LIE ALGEBROIDS AS GAUGE SYMMETRIES IN TOPOLOGICAL FIELD THEORIES

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The Lie algebroids are generalization of the Lie algebras. They arise, in particular, as a mathematical tool in investigations of dynamical systems with the first class constraints. Here we consider canonical symmetries of Hamiltonian systems generated by a special class of Lie algebroids. The "coordinate part" of the Hamiltonian phase space is the Poisson manifold M and the Lie algebroid brackets are defined by means of the Poisson bivector. The Lie algebroid action defined on M can be lifted to the phase space. The main observation is that the classical BRST operator has the same form as in the case of the Lie groups action. Two examples are analyzed. In the first, M is the space of $SL(3,C)$-opers on Riemann curves with the Adler-Gelfand-Dikii brackets. The corresponding Hamiltonian system is the W_3-gravity. Its phase space is the base of the algebroid bundle. The sections of the bundle are the second order differential operators on Riemann curves. They are the gauge symmetries of the theory. The moduli space of W_3 geometry of Riemann curves is the symplectic quotient with respect to their action. It is demonstrated that the nonlinear brackets and the second order differential operators arise from the canonical brackets and the standard gauge transformations in the Chern-Simons field theory, as a result of the partial gauge fixing. The second example is $M = C^4$ endowed with the Sklyanin brackets. The symplectic reduction with respect to the algebroid action leads to a generalization of the rational Calogero-Moser model. As in the previous example the Sklyanin brackets can be derived from a "free theory." In this case it is a "relativistic deformation" of the $SL(2,C)$ Higgs bundle over an elliptic curve.

Contents

1 Introduction

In this paper I analyze the special first class constraints in the classical Hamiltonian systems. This subject was one of the main interests of Misha Marinov. I remember very interesting discussions with him when he prepared with Misha Terentev their paper devoted to dynamics on group manifolds.[1] They used the path integral for the quantization and the Lie group symmetries played essential role in their construction. The aim of this paper is going beyond the group symmetry, though we consider only the classical theory. I hope that Misha would have read to the present paper with interest.[a]

In fact, the group symmetries by no means exhaust all interesting symmetries in classical and quantum Hamiltonian systems. Generic first class constraints generate transformations that generalize the Lie group action. For example, the structure constants can depend on the points of the phase space. A generalization of this situation leads to the notion of the Lie algebroid and Lie groupoid. Batalin is likely to be the first who explicitly used this construction in gauge theories.[3] He called these transformations the quasigroups.

The BRST approach allows to work with arbitrary forms of constraints. In general situation the BRST operator has an infinite rank (the degree of the ghost momenta), while in the Lie group case it has rank one or less.[4] Nevertheless, it is possible to modify the Lie group symmetries in a such way that the BRST operator has the same form as for the Lie groups. We consider one of these cases.

To construct these symmetries we carry out the following steps:

1. Consider a Poisson manifold M. The transformations of M depend on the Poisson structure. The local version of them is a particular type of Lie algebroids $\mathcal{A}_M$. The space M plays the role of the "coordinate subspace" and the algebroid action is the transformations of the coordinates.
2. To come to the field theory we introduce the space of maps $\mathbf{M} = D \to M,\ D = \{|z| \leq 1\}$. The algebroid $\mathcal{A}_{\mathbf{M}}$ is the analogue of the loop algebra. It can be central extended $\hat{\mathcal{A}}_{\mathbf{M}}$ by a one-cocycle as in the loop algebra case.
3. The transformations coming from the Lie algebroid $\hat{\mathcal{A}}_{\mathbf{M}}$ can be lifted to the phase space $\mathcal{R}$. In addition to the coordinate space $\mathbf{M}$ the phase space $\mathcal{R}$ contains the conjugate variables. It is endowed with the canonical symplectic structure. The space $\mathcal{R}$ along with its symmetries defines the so-called Poisson sigma-model.[5,6]
4. The infinitesimal form of the symmetry transformations we call the Hamiltonian algebroid $\mathcal{A}^H_{\mathcal{R}}$ related to the Lie algebroid $\mathcal{A}_M$. The Hamiltonian algebroid

[a] Sections 2-4 of this paper is an updated version of part of Ref. 2.

is a generalization of the Lie algebra of symplectic vector fields with respect to the symplectic structure on $\mathcal{R}$. The special feature of these systems is that the BRST operator has the same structure as for the transformations performed by Lie algebras.

We present two examples of topological field theories with these symmetries. The first example is the W_3-gravity[7,8,9] and related to this theory generalized deformations of complex structures of Riemann curves by the second order differential operators. This theory is a generalization of $2+1$-gravity (W_2-gravity),[10] where the space component has a topology of a Riemann curve Σ_g of genus g. The Lie algebra symmetries in W_2-gravity is the algebra of smooth vector fields on Σ_g. After killing the gauge degrees of freedom one comes to the moduli space of projective structures on Σ_g. These structures can be described by the BRST method which is straightforward in this case. The case of W_N-gravity ($N > 2$) is more subtle. The main reason is that the gauge symmetries do not generate the Lie group action. This property of W_N-gravity was well known.[7]

We consider here in detail the W_3 case. The infinitesimal symmetries are carried out by the second order differential operators on Σ_g without constant terms. The role of the Poisson manifold M is played by the $M_3 = \mathrm{SL}(3, \mathbb{C})$-opers[11,12] endowed with the Adler-Gelfand-Dikii bivector.[13,14] The space M_3 is the configuration space of W_3-gravity. We define in a canonical way the action of the second order differential operators on $\mathrm{SL}(3, \mathbb{C})$-opers. In this way we come to a Lie algebroid $\mathcal{A}_{M_3}$ over $\mathrm{SL}(3, \mathbb{C})$-opers. The space of sections $\Gamma(\mathcal{A}_{M_3})$ is the space of second order differential operators on Σ_g without constant terms, endowed by a Lie bracket. The algebroid $\mathcal{A}_{M_3}$ is lifted to the Hamiltonian algebroid $\mathcal{A}^H_{\mathcal{R}_3}$ over the phase space $\mathcal{R}_3$ of W_3-gravity. The symplectic quotient of the phase space is the so-called $\mathcal{W}_3$-geometry of Σ_g. Roughly speaking, this space is a combination of the moduli of generalized complex structures and the spin 2 and 3 fields as the dual variables. Note that we deform the operator of complex structure $\bar{\partial}$ by symmetric combinations of vector fields $(\varepsilon\partial)^2$, in contrast with Ref. 15, where the deformations of complex structures are carried out by the polyvector fields. To define the W_3-geometry we construct the BRST operator for the Hamiltonian algebroid. As it follows from the general construction, it has the same structure as in the Lie algebra case. It should be noted that the BRST operator for the W_3-algebras was constructed in Ref. 16. But here we construct the BRST operator for the different object - the algebroid symmetries of W_3-gravity. We explain our formulae and the origin of the algebroid by the special gauge procedure of the $\mathrm{SL}(3, \mathbb{C})$ Chern-Simons theory using an approach developed in Ref. 8.

The next example is the Poisson manifold $\mathbb{C}^4$ with Sklyanin algebra structure.[17] As in the previous example, we demonstrate that these algebroid structure can be derived from the canonical brackets. The role of the Chern-Simons theory is played by a free theory in 1+1 dimension. It is a “relativistic” deformation of the Hitchin model for the elliptic Calogero-Moser system. The Sklyanin algebra arises on an intermediate step of the symplectic reduction. Our construction is closed to the scheme proposed in Ref. 18. In conclusion, we perform the further symplectic reduction with respect to the rest algebroid action and obtain an integrable model with two degrees of freedom following Ref. 19.

2 Lie algebroids

2.1 Poisson manifolds

Let M be a Poisson manifold endowed with the Poisson bivector π. In local coordinates $x = (x_1, \ldots, x_n)$ $\pi = \pi^{jk}$ and

$$\pi^{jk}(x) = -\pi^{kj}(x),$$

$$\partial_i \pi^{jk}(x)\pi^{im}(x) + \text{c.p.}(j,k,m) = 0. \tag{2.1}$$

The Poisson brackets are defined on the space of smooth functions $\mathcal{H}(M)$ [b]

$$\{f(x), g(x)\} = \pi^{jk}\partial_j f(x)\partial_k g(x) := \langle \partial f|\pi|\partial g\rangle, \quad (\partial_j = \partial_{x^j})\,.$$

The bivector allows to construct a map from $f \in \mathcal{H}(M)$ to the vector fields $V_f \in \Gamma(TM)$

$$f \rightarrow V_f = \pi^{jk}(x)\partial_j f(x)\partial_k = \langle \partial f|\pi|\partial\rangle. \tag{2.2}$$

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ be a section of T^*M. Consider the vector field

$$V_\varepsilon = \langle \varepsilon|\pi|\partial\rangle. \tag{2.3}$$

It acts on $\mathcal{H}(M)$ in the standard way

$$\delta_\varepsilon f = i_{V_\varepsilon} df = \langle \varepsilon|\pi|df\rangle.$$

In particular,

$$\delta_\varepsilon x^k = \pi^{kj}(x)\varepsilon_j. \tag{2.4}$$

[b] The sums over repeated indices are understood in Sections 2-4. For simplicity we assume here that M is a finite dimensional manifold, so later we shall consider the infinite dimensional case.

