CUBE OR HYPERCUBE OF NATURAL UNITS

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Max Planck introduced four natural units: h, c, G, k. Only the first three of them retained their status, representing the so called cube of theories, after the theory of relativity and quantum mechanics were created and became the pillars of physics. This short note is a little pebble on the tombstone of Michael Samuilovich Marinov.

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Dedication

In the evening of January 20, 2000 an email from ITEP was sent to Serezha Gurvitz and Arnon Dar:

> "Dear Serezha,
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> Dear Arnon,
>
> It is with great sorrow that we learned that Misha passed away. He was a brilliant physicist, an extraordinary person of outstanding integrity and courage, a friend of great warmth. He was like a tuning fork for all of us when he was in Moscow and later in Israel. We recall his lectures on group theory at ITEP, we recall his remarkable lectures on path integral. We recall his deep and original papers. He was a source of great knowledge and great wisdom.
>
> Today, on the Monday seminar, there was a minute of silence to the memory of Mikhail Samuilovich Marinov.
>
> Please, forward our condolences to Lilya, Masha, and Dina.
>
> Lev Okun, Victor Novikov, Mikhail Vysotsky".

Today, two years later, I subscribe to every word written that evening. This short note is a little pebble on the tombstone.

1 Planck units

Max Planck during years 1897–1899 published in the Sitz.-Ber. Preuss. Akad. Wiss. five consecutive reports[1] with the same title "Über irreversible Strahlungsvorgänge" [On irreversible processes of radiation]. Under the same title the article[2] appeared in Annalen der Physik in 1900, which summarized his reports[1] and his talk in Munich. In the article **(e)** in Ref. 1 and in Ref. 2 a special section was added on natural units of measure, in which Planck wrote (in an abridged translation from German):

> "All systems of physical units including the so-called absolute C.G.S.-system, appeared up to now due to accidental circumstances, as the choice of basic units in each of these systems occurred not from a general point of view valid for any place and time, but from the needs of our earthly culture. In this connection it would be interesting to note, that by using both constants a and b, which appear in the equation (41) for the entropy of radiation we get the possibility to establish units of length, mass, time and temperature, which

> would not depend on the choice of special bodies or substances and would be valid for all epochs and all cultures including extraterrestrial and extrahuman ones and could therefore serve as 'natural units of measurements' ".[2]

Let us note that equation (41) in Ref. 2 had the form:

$$S = -\frac{U}{a\nu} \ln \frac{U}{eb\nu} \ ,$$

where S is the total electromagnetic entropy, U is the total energy of the system, ν is the frequency of radiation, e is the base of the system of natural logarithms, $a = h/k$, $b = h$ in modern notations, where h is the Planck constant, while k is Boltzman constant.

Already in 1900 Planck wrote his famous formula for the spectral energy density:

$$\frac{8\pi\nu^3 h}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1}$$

using notations h and k instead of b and b/a respectively.[3] The references[1,2,3] have been reprinted in Ref. 4.

> "... The four units – for length, mass, time and temperature – are expressed in terms of above mentioned constants a and b, velocity of light c, and gravitational constant f. When expressed in centimeters, grams, seconds and Celsius degrees, these four quantities have the following values:
>
> $$\begin{aligned} a &= 0.4818 \cdot 10^{-10}\ \mathrm{sec} \cdot {}^0\mathrm{C} \\ b &= 6.885 \cdot 10^{-27}\ \mathrm{cm}^2 \cdot \mathrm{g/sec} \\ c &= 3.00 \cdot 10^{10}\ \mathrm{cm/sec} \\ f &= 6.885 \cdot 10^{-8}\ \mathrm{cm}^2/\mathrm{g} \cdot \mathrm{sec}^2 \ldots \end{aligned}$$ " [2]

Note that the modern notations and values are (see Review of Particle Physics [5]):

$$\begin{aligned} h/k\ (= a) &= 0.4799237 \cdot 10^{-10}\ \mathrm{s} \cdot \mathrm{K} \\ k\ (= b/a) &= 1.3806503(24) \cdot 10^{-23}\ \mathrm{J} \cdot \mathrm{K}^{-1} \\ h\ (= b\ ) &= 6.62606876(52) \cdot 10^{-34}\ \mathrm{J} \cdot \mathrm{s} \\ c &= 299792458\ \mathrm{m\ s}^{-1} \\ G\ (= f\ ) &= 6.673(10) \cdot 10^{-11}\ \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}\ , \end{aligned}$$

where K is Kelvin degree ($1\mathrm{K} = 1\ {}^0\mathrm{C}$).

