) - \mathcal{W}(\phi_{\rm sol})_{m=m_*}]\,. \tag{116}$$

Having in mind $h_{00} = \mathcal{Z}$ we define the modulus m^0 as

$$m^0 = -\frac{1}{\mathcal{Z}}\int \mathrm{d}z\,[\mathcal{W}(\phi_{\rm sol}) - \mathcal{W}(\phi_{\rm sol})_{m=m_*}]\,. \tag{117}$$

With this definition it is clear that

$$h_{0j} = \frac{\partial m_0}{\partial m^j} = 0\,, \quad (j = 1, ..., n)\,. \tag{118}$$

Thus, the Lagrangian describing the moduli dynamics has the form

$$L\left(M^{n+1}\right) = -\mathcal{Z} + \frac{\mathcal{Z}}{2}\left[(\dot z_0)^2 + i\,\eta\,\dot\eta\right] + L\left(\mathcal{M}^n\right)\,, \tag{119}$$

where $L\left(\mathcal{M}^n\right)$ is the Lagrangian of the internal moduli, both bosonic and fermionic, a sigma-model quantum mechanics on $\mathcal{M}^n$. We see, that the motion of the center of mass (together with its fermionic partner) is factored out, and we only need to consider the dynamics on $\mathcal{M}^n$.

The simplest case $n = 0$ was already analyzed in Sec. 5.1. In this case the quantum moduli dynamics is trivial, and the single state BPS multiplet does exist, $\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1) = 1$. For $n \geq 1$ one must differentiate between even and odd n.

For odd n the total number $n+1$ of the fermion moduli is even. Under quantization these $n+1$ moduli become γ matrices (multiplied by frames, as in Eq. (93)) satisfying the Clifford algebra with an even number of generators. All γ's are multiplied by frames, as in Eq. (93) Taking the product of all these γ matrices we get the matrix $\gamma^{n+2} = \prod \gamma^i$ which anticommutes with all γ^i, $(i = 1, \ldots, n+1)$. This γ^{n+2}, an analog of γ^5 in four dimensions, represents $(-1)^F$, i.e., all multiplets are long.

Consider now a less trivial case of even n — only in this category can one expect to find $\mathrm{Tr}\,Q_1 \neq 0$. Quantization of $L\left(\mathcal{M}^n\right)$ is standard. All operators act in the Hilbert space of the spinor wave functions $\Psi_\alpha(m)$, where $\alpha = 1, \ldots, 2^{n/2}$. The operators m^i act as multiplication, while $\dot m_i$ become matrix-differential operators. The fermion moduli of the reduced moduli space become γ matrices of dimension $2^{n/2} \times 2^{n/2}$. The matrix $\gamma^{n+1} = \prod_{i=1}^{i=n} \gamma^i$ is used to represent the remaining fermion modulus, a partner of translation. On the moduli space $\mathcal{M}^n$ the supercharges (33) take the form

$$Q_1 = \sqrt{2\mathcal{Z}}\,\gamma^{n+1}\,, \qquad Q_2 = -\frac{i}{\sqrt{2}}\,\gamma^j\,\nabla_j\,, \tag{120}$$

where the covariant derivative ∇_j includes spin connection. The expression for Q_2 is in fact the Dirac operator $i\nabla\!\!\!/$ on $\mathcal{M}^n$. Moreover, the Hamiltonian takes the form,

$$H - \mathcal{Z} = Q_2^2 = \frac{1}{2}\,(i\nabla\!\!\!/)^2 = -\frac{1}{2}\,\nabla^j\nabla_j + \frac{1}{8}\,\tilde{R}\,, \tag{121}$$

where $\tilde{R}$ is the curvature in the soliton moduli space, and we again used the Lichnerowicz formula (cf. Eq. (102)).