One can define the brackets on the sections of T^*M

$$\lfloor \varepsilon, \varepsilon' \rfloor = d\langle \varepsilon|\pi(x)|\varepsilon'\rangle + \langle d\varepsilon|\pi|\varepsilon'\rangle + \langle \varepsilon|\pi|d\varepsilon'\rangle, \tag{2.5}$$

or

$$\lfloor \varepsilon, \varepsilon' \rfloor_m = \langle \varepsilon|\partial_m \pi|\varepsilon'\rangle + \delta_\varepsilon \varepsilon' - \delta_{\varepsilon'}\varepsilon.$$

It follows from the Jacobi identity (2.1), that the commutator of the vector fields V_ε satisfies the identity

$$[V_\varepsilon, V_{\varepsilon'}] = V_{\lfloor \varepsilon, \varepsilon' \rfloor}. \tag{2.6}$$

2.2 Lie algebroids and Lie groupoids

Now we can define the Lie algebroid over M. As we already mentioned Lie algebroids is a generalization of bundles of Lie algebras over a base M. We present the definition of the Lie algebroid in general case. Details of this theory can be find in Refs. 20, 21, 22.

Definition 1 *A* Lie algebroid *over a differential manifold M is a vector bundle $\mathcal{A} \to M$ with a Lie algebra structure on the space of section $\Gamma(\mathcal{A})$ defined by the brackets $\lfloor \varepsilon_1, \varepsilon_2 \rfloor$, $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A})$ and a bundle map (*the anchor*) $\delta : \mathcal{A} \to TM$, satisfying the following conditions:*
(i) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A})$

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] = \delta_{\lfloor \varepsilon_1 \varepsilon_2 \rfloor}, \tag{2.7}$$

(ii) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$

$$\lfloor \varepsilon_1, f\varepsilon_2 \rfloor = f\lfloor \varepsilon_1, \varepsilon_2 \rfloor + (\delta_{\varepsilon_1} f)\varepsilon_2. \tag{2.8}$$

In other words, the anchor defines a representation of the space of the algebroid sections to the Lie algebra of vector fields on the base M. The second condition is the Leibniz rule with respect to the multiplication of the sections by smooth functions.

A Lie algebra is a particular case of Lie algebroids if M is a one-point space. If the anchor map $\delta_\varepsilon \equiv 0$, then we have a bundle of Lie algebras over M.

Let $\{e^j(x)\}$ be a basis of local sections $\Gamma(\mathcal{A})$. Then the brackets are defined by the structure functions $f_i^{jk}(x)$ of the algebroid

$$\lfloor e^j, e^k \rfloor = f_i^{jk}(x)e^i, \quad x \in V. \tag{2.9}$$

Using the Jacobi identity for the anchor action, we find

$$C^n_{j,k,m}\delta_{e_n} = 0, \tag{2.10}$$

where
$$C_n^{j,k,m} = (f_i^{jk}(x)f_n^{im}(x) + \delta_{e^m} f_n^{jk}(x) + \text{c.p.}(j,k,m)) \tag{2.11}$$
Thus, (2.10) implies *the anomalous Jacoby identity*

$$f_i^{jk}(x)f_n^{im}(x) + \delta_{e^m} f_n^{jk}(x) + \text{c.p.}(j,k,m) = 0 \tag{2.12}$$

For the Lie algebras bundles the anomalous terms disappear.

In our case $\mathcal{A} = T^*M$ with the brackets (2.5) defined on its sections and the structure functions
$$f_i^{jk}(x) = \partial_i \pi^{jk}(x).$$
The anchor is also has a special form $\delta_\varepsilon = V_\varepsilon$ (2.3) based on the Poisson bivector. The properties (2.7) and (2.8) follow from (2.6) and from (2.5). We denote the algebroid T^*M as $\mathcal{A}_M$. This type of Lie algebroids was introduced in Refs. 24, 23.

In generic case Lie algebroids $\mathcal{A}$ can be integrated to *Lie groupoid*.[3,20,22] Again we present the general definition.

Definition 2 *A Lie groupoid G_M over a manifold M is a pair of differential manifolds (G,M), two differential mappings $l,r\ :\ G_M \to M$ and a partially defined binary operation (a product) $(g,h) \mapsto g\cdot h$ satisfying the following conditions:*
(i) It is defined only when $l(g) = r(h)$.
(ii) It is associative: $(g\cdot h)\cdot k = g\cdot(h\cdot k)$ whenever the products are defined.
(iii) For any $g \in G_M$ there exist the left and right identity elements l_g and r_g such that $l_g \cdot g = g\cdot r_g = g$.
(iv) Each g has an inverse g^{-1} such that $g\cdot g^{-1} = l_g$ and $g^{-1}\cdot g = r_g$.

We denote an element of $g \in G_M$ by the triple $\ll x|g|y \gg$, where $x = l(g)$, $y = r(g)$. Then the product $g\cdot h$ is

$$g\cdot h \to \ll x|g\cdot h|z \gg = \ll x|g|y \gg \ll y|h|z \gg .$$

An orbit of the groupoid in the base M is defined as an equivalence $x \sim y$ if $x = l(g),\ y = r(g)$. There is the isotropy subgroup G_x for $x \in M$.

$$G_x = \{g \in G_M \mid l(g) = x = r(g)\} \sim \{\ll x|g|x \gg\}.$$

2.3 Central extension of Lie algebroids

There is a straightforward generalization of this construction to the infinite dimensional case. Consider a disk $D = |z| \leq 1$ and the space of anti-meromorphic maps

$\mathbf{M} = \{X : D \to M\}$. Define the Lie algebroid $\mathcal{A}_{\mathbf{M}}$ over the space $\mathbf{M}$. Now the base of the algebroid is infinite-dimensional. For simplicity we do not change the notion of the anchor action

$$\delta_\varepsilon X = \pi(X)|\varepsilon\rangle. \tag{2.13}$$

Here ε are the sections of $X^*(T^*M)$.

In the infinite dimensional case it is possible to extend the anchor action (2.13)

$$\hat{\delta}_\varepsilon f(X) = \delta_\varepsilon f(X) + c(X,\varepsilon), \tag{2.14}$$

by the one-cocycle

$$c(X,\varepsilon) = \frac{1}{2\pi}\oint \langle\varepsilon|\bar{\partial}X\rangle. \tag{2.15}$$

Since

$$\delta_{\varepsilon_1} c(X,\varepsilon_2) - \delta_{\varepsilon_2} c(X,\varepsilon_1) - c(X,\lfloor\varepsilon_1,\varepsilon_2\rfloor) = 0, \tag{2.16}$$

(the cocycle property) one has

$$[\hat{\delta}_{\varepsilon_1},\hat{\delta}_{\varepsilon_2}] = \hat{\delta}_{\lfloor\varepsilon_1,\varepsilon_2\rfloor}.$$

We denote the Lie algebroid over $\mathbf{M}$ with the anchor (2.14) as $\hat{\mathcal{A}}_{\mathbf{M}}$. This construction is the generalization of the bundle of the central extended loop Lie algebras with the loop parameter $\bar{z}$.

Consider the two-cocycle $c(X;\varepsilon_1,\varepsilon_2)$ on $\Gamma(\mathcal{A}_{\mathbf{M}})$

$$\delta_{\varepsilon_1} c(X;\varepsilon_2,\varepsilon_3) - \delta_{\varepsilon_2} c(X;\varepsilon_1,\varepsilon_3) = 0. \tag{2.17}$$

$$+\delta_{\varepsilon_3} c(X;\varepsilon_1,\varepsilon_2) - c(X;\lfloor\varepsilon_1,\varepsilon_2\rfloor,\varepsilon_3) + c(X;\lfloor\varepsilon_1,\varepsilon_3\rfloor,\varepsilon_2) - c(X;\lfloor\varepsilon_2,\varepsilon_3\rfloor,\varepsilon_1).$$

It allows to construct the central extensions of brackets on $\Gamma(\mathcal{A})$

$$\lfloor(\varepsilon_1,k_1),(\varepsilon_2,k_2)\rfloor_{c.e.} = (\lfloor\varepsilon_1,\varepsilon_2\rfloor, c(x;\varepsilon_1,\varepsilon_2)). \tag{2.18}$$

The cocycle condition (2.17) means that the new brackets $\lfloor\ ,\ \rfloor_{c.e.}$ satisfies AJI (2.12). The exact cocycles leads to the splitted extensions.

3 Poisson sigma-model

3.1 Poisson sigma-model and Hamiltonian algebroids

The manifold M serves as the target space for the Poisson sigma model. The space-time of the sigma model is the disk D.

Consider the one-form on D taking values in the pull-back by X of the cotangent bundle T^*M:

$$\xi = (\xi_1, \dots, \xi_n)$$

Endow the space of fields $\mathcal{R} = \{X, \xi\}$ with the canonical symplectic form

$$\omega = \frac{1}{2\pi} \oint \langle D\xi \wedge DX \rangle. \tag{3.19}$$

Consider the set of the first-class constraints

$$F^j := \bar{\partial} X^j + (\pi(X)|\xi\rangle)^j = 0. \tag{3.20}$$

They generate the canonical transformations of ω

$$\hat{\delta}_\varepsilon \xi_m = \frac{\delta}{\delta X^m} c(X, \varepsilon) + \langle \varepsilon | \frac{\delta}{\delta X^m} \pi | \xi \rangle = \bar{\partial} \epsilon_m + \langle \varepsilon | \frac{\delta}{\delta X^m} \pi | \xi \rangle, \tag{3.21}$$

and (2.13). It means that we lift the anchor action on $\mathcal{R}$ by means of the cocycle (2.15). Equivalently, the canonical transformations of a smooth functionals can be described as

$$\delta_{\varepsilon_j} f(X, \xi) = \{h_{\varepsilon_j}, f(X, \xi)\}.$$

Here the Poisson brackets are inverse to the symplectic form ω (3.19) and

$$h_{\varepsilon_j} = \frac{1}{2\pi} \int \varepsilon_j F^j, \tag{3.22}$$

(no summation on j). Again, due to (2.16)

$$\{h_\varepsilon, h_{\varepsilon'}\} = h_{\lfloor \varepsilon, \varepsilon' \rfloor}. \tag{3.23}$$

Summarizing, we have defined the symplectic manifold $\mathcal{R}\{\xi, X\}$ and the bundle $\mathcal{A}^H_\mathcal{R}$ over $\mathcal{R}$ with the sections $\varepsilon \in \Gamma(\mathcal{A}^H_\mathcal{R})$.