> "If we chose now the "natural units", so that each of the above constants is put equal to 1, then we will get
>
> $$\begin{array}{lll} \text{for the unit of length:} & \sqrt{\frac{bf}{c^3}}\left(=\sqrt{\frac{hG}{c^3}}\right) & = \quad 4.13\cdot 10^{-33}\ \text{cm}\ , \\ \text{for the unit of mass:} & \sqrt{\frac{bc}{f}}\left(=\sqrt{\frac{hc}{G}}\right) & = \quad 5.56\cdot 10^{-5}\ \text{g}\ , \\ \text{for the unit of time:} & \sqrt{\frac{bf}{c^5}}\left(=\sqrt{\frac{hG}{c^5}}\right) & = \quad 1.38\cdot 10^{-43}\ \text{s}\ , \\ \text{for the unit of temperature:} & a\sqrt{\frac{c^5}{bf}}\left(=\frac{1}{k}\sqrt{\frac{hc^5}{G}}\right) & = \quad 3.50\cdot 10^{32}\ {}^0\text{C}\ . \end{array}$$
>
> These units will have their natural meaning as long as the laws of gravitation and of light propagation as well as both principles of thermodynamics remain valid." [2]

Thus Planck considered four natural constants. This was natural for him because the expression $e^{h\nu/kT}$ in the formula for energy density contains both h and k on equal footing and because quantum mechanics and relativity were unknown at that time, while principles of thermodynamics were considered to be fundamental.

2 The cube of theories

It is interesting that at the beginning of the article by G. Gamov, D. Ivanenko and L. Landau[6] they refer to the natural Planck system of four units, while in the main body of this article they consider as natural only three Planck units without k and temperature. This approach had been taken up by M. Bronstein[7,8] and Zelmanov,[9,10] who developed the idea of the cube of theories and by their followers.[11,12]

The cube is located along three orthogonal axes marked by c (actually by $1/c$), $\hbar$, G. The vertex (000) corresponds to nonrelativistic mechanics,($c00$) – to special relativity, ($0\hbar 0$) – to non-relativistic quantum mechanics, ($c\hbar 0$) – to quantum field theory, ($c0G$) – to general relativity, ($c\hbar G$) – to futuristic quantum gravity and the Theory of Everything, TOE. There is a hope that in the framework of TOE the values of dimensionless fundamental parameters will be ultimately calculated.

3 Temperature and Entropy

Let us note that in "Statistical Physics" by L. Landau and E. Lifshitz[13] temperature is measured in units of energy and hence in all formulas kT is substituted by T (k is put to 1).Mathematically temperature T is defined as a derivative of internal energy U of a system over its entropy S:

$$T = dU/dS \quad .$$

By keeping k Planck contradicted his own definition of natural units, because k is a ratio of "hand-crafted" units of temperature and energy. As temperature is an average energy of an ensemble of particles, it is natural to measure it in units of energy. In fact k is not a natural unit, but a conversion factor from degrees to joules or electronvolts. Therefore one should consider a three-dimensional cube of natural units (as was presented in Ref. 6-12, 14), not a four dimensional hypercube, as was suggested by J. Fröhlich,[15] who following Planck included k into the set of fundamental units.

The way one treats k is intimately connected with the way one treats entropy. According to Planck (and to many modern text-books) entropy looks dimensionful: $[S] = [k]$. The book by Landau and Lifshitz[13], according to which physical entropy is dimensionless, is one of the rare exceptions. On the other hand the informational entropy is usually defined as a dimensionless quantity:

$$S_{\rm inf} = -\sum_{k=1}^{n} P_k \ln P_k \quad ,$$

where P_k is probability of finding x_k from a set $x_1, x_2 ... x_n$ which describes a message. $S_{\rm inf} = 0$, if one of $P_k = 1$, while all other P_k vanish. $S_{\rm inf}$ is maximal when all P_k are equal (in this case the uncertainty of information is maximal). Thus in the case of $k = 1$ the definitions of physical and informational entropy are similar.

Acknowledgements

I am very grateful to Ya.I. Granowsky and V.L. Telegdi for interesting remarks. The work was supported by A. von Humbold Award and grant of RFBR 00-15-96562.

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