From Eqs. (120), (121) it is clear that the BPS soliton states are in correspondence with the zero modes of the Dirac operator $i\nabla\!\!\!/$ on $\mathcal{M}^n$. The index $\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1)$ we defined in Eq. (27) becomes the square of the index of the Dirac operator

$$\begin{aligned} \mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1) &= \left\{\mathrm{Ind}\,(i\nabla\!\!\!/)_{\mathcal{M}^n}\right\}^2\,, \\ \mathrm{Ind}\,(i\nabla\!\!\!/)_{\mathcal{M}^n} &= \mathrm{Tr}\left[\gamma^{n+1}\exp\left(\beta\nabla\!\!\!/^2\right)\right]_{\mathcal{M}^n}\,. \end{aligned} \tag{122}$$

Equation (121) shows that if the curvature $\tilde{R}$ is positive everywhere on the soliton moduli space the Dirac operator has no zero modes, its index vanishes, and so does the index $\mathrm{Ind}_{\mathcal{Z}}\,(Q_2/Q_1)$. Thus, there are no BPS solitons in this case. An explicit example is provided by a sigma model on S^3.

Moreover, the situation turns out to be general in the hybrid models: the geometry of the reduced moduli space is similar to spherical, the index of the Dirac operator vanishes for any compact $\mathcal{M}^n$ with $n \geq 1$. The proof due to P. Pushkar' is presented in Appendix.

A comment is in order here concerning the vanishing of the index of the Dirac operator on even-dimensional reduced moduli spaces $\mathcal{M}^{2\ell}$ (with integer ℓ). Naively one might be tempted to think that since all γ matrices are traceless the index in Eq. (122) vanishes automatically, irrespective of the properties of $\mathcal{M}^{2\ell}$. It is well known that this naive conclusion is wrong — a more careful consideration is necessary.

The moduli dynamics is governed by the Lagrangian

$$L(\mathcal{M}^{2\ell}) = \frac{1}{2}\,h_{ij}(m)\left(\dot{m}^i\dot{m}^j + i\eta^i\mathcal{D}_t\eta^j\right)$$

where the metric h_{ij} is that for the reduced moduli space given in Eq. (114) with $i,j = 1,2,...,2\ell$. As we have already mentioned, after quantization the wave functions become spinors $\Psi_\alpha(m)$, while η's turn out to be γ matrices (times the frames).

The index of the Dirac operator is the regularized trace of $\gamma^{2\ell+1}$ over the space of wave functions, see Eq. (122). Naively, one might think that the space of the wave functions is a product of the spinor representation of the 2ℓ-dimensional Clifford algebra and the space $\mathcal{C}$ of smooth functions on $\mathcal{M}^{2\ell}$. If so, the trace of $\gamma^{2\ell+1}$ over the space of the wave functions would be automatically zero.

In fact, the space of wave functions is a product of the two spaces mentioned above only locally! The manifold $\mathcal{M}^{2\ell}$ should be thought of as covered by open sets, and when we go from one open set to another, generally speaking we need to rotate the spinor representation with the Spin(2ℓ)-valued function of m^i. This means that the wave functions $\Psi_\alpha(m)$ are sections of the spinor bundle that is generically nontrivial. The index of the Dirac operator is one of the characteristics that reflects this nontriviality. It might be nonzero would $\mathcal{M}^{2\ell}$ be similar to CP_2 or $K3$. The central point of the Pushkar' theorem outlined in Appendix is that the geometry of the soliton reduced moduli space is similar to spherical and cannot be similar to that of CP_2 or $K3$.

Thus, for $n \geq 1$ the soliton multiplets are long and generically non-BPS, $M > \mathcal{Z}$. If, for accidental or other reasons, they are still BPS saturated, they form a reducible representation. For example, in $\mathcal{N}=2$ models the index $\text{Ind}_{\mathcal{Z}}\,(Q_2/Q_1)$ vanishes while the BPS states do exist. From the standpoint of $\mathcal{N}=1$ they form a reducible representation for which $(-1)^F$ is well defined.

Our consideration refers to the case of compact $\mathcal{M}^n$. Generally speaking, $\mathcal{M}^n$ may be noncompact. Noncompact geometry of $\mathcal{M}^n$ may emerge, for example, if there is an extra critical point C such that $\mathcal{Z}_{AB} = \mathcal{Z}_{AC} + \mathcal{Z}_{CB}$. Physically it means that there is an infinite degeneracy of the quantized soliton states.