For a general symplectic manifold $\mathcal{R}$ this construction leads to the Hamiltonian algebroid[2]

Definition 3 *$\mathcal{A}^H_\mathcal{R}$ is a* Hamiltonian algebroid *over a symplectic manifold $\mathcal{R}$ if there is a bundle map from $\mathcal{A}^H_\mathcal{R}$ to the Lie algebra on $C^\infty(\mathcal{R})$: $\varepsilon \to h_\varepsilon$, (i.e. $f\varepsilon \to f h_\varepsilon$ for $f \in C^\infty(\mathcal{R})$) satisfying the following conditions:*
(i) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A}^H_\mathcal{R})$ and $x \in \mathcal{R}$

$$\{h_{\varepsilon_1}, h_{\varepsilon_2}\} = h_{\lfloor \varepsilon_1, \varepsilon_2 \rfloor}. \tag{3.24}$$

(ii) For any $\varepsilon_1, \varepsilon_2 \in \Gamma(\mathcal{A}^H_\mathcal{R})$ and $f \in C^\infty(\mathcal{R})$

$$\lfloor \varepsilon_1, f\varepsilon_2 \rfloor = f \lfloor \varepsilon_1, \varepsilon_2 \rfloor + \{h_{\varepsilon_1}, f\} \varepsilon_2. \tag{3.25}$$

The both conditions are similar to the defining properties of the Lie algebroids (2.7),(2.8). For general Hamiltonian algebroids the Jacobi identity takes the form

$$f_i^{jk}(x)f_n^{im}(x) + \{h_{e^m}, f_n^{jk}(x)\} + E_{[ln]}^{j,k,m} h_{\varepsilon^l} + c.p.(j,k,m) = 0, \qquad (3.26)$$

where $E_{[ln]}^{j,k,m}$ is an antisymmetric tensor in $[ln]$. This structure arises in the Hamiltonian systems with the first class constraints and leads to the open algebra of arbitrary rank.[4]

We have constructed a special Hamiltonian algebroid $\mathcal{A}_{\mathcal{R}}^{H}$ with the brackets (2.5), the base $\mathcal{R}$ equipped with the canonical form (3.19) and the anchor (2.13), (3.21). In this case the last term in (3.26) is absent and we come to the anomalous Jacobi identity similar to (2.12)

$$f_i^{jk}(x)f_n^{im}(x) + \{h_m, f_n^{jk}(x)\} = 0, \qquad (3.27)$$

where h_m is defined by (3.22). This identity plays the crucial role in the BRST construction.

3.2 BRST construction for Hamiltonian algebroids

Here we consider the BRST construction for general Hamiltonian algebroids. Let $\mathcal{A}^{H*}$ be the dual bundle. Its sections $\eta \in \Gamma(\mathcal{A}_{\mathcal{R}}^{H*})$ are the odd fields called *the ghosts*. Let

$$h_{e^j} = \frac{1}{2\pi}\oint \langle \eta_j | F(x) \rangle,$$

where $\{\eta_j\}$ is a basis in $\Gamma(\mathcal{A}^{H*})$ and $F(x) = 0$ are the moment constraints, generating the canonical algebroid action on $\mathcal{R}$. Introduce another type of odd variables (*the ghost momenta*) $\mathcal{P}^j \in \Gamma(\mathcal{A}_{\mathcal{R}}^{H})$, $j = 1,2,\ldots$ dual to the ghosts η_k, $k = 1,2,\ldots$. We attribute the ghost number one to the ghost fields $\mathrm{gh}(\eta) = 1$, minus one to the ghost momenta $\mathrm{gh}(\mathcal{P}) = -1$ and $\mathrm{gh}(x) = 0$ for $x \in \mathcal{R}$. Introducing In addition to the non-degenerate Poisson structure on $\mathcal{R}$ we introduce the Poisson brackets

$$\{\eta_j, \mathcal{P}^k\} = \delta_j^k, \quad \{\eta^j, x\} = \{\mathcal{P}_k, x\} = 0. \qquad (3.28)$$

Thus all fields are incorporated in the graded Poisson superalgebra

$$\mathcal{BFV} = \left(\Gamma(\wedge^\bullet(\mathcal{A}_{\mathcal{R}}^{H*} \oplus \mathcal{A}_{\mathcal{R}}^{H})\right) \otimes C^\infty(\mathcal{R}). = \Gamma(\wedge^\bullet \mathcal{A}_{\mathcal{R}}^{H*}) \otimes \Gamma(\wedge^\bullet \mathcal{A}_{\mathcal{R}}^{H}) \otimes C^\infty(\mathcal{R}).$$

(*the Batalin-Fradkin-Vilkovitsky (BFV) algebra*).

There exists a nilpotent operator on Q on $\mathcal{BFV}$ ($Q^2 = 0,\ gh(Q) = 1$) (*the BRST operator*) transforming $\mathcal{BFV}$ into the BRST complex. The cohomology of $\mathcal{BFV}$ complex give rise to the structure of the classical reduced phase space $\mathcal{R}^{red}$. Namely $H^0(Q)$ is identified with the classical observables.

Represent the action of Q as the Poisson brackets:

$$Q\psi = \{\psi, \Omega\}, \quad \psi, \Omega \in \mathcal{BFV}.$$

Due to the Jacobi identity for the Poisson brackets the nilpotency of Q is equivalent to

$$\{\Omega, \Omega\} = 0.$$

Since Ω is odd, the brackets are symmetric. For a generic Hamiltonian systems Ω has the form of infinite series in the ghost $\mathcal{P}^k$ expansion

$$\Omega = h_\eta + \frac{1}{4\pi}\oint \langle \lfloor \eta, \eta' \rfloor | \mathcal{P} \rangle + ...,\ (h_\eta = \oint \langle \eta | F \rangle).$$

The order of $\mathcal{P}$ in Ω is called *the rank* of the BRST operator Q. For generic Hamiltonian algebroids with the anomalous Jacobi identity (3.26) Q can have the infinite rank.

If $\mathcal{A}$ is a Lie algebra bundle defined along with its canonical action on $\mathcal{R}$ then Q has the rank one or less. In this case the BRST operator Q is the extension of the Cartan-Eilenberg operator giving rise to the cohomology of $\mathcal{A}$ with coefficients in $C^\infty(\mathcal{R})$ and the first two terms in the previous expression provide the nilpotency of Q.

The distinguish property of $\mathcal{A}^H_{\mathcal{R}}$ is that Ω has the rank one, though the Jacobi identity has additional terms (3.27) in compare with the Lie algebras[2]

$$\Omega = \frac{1}{2\pi}\oint \langle \eta | F \rangle + \frac{1}{4\pi}\oint \langle \lfloor \eta, \eta' \rfloor | \mathcal{P} \rangle \tag{3.29}$$

4 W_3-gravity

4.1 Lie algebroid over SL(3, $\mathbb{C}$)*-opers*

In our approach the starting point to the W_3-gravity is the space infinite dimensional Poisson space of the third order differential operators with smooth coefficients on a Riemann curve Σ_n (the SL(3, $\mathbb{C}$)-opers). The curve Σ_n play the role of the space part of the W_3-gravity, while the time is just $\mathbb{R}$. Locally SL(3, $\mathbb{C}$)-opers are defined as

$$\partial^3 - T\partial - W :\ \Omega^{(-1,0)}(\Sigma_g) \to \Omega^{(2,0)}(\Sigma_g), \quad (\partial = \partial_z), \tag{4.1}$$

where $\Omega^{(a,b)}(\Sigma_g)$ is the space of section of smooth (a,b) forms on Σ_g. In fact, they can be defined globally starting from $\mathrm{SL}(3,\mathbb{C})$ bundles over Σ_g. The connection on the $\mathrm{SL}(3,\mathbb{C})$ bundle has the special form corresponding to (4.1)

$$\nabla = \partial - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix}. \tag{4.2}$$

The $\mathrm{SL}(N,\mathbb{C})$ opers were introduced in Ref. 11. The general case was considered in Ref. 12.

The space of $\mathrm{SL}(3,\mathbb{C})$-opers plays the role of the Poisson manifold $\mathbf{M} = M_3 = \{W, T\}$, where the brackets are defined as

$$\{T(z), T(w)\} = \left(-2\kappa\partial^3 + 2T(z)\partial + \partial T(z)\right)\delta(z-w), \tag{4.3}$$

$$\{T(z), W(w)\} = \left(-\kappa\partial^4 + T(z)\partial^2 + 3W(z)\partial + \partial W(z)\right)\delta(z-w), \tag{4.4}$$

$$\{W(z), W(w)\} = \tag{4.5}$$

$$+\left(\frac{2}{3}k\partial^5 - \frac{4}{3}T(z)\partial^3 - 2\partial T(z)\partial^2 + \left(\frac{2}{3}(2-\kappa)T(z)^2 - 2\partial^2 T(z) + 2\partial W(z)\right)\partial\right.$$
$$\left. + \left(\partial^2 W(z) - \frac{2}{3}\partial^3 T(z) + \frac{2}{3}(2-\kappa)T(z)\partial T(z)\right)\right)\delta(z-w).$$

These brackets are defined in terms of coordinates in a neighborhood of a noncontractable contour on Σ_g. The parameter κ comes from the central extension of the Lie brackets on the sections of the algebroid $\mathcal{A}_3$ and we have $2g$ nonequivalent extensions (see below (4.20) and Remark 4.1). For $\kappa \neq 2$ the Poisson brackets $\{W(z), W(w)\}$ have two quadratic terms $T(z)^2$ and $\frac{2}{3}T(z)\partial T(z)$). In what follows we take $\kappa = 1$. Note, that $\bar{z}$ dependence is hidden these relations. In this way the Poisson structure on M_3 is just the global version of Adler-Gelfand-Dikii brackets.[13,14]

Define the Lie algebroid $\mathcal{A}_3$ over M_3. Its sections $\mathcal{D}_2 = \Gamma(\mathcal{A}_3)$ are the second order differential operators on Σ_g without constant terms. On a disk $\mathcal{D}_2$ can be trivialized and the sections are represented as

$$\varepsilon^{(1)} = \varepsilon^{(1)}(z,\bar{z})\frac{\partial}{\partial z}, \quad \varepsilon^{(2)} = \varepsilon^{(2)}(z,\bar{z})\frac{\partial^2}{\partial z^2},$$

$$\varepsilon^{(1)} \in \mathcal{D}^1, \quad \varepsilon^{(2)} \in \mathcal{D}^2, \quad \mathcal{D}_2 = \mathcal{D}^1 \oplus \mathcal{D}^2.$$

The second order differential operators do not generate a closed algebra with respect to the standard commutators. Moreover, they cannot be defined invariantly on Riemann curves in contrast with the first order differential operators. We introduce a new brackets that goes around the both disadvantages.

The antisymmetric brackets on $\mathcal{D}_2$ are defined by means the Poisson brackets (4.3),(4.4),(4.5) according to the general prescription (2.5).

$$\lfloor \varepsilon_1^{(1)}, \varepsilon_2^{(1)} \rfloor = \varepsilon_1^{(1)} \partial \varepsilon_2^{(1)} - \varepsilon_2^{(1)} \partial \varepsilon_1^{(1)}. \tag{4.6}$$

$$\lfloor \varepsilon^{(1)}, \varepsilon^{(2)} \rfloor = \left\{ \begin{array}{cc} -\varepsilon^{(2)} \partial^2 \varepsilon^{(1)}, & \in \mathcal{D}^1 \\ -2\varepsilon^{(2)} \partial \varepsilon^{(1)} + \varepsilon^{(1)} \partial \varepsilon^{(2)}, & \in \mathcal{D}^2 \end{array} \right. \tag{4.7}$$

$$\lfloor \varepsilon_1^{(2)}, \varepsilon_2^{(2)} \rfloor = \left\{ \begin{array}{cc} \frac{2}{3}[\partial(\partial^2 - T)\varepsilon_1^{(2)}]\varepsilon_2^{(2)} - \frac{2}{3}[\partial(\partial^2 - T)\varepsilon_2^{(2)}]\varepsilon_1^{(2)}, & \in \mathcal{D}^1 \\ \varepsilon_2^{(2)} \partial^2 \varepsilon_1^{(2)} - \varepsilon_1^{(2)} \partial^2 \varepsilon_2^{(2)}, & \in \mathcal{D}^2 \end{array} \right. \tag{4.8}$$

The brackets (4.6) are the standard Lie brackets of vector fields and therefore $\mathcal{D}^1$ is the Lie subalgebra of $\mathcal{D}_2$. The structure functions in (4.8) depend on the projective connection T.

Now consider the bundle map $\mathcal{A}_3$ to TM_3 defined by the anchor $\delta_{\varepsilon^{(j)}} = \langle \varepsilon^{(j)} | \pi | \frac{\delta}{\delta T} \rangle$, or $\delta_{\varepsilon^{(j)}} = \langle \varepsilon^{(j)} | \pi | \frac{\delta}{\delta W} \rangle$

$$\delta_{\varepsilon^{(1)}} T = -2\partial^3 \varepsilon^{(1)} + 2T\partial \varepsilon^{(1)} + \partial T \varepsilon^{(1)}, \tag{4.9}$$

$$\delta_{\varepsilon^{(1)}} W = -\partial^4 \varepsilon^{(1)} + 3W\partial \varepsilon^{(1)} + \partial W \varepsilon^{(1)} + T\partial^2 \varepsilon^{(1)}, \tag{4.10}$$

$$\delta_{\varepsilon^{(2)}} T = \partial^4 \varepsilon^{(2)} - T\partial^2 \varepsilon^{(2)} + (3W - 2\partial T)\partial \varepsilon^{(2)} + (2\partial W - \partial^2 T)\varepsilon^{(2)}, \tag{4.11}$$

$$\delta_{\varepsilon^{(2)}} W = \frac{2}{3}\partial^5 \varepsilon^{(2)} - \frac{4}{3}T\partial^3 \varepsilon^{(2)} - 2\partial T \partial^2 \varepsilon^{(2)} \tag{4.12}$$

$$+(\frac{2}{3}T^2 - 2\partial^2 T + 2\partial W)\partial \varepsilon^{(2)} + (\partial^2 W - \frac{2}{3}\partial^3 T + \frac{2}{3}T\partial T)\varepsilon^{(2)}.$$

The algebroid structure follows from the identity

$$[\delta_{\varepsilon_1^{(j)}}, \delta_{\varepsilon_2^{(k)}}] = \delta_{\lfloor \varepsilon_1^{(j)}, \varepsilon_2^{(k)} \rfloor}, \quad (j, k = 1, 2).$$

The Jacobi identity (2.12) in $\mathcal{A}_3$ takes the form

$$\lfloor \lfloor \varepsilon_1^{(2)}, \varepsilon_2^{(2)} \rfloor, \varepsilon_3^{(2)} \rfloor^{(1)} - (\varepsilon_1^{(2)} \partial \varepsilon_2^{(2)} - \varepsilon_2^{(2)} \partial \varepsilon_1^{(2)}) \delta_{\varepsilon_3^{(2)}} T + \text{c.p.}(1,2,3) = 0, \tag{4.13}$$

$$\lfloor \lfloor \varepsilon_1^{(2)}, \varepsilon_2^{(2)} \rfloor, \varepsilon_3^{(1)} \rfloor^{(1)} + \text{c.p.}(1,2,3) = (\varepsilon_1^{(2)} \partial \varepsilon_2^{(2)} - \varepsilon_2^{(2)} \partial \varepsilon_1^{(2)}) \delta_{\varepsilon_3^{(1)}} T. \tag{4.14}$$

The brackets here correspond to the product of structure functions in the left hand side of (2.12) and the superscript (1) corresponds to the $\mathcal{D}^1$ component. For the rest brackets the Jacobi identity is the standard one.

The origin of the brackets and the anchor representations follow from the matrix description of $\mathrm{SL}(3,\mathbb{C})$-opers (4.2). Let E_3 be the principle $\mathrm{SL}(3,\mathbb{C})$-bundle over Σ_g. Consider the set $\mathcal{G}_3$ of automorphisms of the bundle E_3

$$A \to f^{-1}\partial f - f^{-1}Af \tag{4.15}$$

that preserve the $\mathrm{SL}(3,\mathbb{C})$-oper structure

$$f^{-1}\partial f - f^{-1}A(W,T)f = A(W',T'), \tag{4.16}$$

$$A(W,T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix}.$$

It is clear that $\mathcal{G}_3$ is the Lie groupoid over $M_3 = \{W,T\}$ with $l(f) = (W,T)$, $r(f) = (W',T')$, $f \to \ll W,T|f|W',T' \gg$. The left identity map is the $\mathrm{SL}(3,\mathbb{C})$ subgroup of $\mathcal{G}_3$

$$P\exp\left(-\int_{z_0}^{z} A(W,T)\right)\cdot C\cdot P\exp\left(\int_{z_0}^{z} A(W,T)\right),$$

where C is an arbitrary matrix from $\mathrm{SL}(3,\mathbb{C})$. The right identity map has the same form with (W,T) replaced by (W',T').

The local version of (4.16) takes the form

$$\partial X - \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ W & T & 0 \end{pmatrix}, X\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta W & \delta T & 0 \end{pmatrix}. \tag{4.17}$$

It is the sixth order linear differential system for the matrix elements of the traceless matrix X. The matrix elements $x_{j,k} \in \Omega^{(j-k,0)}(\Sigma_g)$ depend on two arbitrary fields $x_{23} = \varepsilon^{(1)}$, $x_{13} = \varepsilon^{(2)}$. The solution takes the form

$$X(\varepsilon^{(1)},\varepsilon^{(2)}) = \begin{pmatrix} x_{11} & x_{12} & \varepsilon^{(2)} \\ x_{21} & x_{22} & \varepsilon^{(1)} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \tag{4.18}$$

$$x_{11} = \frac{2}{3}(\partial^2 - T)\varepsilon^{(2)} - \partial\varepsilon^{(1)},\ x_{12} = \varepsilon^{(1)} - \partial\varepsilon^{(2)},$$

$$x_{21} = \frac{2}{3}\partial(\partial^2 - T)\varepsilon^{(2)} - \partial^2\varepsilon^{(1)} + W\varepsilon^{(2)},\ x_{22} = -\frac{1}{3}(\partial^2 - T)\varepsilon^{(2)},$$

$$x_{31} = \frac{2}{3}\partial^2(\partial^2 - T)\varepsilon^{(2)} - \partial^3\varepsilon^{(1)} + \partial(W\varepsilon^{(2)}) + W\varepsilon^{(1)},$$

$$x_{32} = \frac{1}{3}\partial(\partial^2 - T)\varepsilon^{(2)} - \partial^2\varepsilon^{(1)} + W\varepsilon^{(2)} + T\varepsilon^{(1)},$$

$$x_{33} = -\frac{1}{3}(\partial^2 - T)\varepsilon^{(2)} + \partial\varepsilon^{(1)}.$$

The matrix elements of the commutator $[X_1, X_2]_{13}$, $[X_1, X_2]_{23}$ give rise to the brackets (4.6), (4.7), (4.8). Simultaneously, from (4.17) one obtain the anchor action (4.9)-(4.12).