8 Conclusions

We analyzed a wide class of hybrid models with $\mathcal{N}=1$ supersymmetry and central charges in (1+1) dimensions. For the BPS states only one out of two supercharges is realized nontrivially which leads to one-dimensional irreducible representations of the superalgebra. The non-BPS supermultiplets are two-dimensional. Our main topic was the soliton multiplet structure in various models at weak coupling (quasiclassical approach).

We introduced and thoroughly discussed the index $\text{Tr}\, Q_1/\sqrt{2\mathcal{Z}}$ which counts the supershort (single-state) supermultiplets. It was demonstrated that nonvanishing $\text{Tr}\, Q_1$ implies the loss of the fermion parity $(-1)^F$. We showed that $\mathrm{Tr}\, Q_1$ is related to the index of the Dirac operator on the reduced moduli space. The geometry of this space is similar to spherical. It implies that the index vanishes except the very special case when the reduced moduli space is a point.

The vanishing index implies long multiplets which may or may not be BPS saturated. It is clear that the BPS saturation is not protected for long multiplets. We demonstrated that indeed quantum corrections destroy BPS saturation in many cases, by calculating the supercharge Q_2 at one loop. Classically vanishing Q_2 becomes nonzero at one loop. This leads to $M - \mathcal{Z} \neq 0$ at two loops. In special cases, where Q_2 remains zero in perturbation theory, $Q_2 \neq 0$ may be generated nonperturbatively.[31]

What lessons have we learned from the study of the $\mathcal{N}=1$ theories? The main lesson is that of the fermion quantization in the case when the number of fermion zero modes is odd. Let us remind that the only consistent approach to fermions in field theory is based on Berezin's holomorphic quantization which implies the number of the fermion degrees of freedom is even.

How can it be consistent with the fact that the BPS-saturated irreducible representations of $\mathcal{N} = 1$ centrally extended superalgebra are one-dimensional? The resolution is as follows: if the theory is explicitly regularized both in the ultraviolet and infrared it becomes a quantum mechanics of a large number of variables; the number of the fermion variables is necessarily even, the supermultiplets are reducible and $(-1)^F$ is preserved. Thus, in fully regularized theory the number of the fermion zero modes is always even, and so is the number of states in the supermultiplet. Moreover, the BPS saturation is not protected so we can say that there are no BPS states in the fully regularized theory.

We gave clear-cut illustrations to this point. In Sec. 6 the spatial dimension was compactified onto a circle. The soliton had two zero modes – one localized on the soliton, another on the other side of the circle. Another example is given by the softly broken $\mathcal{N}=2$ theory, discussed in Sec. 5.2. There we could see how the second zero mode becomes delocalized once the $\mathcal{N} = 2$ breaking parameter becomes large enough (more exactly the localization of the second zero mode shifts away from the soliton, to the edge of the box).

In spite of the "evenness" of the total number of the fermion modes, in the limit $L \to \infty$ it may (and actually does) happen that in the physical subsector of the Hilbert space there remains an odd number of the fermion moduli. If some of the modes are localized at the boundaries of the large box, the corresponding states are unobservable in any physical local measurement. What is observable are the localized states associated with the soliton. If the number of fermion modes localized on the soliton is odd, we arrive to an abnormal situation. An explicit example was given in Sec. 5. In this case we get the multiplet shortening, and the BPS saturation is implemented. The number of such short multiplets in the physical subsector is counted by $\mathrm{Tr}\, Q_1/\sqrt{2\mathcal{Z}}$.

Our results naturally "blend in" into a general picture. The BPS saturation was studied in detail in the $\mathcal{N}=2$ theories in (1+1) dimensions[24] and in the $\mathcal{N}=2$, $\mathcal{N}=4$ theories in (3+1)-dimensions. The assertion that under the full regularization (UV and IR) there are no short multiplets is general, it is applicable to higher dimensions and higher supersymmetries. Extra states are localized away from the soliton center. What is specific for $\mathcal{N}=1$ in two dimensions, where the number of supercharges is minimal, is the odd number of the soliton fermion moduli (in higher dimension it is always even). This may lead to the loss of $(-1)^F$ in the physical sector.

In a broader context, we found another example of a remarkable phenomenon first discussed by Witten[32] – supersymmetry without the full fermion-boson degeneracy. If such theories could be found in four dimensions, this would be "a dream came true."