There is a nontrivial cocycle corresponding to $H^1(\mathcal{A}_3)$ with two components

$$c_\alpha(\varepsilon_1^{(j)}, \varepsilon_2^{(k)}) = \oint_{\gamma_\alpha} \lambda(\varepsilon_1^{(j)}, \varepsilon_2^{(k)}), \quad (j,k = 1,2), \tag{4.19}$$

The cocycle allows to shift the anchor action

$$\hat{\delta}_{\varepsilon^{(j)}} f(W,T) = \int_{\Sigma_g} \langle \delta_{\varepsilon^{(j)}} W | \frac{\delta f}{\delta W} \rangle + \int_{\Sigma_g} \langle \delta_{\varepsilon^{(j)}} T | \frac{\delta f}{\delta T} \rangle + c^{(j)}, \quad (j = 1,2).$$

There exists the $2g$ central extensions c_α of the algebra $\mathcal{D}_2$, provided by the nontrivial cocycles from $H^2(\mathcal{A}_3, M_3)$. They are the non-contractable contour integrals γ_α

$$c_\alpha(\varepsilon_1^{(j)}, \varepsilon_2^{(k)}) = \oint_{\gamma_\alpha} \lambda(\varepsilon_1^{(j)}, \varepsilon_2^{(k)}), \quad (j,k = 1,2), \tag{4.20}$$

where

$$\lambda(\varepsilon_1^{(1)}, \varepsilon_2^{(1)}) = \varepsilon_1^{(1)}\partial^3\varepsilon_2^{('1)}, \quad \lambda(\varepsilon_1^{(1)}, \varepsilon_2^{(2)}) = \varepsilon_1^{(1)}\partial^4\varepsilon_2^{(2)},$$

$$\lambda(\varepsilon_1^{(2)}, \varepsilon_2^{(2)}) =$$

$$\frac{2}{3}\left(-(\partial^2 - T)\varepsilon_1^{(2)}\partial(\partial^2 - T)\varepsilon_2^{(2)} + \partial^2\varepsilon_1^{(2)}\partial(\partial^2 - T)\varepsilon_2^{(2)} + \partial^2\varepsilon_2^{(2)}\partial(\partial^2 - T)\varepsilon_1^{(2)}\right).$$

It can be proved that $sc^j = 0$ (2.17) and that c^j is not exact. These cocycles allow to construct the central extensions of $\hat{\mathcal{A}}_3$:

$$\lfloor(\varepsilon_1^{(j)}, \sum_\alpha k_\alpha^{(j)}), (\varepsilon_2^{(m)}, \sum_\alpha k_\alpha^{(m)})\rfloor_{c.e.} = (\lfloor\varepsilon_1^{(j)}, \varepsilon_2^{(m)}\rfloor, \sum_\alpha c_\alpha(\varepsilon_1^{(j)}, \varepsilon_2^{(m)})).$$

Remark 4.1 *As in the case of the Lie algebras the Poisson structure (4.3), (4.4) and (4.5) can be read off from the Lie brackets (4.6), (4.7), (4.8) along with their central extensions (4.20) according with the Lie brackets for* $H_1, H_2 \in C^\infty(M_3)$

$$\{H_1, H_2\} = \oint_{\gamma_\alpha} \langle \lfloor \delta H_1, \delta H_2 \rfloor X \rangle + k \oint_{\gamma_\alpha} \lambda(\delta H_1, \delta H_2),$$

where $X = W, T$ *and* $\delta H = \delta H/\delta X \in \mathcal{D}_2$.

4.2 Phase space of W_3-gravity and BRST operator

The dual fields to T and W are the Beltrami differentials $\mu \in \Omega^{(-1,1)}(\Sigma_g)$ and the differentials $\rho \in \Omega^{(-2,1)}(\Sigma_g)$. Along with T and W they generate the space $\mathcal{R}_3$. It is the classical phase space for the W_3-gravity.[7,8,9] The symplectic form on $\mathcal{R}_3$ has the canonical form

$$\omega = \int_{\Sigma_g} DT \wedge D\mu + DW \wedge D\rho. \tag{4.21}$$

According with the general theory the anchor (4.9)-(4.12) can be lifted from M_3 to $\mathcal{R}_3$. This lift is nontrivial owing to the cocycle (4.19). It follows from (3.21) that the anchor action on μ and ρ takes the form

$$\delta_{\varepsilon^{(1)}}\mu = -\bar{\partial}\varepsilon^{(1)} - \mu\partial\varepsilon^{(1)} + \partial\mu\varepsilon^{(1)} - \rho\partial^2\varepsilon^{(1)}, \tag{4.22}$$

$$\delta_{\varepsilon^{(1)}}\rho = -2\rho\partial\varepsilon^{(1)} + \partial\rho\varepsilon^{(1)}, \tag{4.23}$$

$$\delta_{\varepsilon^{(2)}}\mu = \partial^2\mu\varepsilon^{(2)} - \frac{2}{3}\left[(\partial(\partial^2 - T)\rho)\varepsilon^{(2)} - (\partial(\partial^2 - T)\varepsilon^{(2)})\rho\right], \tag{4.24}$$

$$\delta_{\varepsilon^{(2)}}\rho = -\bar{\partial}\varepsilon^{(2)} + (\rho\partial^2\varepsilon^{(2)} - \partial^2\rho\varepsilon^{(2)}) + 2\partial\mu\varepsilon^{(2)} - \mu\partial\varepsilon^{(2)}. \tag{4.25}$$

In this way we have defined the Hamiltonian Lie algebroid over $\mathcal{R}_3$ with the section $\mathcal{D}_2$, the brackets (4.6),(4.7),(4.8), and the anchor (4.9)-(4.12), (4.22)-(4.25).

There are two Hamiltonians, defining by the anchor and by the cocycle

$$h^{(1)} = \int_{\Sigma_g} (\mu\delta_{\varepsilon^{(1)}}T + \rho\delta_{\varepsilon^{(1)}}W) + c^{(1)}, \quad h^{(2)} = \int_{\Sigma_g} (\mu\delta_{\varepsilon^{(2)}}T + \rho\delta_{\varepsilon^{(2)}}W) + c^{(2)}.$$

After the integration by part they take the form

$$h^{(1)} = \int_{\Sigma_{g,n}} \varepsilon^{(1)}F^{(1)},$$

$$h^{(2)} = \int_{\Sigma_{g,n}} \varepsilon^{(2)}F^{(2)},$$

where $F^{(1)} \in \Omega^{(2,1)}(\Sigma_{g,n})$, $F^2 \in \Omega^{(3,1)}(\Sigma_{g,n})$

$$F^{(1)} = -\bar{\partial}T - \partial^4\rho + T\partial^2\rho - (3W - 2\partial T)\partial\rho - \tag{4.26}$$

$$-(2\partial W - \partial^2 T)\rho + 2\partial^3\mu - 2T\partial\mu - \partial T\mu,$$

$$F^{(2)} = -\bar{\partial}W - \frac{2}{5}\partial^5\rho + \frac{4}{3}T\partial^3\rho + 2\partial T\partial^2\rho(-\frac{2}{3}T^2 + 2\partial^2 T - 2\partial W)\partial\rho + \quad (4.27)$$
$$+(-\partial^2 W + \frac{2}{3}\partial^3 T - \frac{2}{3}T\partial T)\rho + \partial^4\mu - 3W\partial\mu - \partial W\mu - T\partial^2\mu.$$

They carry out the moment map

$$m = (m^{(1)} = F^{(1)}, m^{(2)} = F^{(2)}) : \mathcal{R}_3 \to \Gamma^*(\mathcal{A}_3).$$

We assume that $m^{(1)} = 0,\ m^{(2)} = 0$. Then the coadjoint action of $\mathcal{D}_2$ preserve $m = (0,0)$. The moduli space $\mathcal{W}_3$ of the W_3-gravity (W_3-geometry) is the symplectic quotient with respect to the groupoid $\mathcal{G}_3$ action

$$\mathcal{W}_3 = \mathcal{R}_3//\mathcal{G}_3 = \{F^1 = 0, F^2 = 0\}/\mathcal{G}_3.$$

The space $\mathcal{W}_3$ is finite-dimensional. It has dimension

$$\dim \mathcal{W}_3 = 16(g-1).$$

The prequantization of $\mathcal{W}_3$ can be realized in the space of sections of a linear bundle $\mathcal{L}$ over the space of orbits $\tilde{M}_3 \sim M_3/\mathcal{G}_3$. The sections are functionals $\Psi(T,W)$ on M_3 satisfying the following conditions

$$\hat{\delta}_{\varepsilon^{(j)}}\Psi(T,W) := \langle\delta_{\varepsilon^{(j)}}W|\frac{\delta\Psi}{\delta W}\rangle + \langle\delta_{\varepsilon^{(j)}}T|\frac{\delta\Psi}{\delta T}\rangle + c^{(j)}\Psi = 0, \quad (j = 1,2).$$

Presumably, the bundle $\mathcal{L}$ can be identified with the determinant bundle $\det(\partial^3 - T\partial - W)$.

The moment equations $F^{(1)} = 0,\ F^{(2)} = 0$ are the consistency conditions for the linear system

$$\begin{cases} (\partial^3 - T\partial - W)\psi(z,\bar{z}) = 0, \\ \left(\bar{\partial} + (\mu - \partial\rho)\partial + \rho\partial^2 + \frac{2}{3}(\partial^2 - T)\rho - \partial\mu\right)\psi(z,\bar{z}) = 0, \end{cases} \quad (4.28)$$

where $\psi(z,\bar{z}) \in \Omega^{(-1,0)}(\Sigma_g)$. The last equation represents the deformation of the antiholomorphic operator $\bar{\partial}$ by the first order operator $\mu\partial$ and by the second order differential operator $\rho\partial^2$. The left hand side is the exact form of the deformed operator when it acts on $\Omega^{(-1,0)}(\Sigma_g)$. This deformation cannot be supported by the structure of a Lie algebra and one leaves with the Hamiltonian algebroid symmetries.

Instead of the symplectic reduction one can apply the BRST construction. Then cohomology of the moduli space $\mathcal{W}_3$ are isomorphic to $H^j(Q)$. To construct the BRST complex we introduce the ghosts fields $\eta^{(1)}, \eta^{(2)}$ and their momenta $\mathcal{P}^{(1)}, \mathcal{P}^{(2)}$. Then it follows from the general construction that for

$$\Omega = \sum_{j=1,2} h^{(j)}(\eta^{(j)}) + \frac{1}{2}\sum_{j,k,l=1,2}\int_{\Sigma_{g,n}}(\lfloor\eta^{(j)}, \eta^{(k)}\rfloor\mathcal{P}^{(l)}).$$

The operator $QF = \{F, \Omega\}$ is nilpotent and defines the BRST cohomology in the complex
$$\bigwedge{}^{\bullet}(\mathcal{D}_2 \oplus \mathcal{D}_2^*) \otimes C^\infty(\mathcal{R}_3).$$

4.3 Chern-Simons description

Here we follow the approach proposed in Ref. 8. Consider the Chern-Simons functional on $\Sigma_g \oplus \mathbf{R}^+$
$$S = \int_{\Sigma_g \oplus \mathbf{R}} \mathrm{tr}\left(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right), \quad \mathcal{A} = (A, \bar{A}, A_t).$$

In the hamiltonian picture the components $A, \bar{A}$ are elements of the phase space $\mathcal{R}_{\mathrm{SL}(3,\mathbb{C})}$ with the symplectic form
$$\int_{\Sigma_g} DA \wedge D\bar{A},$$
while A_t is the Lagrange multiplier for the first class constraints
$$F(A, \bar{A}) := \bar{\partial}A - \partial\bar{A} + [\bar{A}, A] = 0. \tag{4.29}$$

The phase space $\mathcal{R}_3$ can be derived from the phase space of the Chern-Simons theory $\mathcal{R}_{\mathrm{SL}(3,\mathbb{C})}$. The flatness condition (4.29) generates the gauge transformations
$$A \to f^{-1}\partial f + f^{-1}Af, \quad \bar{A} \to f^{-1}\bar{\partial} f + f^{-1}\bar{A}f, \quad f \in G_{\mathrm{SL}(3,\mathbb{C})} \tag{4.30}$$
The result of the gauge fixing with respect to the whole gauge group $G_{\mathrm{SL}(3,\mathbb{C})}$ is the moduli space $\mathcal{M}_3^{flat}$ of the flat $\mathrm{SL}(3,\mathbb{C})$ bundles over Σ_g.

Let P be the maximal parabolic subgroup of $\mathrm{SL}(3,\mathbb{C})$ of the form
$$P = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix},$$
and G_P be the corresponding gauge group. We partly fix the gauge with respect to G_P. A generic connection ∇ can be gauge transformed by $f \in G_P$ to the form (4.2).

The form of $\bar{A}$ can be read off from (4.29)
$$\bar{A} = \begin{pmatrix} a_{11} & a_{12} & -\rho \\ a_{21} & a_{22} & -\mu \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{4.31}$$

$$a_{11} = -\frac{2}{3}(\partial^2 - T)\rho + \partial\mu, \quad a_{12} = -\mu + \partial\rho,$$

$$a_{21} = -\frac{2}{3}\partial(\partial^2 - T)\rho + \partial^2\mu - W\rho, \quad a_{22} = \frac{1}{3}(\partial^2 - T)\rho,$$

$$a_{31} = -\frac{2}{3}\partial^2(\partial^2 - T)\rho + \partial^3\mu - \partial(W\rho) - W\mu,$$

$$a_{32} = -\frac{1}{3}\partial(\partial^2 - T)\rho + \partial^2\mu - W\rho - T\mu, \quad a_{33} = \frac{1}{3}(\partial^2 - T)\rho - \partial\mu.$$

The flatness (4.29) for the special choice A (4.2) and $\bar{A}$ (4.31) gives rise to the moment constraints $F^{(2)} = 0$, $F^{(1)} = 0$ (4.26),(4.27). Namely, one has $F(A, \bar{A})|_{(3,1)} = F^{(2)}$ (4.26), $F(A, \bar{A})|_{(2,1)} = F^{(1)}$ (4.27), while the other matrix elements of $F(A, \bar{A})$ vanish identically. In this way, we come to the matrix description of the moduli space $\mathcal{W}_3$.

The groupoid action on $A, \bar{A}$ plays the role of the rest gauge transformations that complete the G_P action to the $G_{\mathrm{SL}(3,\mathbb{C})}$ action. The algebroid symmetry arises in this theory as a result of the partial gauge fixing by G_P. Thus we come to the following diagram

$$\begin{tikzcd}
\boxed{\mathcal{R}_{\mathrm{SL}(3,\mathbb{C})}} \arrow[dd, dashed, "G_{\mathrm{SL}(3,\mathbb{C})}"'] \arrow[dr, "G_P"] & \\
& \boxed{\mathcal{R}_3} \arrow[d, "\Gamma(\mathcal{G}_3)"] \\
\boxed{\mathcal{M}^{flat}_{\mathrm{SL}(3,\mathbb{C})}} & \boxed{\mathcal{W}_3}
\end{tikzcd}$$

The tangent space to $\mathcal{M}^{flat}_{\mathrm{SL}(3,\mathbb{C})}$ at the point $(A = 0, \bar{A} = 0)$ coincides with the tangent space to $\mathcal{W}_3$ at the point $(W = 0, T = 0, \mu = 0, \rho = 0)$. Their dimension is $16(g - 1)$. But their global structure is different and the diagram cannot be closed by the horizontal isomorphisms. The interrelations between $\mathcal{M}^{flat}_{\mathrm{SL}(3,\mathbb{C})}$ and $\mathcal{W}_3$ were analyzed by Hitchin.[25]

5 Sklyanin algebra

5.1 Hamiltonian algebroid for Sklyanin algebra

We start from the Poisson bivector defined on a $\mathbb{C}^4$ with generators $(S_0, \vec{S} = (S_\alpha),\ \alpha = 1, 2, 3)$ [c]

$$\{S_\alpha, S_0\} = 2J_{\beta\gamma} S_\beta S_\gamma, \quad \{S_\alpha, S_\beta\} = -2\epsilon_{\alpha\beta\gamma} S_0 S_\gamma, \tag{5.1}$$

where $J_{\beta\gamma} = \wp_\beta - \wp_\gamma$, $\wp_\alpha = e_\alpha$, [d] and

$$e_1 = \wp\left(\frac{1}{2}\right), \ e_2 = \wp\left(\frac{1+\tau}{2}\right), \ e_3 = \wp\left(\frac{\tau}{2}\right), \tag{5.2}$$

and $\wp = \wp(u;\tau)$ is the Weierstrass function. In this way the algebra is parameterized by the modular parameter τ. The Jacobi identity is provided by the identity $e_1 + e_2 + e_3 = 0$. It is the Sklyanin algebra $\mathcal{S}$. [17] It has two Casimirs

$$K_0 = \sum S_\alpha^2 \quad , K_1 = S_0^2 + \sum \wp_\alpha S_\alpha^2. \tag{5.3}$$

The Lie algebroid brackets (2.5) for $(\varepsilon_0, \vec{\varepsilon}) \in T^*\mathcal{S}$ take the form

$$\begin{aligned}
\lfloor \varepsilon_\beta, \varepsilon_0 \rfloor_\alpha &= 2J_{\beta\alpha} S_\beta \varepsilon_\alpha, \\
\lfloor \varepsilon_\alpha, \varepsilon_\beta \rfloor_0 &= -2\epsilon_{\alpha\beta\gamma} S_\gamma \varepsilon_\alpha \varepsilon_\beta, \\
\lfloor \varepsilon_\alpha, \varepsilon_\beta \rfloor_\gamma &= -2\epsilon_{\alpha\beta\gamma} S_0 \varepsilon_\alpha \varepsilon_\beta.
\end{aligned} \tag{5.4}$$