Witten's example is in the context of 2+1 supergravity with the conic geometry. Out of four supercharges of the model two supercharges annihilate the BPS solitons. The other two supercharges produce the fermion zero modes. Without gravity these modes are normalizable which leads to two-component short supermultiplet of $\mathcal{N}=2$. With gravity switched on the fermion modes become non-normalizable, implying the single-state supermultiplet. This means that in the physical sector of the localized states all supercharges act on the soliton trivially.

In our $\mathcal{N}=1$ examples of the single-state supermultiplet one of two supercharges is realized nontrivially, $Q_1 = \pm\sqrt{2\mathcal{Z}}$. In terms of modes there is one normalizable fermion mode. In Witten's case all fermion zero modes run away to the boundary, while in our case one mode is localized on the soliton, and the other at the boundary. Similar run-away behavior of the modes occurs in the phenomenon of the fractional charge and other phenomena known in solid state physics.

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Appendix (by P. Pushkar'): Vanishing of the index of the Dirac operator on compact reduced moduli space

The proof presented in this Appendix belongs to P. Pushkar'.

The index of the Dirac operator on the reduced moduli space $\mathcal{M}^n$ is known to be equal to $\hat{A}$-genus and can be expressed as an integral of polynomial of Pontriagin classes along $\mathcal{M}^n$. In order to show that the index vanishes we will show (in *Statement 3*) that tangent bundle to the space $\mathcal{M}^n$ of nonparametrized trajectories is such, that its sum with the trivial bundle is a trivial bundle. Really, then its Pontriagin classes (of nonzero degree) should vanish.

Here we assume that there are no other critical values between values of the initial and final critical points of superpotential, thus, the space $\mathcal{M}^n$ is compact. We will also assume that dimension of $\mathcal{M}^n$ is nonzero. Let us take the equi(super)potential surface L defined by $\mathcal{W} = c$ where the constant c is between the values of $\mathcal{W}$ at the initial and final points. The gradient trajectories coming out of the initial point intersect the surface L and produce a sphere $S^{\rm initial}$. The antigradient trajectories coming out of the final point also intersect L and produce another sphere $S^{\rm final}$. The space $\mathcal{M}^n$ is an intersection of these two spheres.

Statement 1: The normal bundles of $S^{\rm initial}$ and $S^{\rm final}$ in L are trivial.

For instance, to show this for $S^{\rm initial}$ let us move L close to the initial critical point (it would be a homotopy that should not change the triviality of the bundle). In the vicinity of the critical point we can replace the function by its quadratic approximation – then the triviality of the normal bundle becomes obvious.

Statement 2: The normal bundle to $\mathcal{M}^n$ in $S^{\rm initial}$ is trivial.

Indeed, this bundle is a restriction (to $\mathcal{M}^n$) of the normal bundle of $S^{\rm final}$ in L, and the latter is trivial due to *Statement 1.* Since the restriction of the trivial bundle is trivial, we have proved *Statement 2.*

Statement 3: The tangent bundle to $\mathcal{M}^n$ plus the trivial bundle is a trivial bundle.

Suppose that $\mathcal{M}^n$ is different from the total sphere $S^{\rm initial}$, then it is a submanifold of the Euclidean space. The tangent bundle to $\mathcal{M}^n$ plus the normal bundle (that is trivial due to *Statement 2*) gives a restriction on $\mathcal{M}^n$ of the tangent bundle to the Euclidean space (that is obviously trivial). Since the restriction of the trivial bundle is trivial we have proved *Statement 3* for $\mathcal{M}^n \neq S^{\rm initial}$. If $\mathcal{M}^n$ is a sphere $S^{\rm initial}$, then it could be obviously embedded in the Euclidean space with the trivial normal bundle. This completes the proof of *Statement 3.*

The generation function for Pontriagin classes for a trivial bundle is a class of degree zero. The generation function for Pontriagin classes of the sum of bundles is the product of the generation functions of each bundle. Thus, the only nonvanishing Pontriagin class on $\mathcal{M}^n$ is of degree zero, and the integral of the polynomial of Pontriagin classes along $\mathcal{M}^n$ will give zero.

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