It is convenient to arrange the Poisson brackets on $\mathcal{S}$ in the form of the classical Yang-Baxter equation. Introduce three functions $\varphi_\alpha(z)$ depending on a formal (spectral) parameter z living on the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$

$$\varphi_1(z) = \phi\left(\frac{1}{2}, z\right), \ \varphi_2(z) = -e^{\pi i\tau}\phi\left(\frac{1+\tau}{2}, z\right), \ \varphi_3(z) = e^{\pi i\tau}\phi\left(\frac{\tau}{2}, z\right), \tag{5.5}$$

where

$$\phi(u, z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)},$$

and

$$\vartheta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i(n(n+1)\tau + 2nz)}$$

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[c] In this section to reconcile with the accepted notations of the Sklyanin algebra we do not comply with the positions of the upper and lower indices.
[d] We use the standard notations of the elliptic functions.[26]

is the odd theta-function. Consider the Lax operator

$$L(z) = S_0\mathrm{Id} + i\mathbf{S}(z), \tag{5.6}$$

where

$$\mathbf{S}(z) = \sum S_\alpha \varphi_\alpha(z)\sigma_\alpha$$

and σ_α are the Hermitian sigma matrices $\sigma_\alpha\sigma_\beta = -i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$. Then (5.1) is equivalent to the matrix relation

$$\{L_1(z), L_2(w)\} = [r(z-w), L_1 \otimes L_2], \tag{5.7}$$

where r is the classical $\mathrm{SL}(2,\mathbb{C})$ elliptic r-matrix

$$r(z) = \varphi_\alpha(z)\sigma_\alpha \otimes \sigma_\alpha. \tag{5.8}$$

Note that

$$\det(L^2) = K_0\wp(z) - K_1. \tag{5.9}$$

5.2 Sklyanin algebra from canonical brackets

Let E_2 be the principle $\mathrm{GL}(2,\mathbb{C})$-bundle over Σ_τ with $\deg(E_2) = 1$. Consider the group element $g \in \Omega^{(0)}(\Sigma_\tau, \mathrm{GL}(2,\mathbb{C}))$

$$g(z,\bar{z}) = g_0(z,\bar{z}) + ic\sum_\alpha g_\alpha(z,\bar{z})\sigma_\alpha,$$

where c is a relativistic parameter. In what follows for brevity we omit the $\bar{z}$ dependence. To be a section of the degree one bundle the functions g_α should satisfy the quasi-periodicity conditions

$$g(z+1) = \sigma_3 g(z)\sigma_3^{-1}, \quad g(z+\tau) = \Lambda g(z)\Lambda^{-1}, \tag{5.10}$$

$$\Lambda = \Lambda(z;\tau) = -\sigma_1 \exp -\pi(\frac{1}{2}\tau + z).$$

The holomorphic structure on E_2 is defined by the operator

$$d_{\bar{A}} = k\bar{\partial} + \bar{A}: \ \Omega^{(j,k)}(\Sigma_\tau, \mathrm{sl}(2,\mathbb{C})) \to \Omega^{(j,k+1)}(\Sigma_\tau, \mathrm{sl}(2,\mathbb{C})),$$

$$\bar{A} = A_0 + i\sum_\alpha \bar{A}_\alpha\sigma_\alpha.$$

Here we have introduced the central charge k. The moduli space $\mathcal{M}$ of stable holomorphic bundles is the factor space

$$\mathcal{M} = \{k\bar{\partial} + \bar{A}\}/\mathcal{G},$$

where $\mathcal{G} = \{f \in \Omega^{(0)}(\Sigma_\tau, \mathrm{GL}(2,\mathbb{C}))\}$ is the gauge group

$$\bar{A} \to f^{-1}k\bar{\partial}f + f^{-1}\bar{A}f. \tag{5.11}$$

There exists the canonical symplectic structure on the pair of fields $\mathcal{R}' = \{g, \bar{A}\}$

$$\omega' = \int_{\Sigma_\tau} \mathrm{tr}\left(D(\bar{A}g^{-1}) \wedge Dg\right) + \frac{k}{2}\int_{\Sigma_\tau}\left(g^{-1}Dg \wedge \bar{\partial}(g^{-1}Dg)\right). \tag{5.12}$$

The canonical class of Σ_τ is incorporated in the integrals. We fix it in what follows.

If $c \to 0$ then in the first order in c we come to the symplectic structure on the cotangent bundle to the holomorphic structures on E_2.[27] The element $\sum_\alpha g_\alpha\sigma_\alpha$ being multiply on the canonical class plays the role of the Higgs field in the Hitchin construction.

The form generates the following Poisson brackets in the space of smooth functionals over $\mathcal{R}'$

$$\{H_1(\bar{A},g), H_2(\bar{A},g)\} =$$
$$\int_{\Sigma_\tau} \mathrm{tr}\left(\left[\frac{\delta H_1(\bar{A},g)}{\delta\bar{A}}, \frac{\delta H_2(\bar{A},g)}{\delta\bar{A}}\right]\bar{A}\right) + k\int_{\Sigma_\tau}\mathrm{tr}\left(\frac{\delta H_1(\bar{A},g)}{\delta\bar{A}}\bar{\partial}\frac{\delta H_2(\bar{A},g)}{\delta\bar{A}}\right)$$
$$+ \int_{\Sigma_\tau}\mathrm{tr}\left(\frac{\delta H_2(\bar{A},g)}{\delta\bar{A}}\frac{\delta H_1(\bar{A},g)}{\delta g}g\right) - \int_{\Sigma_\tau}\mathrm{tr}\left(\frac{\delta H_1(\bar{A},g)}{\delta\bar{A}}\frac{\delta H_2(\bar{A},g)}{\delta g}g\right).$$

In particular,

$$\begin{aligned}\{\bar{A}_\alpha(z), \bar{A}_\beta(w)\} &= \epsilon_{\alpha\beta\gamma}\bar{A}_\gamma(z)\delta^{(2)}(z-w) + \frac{k}{2}\delta_{\alpha,\beta}\bar{\partial}\delta^{(2)}(z-w),\\ \{\bar{A}_0(z), \bar{A}_0(w)\} &= \frac{k}{2}\delta_{\alpha,\beta}\bar{\partial}\delta^{(2)}(z-w),\end{aligned} \tag{5.13}$$

$$\begin{aligned}\{\bar{A}_0(z), g_\alpha(w)\} &= \frac{1}{2}g_\alpha(z)\delta^{(2)}(z-w),\\ \{\bar{A}_0(z), g_0(w)\} &= \frac{1}{2}g_0(z)\delta^{(2)}(z-w),\\ \{\bar{A}_\alpha(z), g_\beta(w)\} &= -\frac{1}{2}\epsilon_{\alpha\beta\gamma}g_\gamma(z)\delta^{(2)}(z-w) - \frac{1}{2}\delta_{\alpha,\beta}g_0(z)\delta^{(2)}(z-w),\\ \{\bar{A}_\alpha(z), g_0(w)\} &= \frac{1}{2}g_\alpha(z)\delta^{(2)}(z-w),\end{aligned} \tag{5.14}$$

$$\{g_\alpha, g_\beta\} = \{g_0, g_\beta\} = 0, \tag{5.15}$$

where $\delta^{(2)}(z-w)$ is the functional $f(0) = \int_{\Sigma_\tau} f(w)\delta^{(2)}(z-w)$.

Consider the canonical transformation of the fields

$$\bar{A}_0 \to \bar{A}_0 + c(z,\bar{z}), \quad \bar{A}_\alpha \to \bar{A}_\alpha, \quad g \to g,$$

where $c(z,\bar{z}) \in \Omega^{(0,1)}(\Sigma_\tau)$. The corresponding moment map is the functional

$$\log\det g(z) = \log(g_0^2 + \sum_\alpha g_\alpha^2). \tag{5.16}$$

Let fix the gauge as $\bar{A}_0 = 0$. The form ω' being pushed down on the reduced phase space

$$\mathcal{R}'' = \{g, \bar{A} \mid \bar{A}_0 = 0,\ g_0^2 + \sum_\alpha g_\alpha^2 = \text{const}\} \tag{5.17}$$

is non-degenerate. The Poisson structure on $\mathcal{R}''$ has the form (5.13), (5.14), (5.15) where $\bar{A}_0 = 0$. The determinant

$$\det g(z) = g_0^2 + \sum_\alpha g_\alpha^2 \tag{5.18}$$

is the Casimir functional with respect to the new brackets.

To complete the model we add some finite degrees of freedom. Consider the cotangent bundle $T^*\mathrm{SL}(2,\mathbb{C})$ represented by $\mathbf{T} \in \mathrm{sl}(2,\mathbb{C})$, $h \in \mathrm{SL}(2,\mathbb{C})$ with the canonical form

$$\omega_0 = \mathrm{tr}\left(D(\mathbf{T}h^{-1}) \wedge Dh\right). \tag{5.19}$$

The whole upstairs phase space is

$$\mathcal{R}_2 = \mathcal{R}'' \cup T\mathrm{SL}(2,\mathbb{C}) \sim \{d_{\bar{A}}, g, \mathbf{T}, h\}$$

with the symplectic structure

$$\omega = \omega' + \int_{\Sigma_\tau} \omega_0 \delta^{(2)}(z,\bar{z}). \tag{5.20}$$

The form ω is invariant under the gauge transformation (5.11) accompanied by

$$g \to f^{-1} g f, \quad (f \in \mathcal{G}), \tag{5.21}$$
$$\mathbf{T} \to f_0^{-1}\mathbf{T} f_0, \quad h \to f_0^{-1} h f_0, \quad f_0 = f(0),$$

These transformations generate the moment map

$$\mu : \mathcal{R}_2 \to \mathrm{Lie}^*(\mathcal{G})\mu = -\bar{A} + g\bar{A}g^{-1} - k\bar{\partial}gg^{-1} - \mathbf{T}\delta^2(z).$$

In the case of degree one bundles on elliptic curves the connection $\bar{A}$ can be represented as the pure gauge $\bar{A} = f^{-1}\bar{\partial} f$. The same gauge transform brings g in the form

$$g = f^{-1} L f. \tag{5.22}$$

We preserve the same notations for the transformed variables $\mathbf{T}$ and h. Then the moment condition $\mu = 0$ takes the form

$$k\bar{\partial} L L^{-1} = \mathbf{T}\delta^2(z, \bar{z}). \tag{5.23}$$

Solutions of this equation should satisfy the consistency condition

$$k(\mathrm{Res}|_{z=0} L) = \mathbf{T}(\mathrm{Res}|_{z=0} L), \tag{5.24}$$

(see Ref. 18). Taking into account the quasi-periodicity of g (5.10) we can write the general solution in the form (5.6)

$$L(z) = S_0 + i\sum_\alpha S_\alpha \varphi_\alpha(z)\sigma_\alpha.$$

Then (5.24) takes the form

$$-ikS_\alpha = \epsilon_{\alpha\beta\gamma} T_\beta S_\gamma, \tag{5.25}$$

$$\sum_\alpha S_\alpha T_\alpha = 0.$$

The last equation is consistent with the system (5.25). To come to nontrivial solutions $\vec{S}$ of (5.25) we should assumed that its determinant vanishes. It is happened in the case $k = 0$, or

$$k^2 = \sum_\alpha T_\alpha^2. \tag{5.26}$$

This condition restricts the space $T^*\mathrm{SL}(2,\mathbb{C})$ to the coadjoint orbit of $\mathrm{SL}(2,\mathbb{C})$. Fortunately, we still have at hand the constant gauge transformations preserving the gauge condition $\bar{A} = 0$. The condition (5.26) is the moment constraint in the symplectic reduction of $T^*\mathrm{SL}(2,\mathbb{C})$ generating by the constant matrices. In this way we come to the Lax operator L (5.6), where S_α are related to the elements of the coadjoint orbit via the solutions of (5.25).

Finally, we should demonstrate that the brackets (5.13), (5.14), (5.15) being restricted on shell (5.23) after the gauge fixing $\bar{A} = 0$ lead to the Sklyanin algebra (5.1). Using the Dirac procedure we define

$$\{g(z), g(w)\}^*|_{\text{on shell}} = -\int_{\Sigma_\tau}\int_{\Sigma_\tau} dv du \{g(z), \bar{A}(v)\} C(v,u) \{g(w), \bar{A}(u)\}|_{\text{on shell}}\,. \tag{5.27}$$

On shell only the second term in the r.h.s. of (5.13) is essential. In this way $C(v,u)$ is the inverse operator to

$$k\bar{\partial} : \Omega^{(0)}(\Sigma_\tau, \mathrm{sl}(2,\mathbb{C})) \to \Omega^{(0,1)}(\Sigma_\tau, \mathrm{sl}(2,\mathbb{C})),$$

$$C(v,u) : \eta(u) \in \Omega^{(0,1)}(\Sigma_\tau, \mathrm{sl}(2,\mathbb{C})) \to \int_{\Sigma_\tau} G(v-u)\eta(v) \in \Omega^{(0)}(\Sigma_\tau, \mathrm{sl}(2,\mathbb{C})).$$

Note, that its action is diagonal

$$\eta(u)_\alpha \to \int_{\Sigma_\tau} G_\alpha(v-u)\eta_\alpha(v)$$

Taking into account the quasi-periodicity (5.10) we find

$$G_\alpha = \varphi_\alpha(v).$$

Substituting in (5.27) $g(z)|_{\text{on shell}} = L(z)$ and $C(v-u)$ we come to the Yang-Baxter equation (5.7) with $r(z-w)$ replaced by $\frac{1}{k}r(z-w)$. In this way, after the symplectic reduction of the upstairs space $\mathcal{R}_2$ we come to the two-dimensional phase space $\mathcal{R}^{\mathcal{S}} = \{S_0, \mathbf{S}\}/(K_0, K_1)$.

Remark 5.1 *The upstairs space $\mathcal{R}' = \{g, \bar{A}\}$ is similar to the upstairs space for the elliptic Ruijsenaars-Schneider (ERS) system.*[18] *The only difference is that for ERS system the bundle E_2 is trivial (deg$(E_2) = 0$). There exists the symplectic map from the phase space $\mathcal{R}^{ERS}$ of the two-body ERS to $\mathcal{R}^{\mathcal{S}}$. In fact, it is the same map that works for the two-body elliptic Calogero-Moser (ECM) system -* SL$(2,\mathbb{C})$*-elliptic rotator correspondence.*[28] *It follows from the statement that ERS and ECM are governed by the same r-matrix.*[29] *This symplectic map is a twist of the r-matrices. Note that the Hamiltonian of the elliptic rotator H^{rot} comes from the representation of the Casimir*

$$\frac{1}{2}K_1 = \frac{1}{2}S_0^2 + H^{rot}.$$

5.3 Symplectic reduction in the Sklyanin algebra

Here reproduce part of results from Ref. 19. To accomplished our general scheme we introduce the phase space

$$\mathcal{R}_{\mathcal{S}} = \{S_0, \vec{S}; \xi_0, \vec{\xi}\},$$

and the symplectic form (3.19)

$$\omega = D\xi_0 \wedge DS_0 + D\vec{\xi} \wedge D\vec{S}.$$

Note that we put here the central charge $k = 0$ and therefore the Poisson sigma-model is one-dimensional. The form ω is invariant with respect the algebroid action (2.4), (3.21)

$$\delta_\varepsilon S_0 = -2J_{\beta\gamma}S_\beta S_\gamma \varepsilon_\alpha, \quad \delta_\varepsilon S_\alpha = -2\epsilon_{\alpha\beta\gamma}S_0 S_\gamma \varepsilon_\beta,$$
$$\delta_\varepsilon \xi_0 = -2\epsilon_{\alpha\beta\gamma}S_\gamma \xi_\beta \varepsilon_\alpha$$
$$\delta_\varepsilon \xi_\alpha = 2J_{\alpha\beta}S_\beta \xi_\gamma \varepsilon_0 - 2\epsilon_{\alpha\beta\gamma}S_0\xi_\gamma\varepsilon_\beta.$$

The moment map takes the form

$$\mu_0 = 2J_{\beta\gamma}S_\beta S_\gamma \xi_\alpha, \quad \mu_\alpha = 2\epsilon_{\alpha\beta\gamma}S_0 S_\gamma \xi_\beta - 2J_{\beta\gamma}S_\beta S_\gamma \xi_0.$$

Assume that

$$\mu_0 = \mu_2 = \mu_3 = 0, \quad \mu_1 = 2\nu. \tag{5.28}$$

The Hamiltonian vector field V_ε preserving this form of the moment has the form

$$\begin{gathered}\delta_\varepsilon S_1 = 0, \quad \delta\xi_1 = 0, \\ \delta_\varepsilon S_0 = J_{23}S_2S_3\varepsilon, \quad \delta_\varepsilon\xi_0 = (\xi_2S_3 - \xi_3S_2)\varepsilon, \\ \delta_\varepsilon S_2 = -S_0S_3\varepsilon, \quad \delta_\varepsilon\xi_2 = -(S_0\xi_3 + J_{23}S_3\xi_0)\varepsilon, \\ \delta_\varepsilon = S_0S_2\varepsilon, \quad \delta_\varepsilon = (S_0\xi_2 - J_{23}S_2\xi_0)\varepsilon.\end{gathered} \tag{5.29}$$

The reduced phase space

$$\mathcal{R}_{\mathcal{S}}^{red} = \{\mu_0 = \mu_2 = \mu_3 = 0, \mu_1 = 2\nu\}/\delta_\varepsilon \text{ action } (5.29)$$

is parameterized by the canonical coordinates $(v_1, v_2; u_1, u_2)$ such that

$$\xi_0 = u_1, \quad \xi_1 = 0, \quad \xi_2 = 0, \quad \xi_3 = u_2,$$
$$S_0 = v_1, \quad S_1 = 0, \quad S_2 = -\frac{\nu}{v_1u_2 - J_{32}v_2u^1}, \quad S_3 = v_2,$$

and

$$\omega|_{\mathcal{R}_{\mathcal{S}}^{red}} = \omega^{red} = Dv_1 \wedge Du_1 + Dv_2 \wedge Du_2.$$

In these coordinates the Casimirs (5.3) being reduced on $\mathcal{R}_{\mathcal{S}}^{red}$ take the form

$$K_0 = \frac{v_1^2}{2} + \frac{\nu^2}{2(v_1u_2 - J_{32}u_1v_2)^2}, \quad K_1 = \frac{1}{2}(v_1^2 + J_{32}v_2^2).$$

Since they commute this two-dimensional system is completely integrable. In the trigonometric limit ($Im\tau \to \infty$) $e_2 = e_3$ and thereby $J_{32} = 0$. Then $v_1 = const$ and K_0 describes the rational two-body Calogero system with the coupling constant depending on v_1. The Sklyanin algebra is still quadratic in this limit. Nevertheless, the result of the symplectic reduction is essentially the same as when the upstairs space has the linear $\mathrm{gl}(2,\mathbb{C})$ brackets.[30]

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